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SIGNED TOTAL k-DOMATIC NUMBERS OF GRAPHS

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Abstract. Let k be a positive integer and let G be a simple graph with vertex set $V(G)$. A function $f : V(G) \longrightarrow \{-1,1\}$ is called a *signed total k-dominating function* if $\sum_{u\in N(v)} f(u) \ge k$ for each vertex $v \in V(G)$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed total k-dominating functions of G with the property that $\sum_{i=1}^{d} f_i(v) \le 1$, for each $v \in V(G)$, is called a *signed total k-dominating family* (of functions) of G. The maximum number of functions in a signed total k-dominating family of G is the *signed total k-domatic number* of G, denoted by $d_{kS}^t(G)$. In this note we initiate the study of the signed total k-domatic numbers of graphs and present some sharp upper bounds for this parameter. We also determine the signed total k-domatic numbers of complete graphs and complete bipartite graphs.

1. Introduction

In this paper, G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *open neighborhood* $N(S)$ of a set $S \subseteq V(G)$ is the set $\bigcup_{v\in S} N(v)$. The minimum degree of G, denoted by $\delta(G)$, is $\min\{|N(v)| \mid v \in V(G)\}$. Consult [5] for the notation and terminology which are not defined here.

For a real-valued function $f : V(G) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v\in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$. So $w(f) = f(V)$. Let $k \ge 1$ be an integer and let G be a graph with $\delta(G) \ge k$. A *signed total k-dominating function* (STkDF) is a function $f : V(G) \to \{-1,1\}$ satisfying $\sum_{u\in N(v)} f(u) \ge k$ for every $v \in V(G)$. The minimum of the values of $\sum_{v\in V(G)} f(v)$, taken over all signed total k-dominating functions f, is called the signed total k-domination number and is denoted by $\gamma_{kS}^t(G)$. As assumption $\delta(G) \ge k$ is clearly necessary, we will always assume that when we discuss $\gamma_{kS}^t(G)$ all graphs involved satisfy $\delta(G) \ge k$. In the special case when $k = 1$,

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$\gamma^t_{kS}(G)$ is the *signed total domination number* investigated in [2, 6]. The signed total k-domination numbers of graphs was introduced by Wang [4].

A set $\{f_1, f_2, \ldots, f_d\}$ of signed total k-dominating functions on G with $\sum_{i=1}^{d} f_i(v) \leq 1$, for each $v \in V(G)$, is called a *signed total k-dominating family* (STkD family) of G. The maximum number of functions in a signed total k-dominating family on G is the *signed total k-domatic number* of G, denoted by $d^t_{kS}(G)$. The signed total k-domatic number is well-defined and $d^t_{kS}(G) \geq 1$ for all graphs G in which $\delta(G) \geq k$, since the set consisting of any one STkD function forms a STkD family of G. A d^t_{kS}*-family* of a graph G is a STkD family consists of $d^t_{kS}(G)$ STkD functions. The signed total 1-domatic number $d^t_{1S}(G)$ is the usual *signed total domatic number* $d^t_S(G)$, which was introduced by Henning in [3] and has been studied by several authors (see for example [1]).

In this note, we first study some basic properties of $d^t_{kS}(G)$ and find some sharp upper bounds for this parameter. Then we determine the signed total k-domatic numbers of complete graphs and of complete bipartite graphs, generalizing Propositions A and B.

We make use of the following results and observations in this paper.

Proposition A ([3]). *If $G = K_n$ is the complete graph of order $n \geq 2$, then*

$$d^t_S(K_n) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

Proposition B ([3]). *For $m \geq n \geq 1$,*

$$d^t_S(K_{m,n}) = \begin{cases} n & \text{if } n \text{ and } m \text{ are odd,} \\ \min\{n, \frac{m}{2} - \lceil \frac{m+2}{4} \rceil + \lfloor \frac{m+2}{4} \rfloor\} & \text{if } n \text{ is odd and } m \text{ is even,} \\ \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Observation 1. *Let G be a graph of order n and $k \in \{n-2, n-1\}$. Then $\gamma^t_{kS}(G) = n$ and hence, $d^t_{kS}(G) = 1$.*

Observation 2. *Let G be a graph of order n. Then $\gamma^t_{kS}(G) = n$ if and only if $k \leq \delta(G) \leq k+1$ and for each $v \in V(G)$ there exists a vertex $u \in N(v)$ such that* $\deg(u) = k$ *or* $\deg(u) = k+1$.

Proof. If $k \leq \delta(G) \leq k+1$ and for each $v \in V(G)$ there exists a vertex $u \in N(v)$ such that $\deg(u) = k$ or $\deg(u) = k+1$, then trivially $\gamma^t_{kS}(G) = n$.

Conversely, assume that $\gamma^t_{kS}(G) = n$. By assumption $k \leq \delta(G)$. Let, to the contrary, $\delta(G) > k+1$ or there exists a vertex $v \in V(G)$ such that $\deg(u) \geq k+2$ for each $u \in N(v)$. If $\delta(G) > k+1$, define $f : V(G) \to \{-1, 1\}$ by $f(v) = -1$ for some fixed v and $f(x) = 1$ for $x \in V(G) \setminus \{v\}$. Obviously, f is a signed total k-dominating function of G with weight less than n, which is a contradiction. Thus $k \leq \delta(G) \leq k+1$. Now let $v \in V(G)$ and $\deg(u) \geq k+2$ for each $u \in N(v)$.

Define $f : V(G) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(G) \setminus \{v\}$. Again, f is a signed total k-dominating function of G, which is a contradiction. This completes the proof. □

The following theorem generalizes the result on $\gamma^t_{kS}(K_{n,n})$ obtained in [4].

Theorem 3. *Let $k \geq 1$ be an integer. Then for every integers $m, n \geq k$,*

$$(3) \qquad \gamma^t_{kS}(K_{m,n}) = \begin{cases} 2k & \text{if} \quad m \equiv n \equiv k \pmod 2 \\ 2k+1 & \text{if} \quad m \equiv k+1 \pmod 2, n \equiv k \pmod 2 \\ 2k+2 & \text{if} \quad m \equiv n \equiv k+1 \pmod 2. \end{cases}$$

Proof. Let the partite sets of a $K_{m,n}$ be $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. We consider three cases.

Case 1. $m \equiv n \equiv k \pmod 2$. First note that if f is a STkDF of $K_{m,n}$, then $\sum_{i=1}^m f(x_i) \geq k$ and $\sum_{i=1}^n f(y_i) \geq k$, which implies $\gamma^t_{kS}(K_{m,n}) \geq 2k$. Label $\frac{m+k}{2}$ vertices of X with $+1$ and the rest with -1. Similarly, label $\frac{n+k}{2}$ vertices of Y with $+1$ and the rest with -1. It is clear that this labeling defines a STkDF, say f, of $K_{m,n}$. Since $f(N(v)) = k$ for every vertex $v \in (X \cup Y)$, it follows that $\gamma^t_{kS}(K_{m,n}) = w(f) = 2k$.

Case 2. $m \equiv k+1 \pmod 2$, $n \equiv k \pmod 2$. First note that if f is a STkDF of $K_{m,n}$, then $\sum_{i=1}^m f(x_i) \geq k+1$ and $\sum_{i=1}^n f(y_i) \geq k$ since $m \equiv k+1 \pmod 2$. Label $\frac{m+k+1}{2}$ vertices of X with $+1$ and the rest with -1. Similarly, label $\frac{n+k}{2}$ vertices of Y with $+1$ and the rest with -1. It is clear that this labeling defines a STkDF, say f, of $K_{m,n}$. Since $f(N(y)) = k+1$ for every vertex $y \in Y$ and $f(N(x)) = k$ for every $x \in X$, it follows that $\gamma^t_{kS}(K_{m,n}) = w(f) = 2k+1$.

Case 3. $m \equiv n \equiv k+1 \pmod 2$. First note that if f is a STkDF of $K_{m,n}$, since $m \equiv n \equiv k+1 \pmod 2$, $\sum_{i=1}^m f(x_i) \geq k+1$ and $\sum_{i=1}^n f(y_i) \geq k+1$. Label $\frac{m+k+1}{2}$ vertices of X with $+1$ and the rest with -1. Similarly, label $\frac{n+k+1}{2}$ vertices of Y with $+1$ and the rest with -1. It is clear that this labeling defines a STkDF, say f, of $K_{m,n}$. Since $f(N(v)) = k+1$ for every vertex $v \in (X \cup Y)$, it follows that $\gamma^t_{kS}(K_{m,n}) = w(f) = 2k+2$. □

2. Basic properties and upper bounds

In this section we present basic properties of the signed total k-domatic number and find some sharp upper bounds for this parameter. Our first result is obtained by the definition of the signed total k-domatic number.

Theorem 4. *Let G be a graph of order n and $\delta(G) \geq k > 0$. Then $\gamma^t_{kS}(G) \cdot d^t_{kS}(G) \leq n$. Moreover if $\gamma^t_{kS}(G) \cdot d^t_{kS}(G) = n$, then for each $d = d^t_{kS}$-family $\{f_1, \ldots, f_d\}$ of G each function f_i is a γ^t_{kS}-function and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a STkD family of G such that $d = d^t_{kS}(G)$ and let $v \in V$. Then

$$\begin{array}{rcl} d \cdot \gamma^t_{kS}(G) & = & \sum_{i=1}^d \gamma^t_{kS}(G) \\ & \leq & \sum_{i=1}^d \sum_{v\in V} f_i(v) \\ & = & \sum_{v\in V} \sum_{i=1}^d f_i(v) \\ & \leq & \sum_{v\in V} 1 \\ & = & n. \end{array}$$

If $\gamma^t_{kS}(G) \cdot d^t_{kS}(G) = n$, then the two inequalities occurring in the proof become equalities. Hence, for the d^t_{kS}-family $\{f_1, \ldots, f_d\}$ of G and for each i, $\sum_{v\in V} f_i(v) = \gamma^t_{kS}(G)$, thus each function f_i is a γ^t_{kS}-function, and $\sum_{i=1}^d f_i(v) = 1$ for all v. □

Corollary 5. *If G is a graph of order n, then $\gamma^t_{kS}(G) + d^t_{kS}(G) \leq n + 1$.*

Proof. By Theorem 4,

$$\gamma^t_{kS}(G) + d^t_{kS}(G) \leq d^t_{kS}(G) + \frac{n}{d^t_{kS}(G)}. \tag{4}$$

Using the fact that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$, this inequality leads to the desired bound immediately. □

Corollary 6. *Let G be a graph of order $n \geq 4$. If $2 \leq \gamma^t_{kS}(G) \leq n - 1$, then*

$$\gamma^t_{kS}(G) + d^t_{kS}(G) \leq n.$$

Proof. Theorem 4 implies that

$$\gamma^t_{kS}(G) + d^t_{kS}(G) \leq \gamma^t_{kS}(G) + \frac{n}{\gamma^t_{kS}(G)}. \tag{5}$$

If we define $x = \gamma^t_{kS}(G)$ and $g(x) = x + n/x$ for $x > 0$, then because $2 \leq \gamma^t_{kS}(G) \leq n - 1$, we have to determine the maximum of the function g on the interval $I : 2 \leq x \leq n - 1$. It is easy to see that

$$\begin{array}{rcl} \max\limits_{x\in I}\{g(x)\} & = & \max\{g(2), g(n-1)\} \\ & = & \max\{2 + \frac{n}{2}, n - 1 + \frac{n}{n-1}\} \\ & = & n - 1 + \frac{n}{n-1} < n + 1, \end{array}$$

and we obtain $\gamma^t_{kS}(G) + d^t_{kS}(G) \leq n$. This completes the proof. □

Corollary 7. *Let G be a graph of order n and let $k \geq 1$ be an integer. If $\min\{\gamma^t_{kS}(G), d^t_{kS}(G)\} \geq 2$, then*

$$\gamma^t_{kS}(G) + d^t_{kS}(G) \leq \frac{n}{2} + 2.$$

Proof. Since $\min\{\gamma^t_{kS}(G), d^t_{kS}(G)\} \geq 2$, it follows by Theorem 4 that $2 \leq d^t_{kS}(G) \leq \frac{n}{2}$. By (4) and the fact that the maximum of $g(x) = x + n/x$ on the interval $2 \leq x \leq n/2$ is $g(2) = g(n/2)$, we see that

$$\gamma^t_{kS}(G) + d^t_{kS}(G) \leq d^t_{kS}(G) + \frac{n}{d^t_{kS}(G)} \leq \frac{n}{2} + 2.$$

□

Observation 1 shows that Corollary 7 is no longer true if $\min\{\gamma^t_{kS}(G), d^t_{kS}(G)\} = 1$.

Theorem 8. *The signed total k-domatic number of a graph is an odd integer.*

Proof. Let G be a graph, and suppose that $d = d^t_{kS}(G)$ is even. Let $\{f_1, f_2, \ldots, f_d\}$ be the corresponding signed total k-dominating family of G. If $u \in V(G)$, then $\sum_{i=1}^{d} f_i(u) \leq 1$. But on the left-hand side of this inequality, a sum of an even number of odd summands occurs. Therefore it is an even number, and we obtain $\sum_{i=1}^{d} f_i(u) \leq 0$ for each $u \in V(G)$. This forces

$$\begin{aligned} d &= \textstyle\sum_{i=1}^{d} 1 \\ &\leq \textstyle\sum_{i=1}^{d} (\frac{1}{k} \sum_{u \in N(v)} f_i(u)) \\ &= \textstyle\frac{1}{k} \sum_{u \in N(v)} \sum_{i=1}^{d} f_i(u) \\ &\leq 0, \end{aligned}$$

which is a contradiction. □

Theorem 9. *Let G be a graph and $v \in V(G)$. Then*

$$d^t_{kS}(G) \leq \begin{cases} \dfrac{\deg(v)}{k} & \text{if} \quad \deg(v) \equiv k \pmod 2 \\ \dfrac{\deg(v)}{k+1} & \text{if} \quad \deg(v) \equiv k+1 \pmod 2. \end{cases}$$

Moreover, if the equality holds, then for each function f_i of a STkD family $\{f_1, f_2, \ldots, f_d\}$ and for every $u \in N(v)$, $\sum_{u \in N(v)} f_i(u) = k$ if $\deg(v) \equiv k \pmod 2$, $\sum_{u \in N(v)} f_i(u) = k+1$ if $\deg(v) \equiv k+1 \pmod 2$ and $\sum_{i=1}^{d} f_i(u) = 1$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a STkD family of G such that $d = d^t_{kS}(G)$. If $\deg(v) \equiv k \pmod 2$, then

$$\begin{aligned} d &= \textstyle\sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \frac{1}{k} \sum_{u \in N(v)} f_i(u) \\ &= \textstyle\frac{1}{k} \sum_{u \in N(v)} \sum_{i=1}^{d} f_i(u) \leq \frac{1}{k} \sum_{u \in N(v)} 1 \\ &= \textstyle\frac{\deg(v)}{k}. \end{aligned}$$

Similarly, if $\deg(v) \equiv k+1 \pmod 2$, then

$$\begin{aligned} d &= \textstyle\sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \frac{1}{k+1} \sum_{u \in N(v)} f_i(u) \\ &= \textstyle\frac{1}{k+1} \sum_{u \in N(v)} \sum_{i=1}^{d} f_i(u) \leq \frac{1}{k+1} \sum_{u \in N(v)} 1 \\ &= \textstyle\frac{\deg(v)}{k+1}. \end{aligned}$$

If $d^t_{kS}(G) = \frac{\deg(v)}{k}$ when $\deg(v) \equiv k \pmod 2$ or $d^t_{kS}(G) = \frac{\deg(v)}{k+1}$ when $\deg(v) \equiv k+1 \pmod 2$, then the two inequalities occurring in the proof of each corresponding case become equalities, which gives the properties given in the statement. □

Corollary 10. *Let G be a graph and $1 \le k \le \delta(G)$. Then*

$$d^t_{kS}(G) \le \begin{cases} \dfrac{\delta(G)}{k} & \text{if } \delta(G) \equiv k \pmod 2 \\ \dfrac{\delta(G)}{k+1} & \text{if } \delta(G) \equiv k+1 \pmod 2. \end{cases}$$

Corollary 11. *Let $1 \le k \le n \le m$ be integers. Then*

$$d^t_{kS}(K_{n,m}) \le \begin{cases} \min\{\frac{n}{k}, \frac{m}{k+1}\} & \text{if } n \equiv k \pmod 2 \text{ and } m \not\equiv n \pmod 2 \\ \frac{n}{k} & \text{if } n \equiv k \pmod 2 \text{ and } m \equiv n \pmod 2 \\ \frac{n}{k+1} & \text{if } n \equiv k+1 \pmod 2. \end{cases}$$

The next two results are immediate consequences of Theorems 8 and 9.

Corollary 12. *For any tree T, $d^t_S(T) = 1$.*

Corollary 13. *If C_n is the cycle on n vertices, then $d^t_S(C_n) = d^t_{2S}(C_n) = 1$.*

Corollary 14. *Let G be a graph of order n. Then $\gamma^t_{kS}(G) + d^t_{kS}(G) = n+1$ if and only if $k \le \delta(G) \le k+1$ and for each $v \in V(G)$ there exists a vertex $u \in N(v)$ such that $\deg(u) = k$ or $\deg(u) = k+1$.*

Proof. If $k \le \delta(G) \le k+1$ and for each $v \in V(G)$ there exists a vertex $u \in N(v)$ such that $\deg(u) = k$ or $\deg(u) = k+1$, then $\gamma^t_{kS}(G) = n$ by Observation 2. Hence, $d_{kS}(G) = 1$ and the result follows.

Conversely, let $\gamma^t_{kS}(G) + d^t_{kS}(G) = n+1$. The result is obviously true for $n = 2, 3$. Assume $n \ge 4$. By Corollary 7, we may assume $\min\{\gamma^t_{kS}(G), d^t_{kS}(G)\} = 1$. If $\gamma^t_{kS}(G) = 1$, then $d^t_{kS}(G) = n$, which is a contradiction by Theorem 9. If $d^t_{kS}(G) = 1$, then $\gamma^t_{kS}(G) = n$ and the result follows by Observation 2. □

Theorem 15. *For every graph G of order n and $1 \le k \le \min\{\delta(G), \delta(\overline{G})\}$,*

$$d^t_{kS}(G) + d^t_{kS}(\overline{G}) \le \frac{n-1}{k}, \tag{6}$$

and equality in (6) *implies that G is a regular graph.*

Proof. Since $\delta(G) + \delta(\overline{G}) \le n-1$, by Corollary 10

$$d_{kS}(G) + d_{kS}(\overline{G}) \le \frac{\delta(G)}{k} + \frac{\delta(\overline{G})}{k} \le \frac{n-1}{k}.$$

If G is not regular, then $\delta(G) + \delta(\overline{G}) \le n-2$, hence $d^t_{kS}(G) + d^t_{kS}(\overline{G}) \le \frac{n-2}{k}$. □

3. The signed total k-domatic number of complete graphs

In this section, we determine the value of signed total k-domatic number of a complete graph. First we determine the signed total k-domination number of K_n.

Lemma 16 ([4]). *For $n \geq 2$,*

$$\gamma^t_{kS}(K_n) = \begin{cases} k+2 & n \equiv k \pmod 2 \\ k+1 & n \equiv k+1 \pmod 2. \end{cases} \tag{7}$$

Proof. Assume $V(K_n) = \{v_1, \ldots, v_n\}$ is the vertex set of K_n. Suppose that f is a signed total k-dominating function of K_n and $f(v) = 1$ for some $v \in V(G)$. If $n \equiv k \pmod 2$, then $f(N(v)) \geq k+1$ and hence, $f(V(G)) = f(v) + f(N(v)) \geq k+2$. Thus $\gamma^t_{kS}(K_n) \geq k+2$. Now define $f : V(K_n) \to \{-1, 1\}$ by $f(v_i) = -1$ for $1 \leq i \leq \frac{n-k}{2} - 1$ and $f(v_i) = 1$ when $\frac{n-k}{2} \leq i \leq n$. It is easy to see that f is a signed total k-dominating function on K_n with $f(V(K_n)) = k+2$.

If $n \equiv k+1 \pmod 2$, then $f(V(G)) = f(v) + f(N(v)) \geq k+1$ and hence, $\gamma^t_{kS}(K_n) \geq k+1$. Now define $f : V(K_n) \to \{-1, 1\}$ by $f(v_i) = -1$ for $1 \leq i \leq \frac{n-k-1}{2}$ and $f(v_i) = 1$ when $\frac{n-k+1}{2} \leq i \leq n$. It is easy to see that f is a signed total k-dominating function on K_n with $f(V(K_n)) = k+1$. This completes the proof. □

The next result is a generalization of Proposition A.

Theorem 17. *For $n \geq 2$,*

$$d^t_{kS}(K_n) = \begin{cases} \lfloor \frac{n}{k+2} \rfloor & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k+2} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+2} \rfloor - 1 & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k+2} \rfloor \text{ is even} \\ \lfloor \frac{n}{k+1} \rfloor & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+1} \rfloor - 1 & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is even.} \end{cases}$$

Proof. By Lemma 16 and Observation 1 we may assume $k \leq n-3$. Let $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$ be the vertex set of K_n. We consider two cases.

Case 1. $n \equiv k \pmod 2$. Suppose that $n = (k+2)q + r$, where q is a positive integer and $0 \leq r \leq k+1$. By Lemma 16, $\gamma^t_{kS}(K_n) = k+2$. Hence, by Theorems 4 and 8, $d^t_{kS}(K_n) \leq q$ if q is odd and $d^t_{kS}(K_n) \leq q-1$ if q is even.

Subcase 1.1 q is odd. Then r is even. Define the functions $f_1, \ldots, f_q$ as follows.

$$\begin{aligned} f_1(x_i) &= 1 \quad \text{if} \quad 0 \leq i \leq \tfrac{(k+2)(q-1)}{2} + k + 1, \\ f_1(x_i) &= -1 \quad \text{if} \quad \tfrac{(k+2)(q-1)}{2} + k + 2 \leq i \leq (k+2)q - 1 \end{aligned}$$

and for $2 \leq j \leq q$ and $0 \leq i \leq (k+2)q - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+2)}),$$

where the sum is taken modulo $(k+2)q$. In addition, if $r > 0$,

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \leq j \leq q \text{ and } (k+2)q \leq i \leq n-1.$$

It is easy to see that f_j is a signed total k-dominating function of K_n for each $1 \le j \le q$ and $\{f_1, \dots, f_q\}$ is a signed total k-dominating family of K_n. Hence $d^t_{kS}(K_n) \ge q$. Therefore $d^t_{kS}(K_n) = q$, as desired.

Subcase 1.2 q is even. Then $r + k + 2$ is even. Define the functions $f_1, \dots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \le i \le \frac{(k+2)(q-2)}{2} + k + 1, \\ f_1(x_i) = -1 & \text{if} & \frac{(k+2)(q-2)}{2} + k + 2 \le i \le (k+2)(q-1) - 1 \end{array}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+2)(q-1) - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+2)}),$$

where the sum is taken modulo $(k+2)(q-1)$. In addition,

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+2)(q-1) \le i \le n-1.$$

It is easy to see that f_j is a signed total k-dominating function of K_n for each $1 \le j \le q-1$ and $\{f_1, \dots, f_{q-1}\}$ is a signed total k-dominating family on K_n. Hence, $d^t_{kS}(K_n) \ge q-1$ and so $d^t_{kS}(K_n) = q-1$, as desired.

Case 2. $n \equiv k+1 \pmod 2$. Suppose that $n = (k+1)q + r$, where q is a positive integer and $0 \le r \le k$. By Lemma 16, $\gamma^t_{kS}(K_n) = k+1$. Hence, by Theorems 4 and 8, $d^t_{kS}(K_n) \le q$ if q is odd and $d^t_{kS}(K_n) \le q-1$ if q is even.

Subcase 2.1 q is odd. Then r is even. Define the functions $f_1, \dots, f_q$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \le i \le \frac{(k+1)(q-1)}{2} + k, \\ f_1(x_i) = -1 & \text{if} & \frac{(k+1)(q-1)}{2} + k + 1 \le i \le (k+1)q - 1 \end{array}$$

and for $2 \le j \le q$ and $0 \le i \le (k+1)q - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)q$. In addition, if $r > 0$,

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)q \le i \le n-1.$$

It is easy to see that f_j is a signed total k-dominating function of K_n for each $1 \le j \le q$ and $\{f_1, \dots, f_q\}$ is a signed total k-dominating family of K_n. Hence, $d^t_{kS}(K_n) \ge q$. Therefore $d^t_{kS}(K_n) = q$, as desired.

Subcase 2.2 q is even. Then $r + k + 1$ is even. Define the functions $f_1, \dots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \le i \le \frac{(k+1)(q-2)}{2} + k, \\ f_1(x_i) = -1 & \text{if} & \frac{(k+1)(q-2)}{2} + k + 1 \le i \le (k+1)(q-1) - 1 \end{array}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+1)(q-1) - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)(q-1)$. In addition,

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)(q-1) \le i \le n-1.$$

It is easy to see that f_j is a signed total k-dominating function of K_n for each $1 \leq j \leq q-1$ and $\{f_1, \ldots, f_{q-1}\}$ is a signed total k-dominating family of K_n. Hence, $d^t_{kS}(K_n) \geq q-1$ and so $d^t_{kS}(K_n) = q-1$, as desired. □

4. The signed total k-domatic number of $K_{n,m}$

In this section, we first determine the signed total k-domatic numbers for complete bipartite graphs $K_{n,n}$. Then we use this result to find the signed total k-domatic numbers for complete bipartite graphs $K_{n,m}$. This generalizes Proposition B for $k \geq 1$.

Theorem 18. *For $n \geq 1$,*

$$d^t_{kS}(K_{n,n}) = \begin{cases} \lfloor \frac{n}{k} \rfloor & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k} \rfloor \text{ is odd,} \\ \lfloor \frac{n}{k} \rfloor - 1 & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k} \rfloor \text{ is even,} \\ \lfloor \frac{n}{k+1} \rfloor & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is odd,} \\ \lfloor \frac{n}{k+1} \rfloor - 1 & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is even.} \end{cases}$$

Proof. By Theorem 3 and Observation 1, we may assume $k \leq n-2$. Let $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$ be the partite sets of $K_{n,n}$. We consider two cases.

Case 1. $n \equiv k \pmod 2$. Suppose that $n = kq + r$, where q is a positive integer and $0 \leq r < k$. By Theorem 3, $\gamma^t_{kS}(K_{n,n}) = 2k$. Hence, by Theorems 4 and 8, $d^t_{kS}(K_{n,n}) \leq q$ if q is odd and $d^t_{kS}(K_{n,n}) \leq q-1$ if q is even.

Subcase 1.1 q is odd. Then r is even. Define the functions $f_1, \ldots, f_q$ as follows.

$$\begin{array}{lll} f_1(x_i) = f_1(y_i) = 1 & \text{if} & 0 \leq i \leq \frac{k(q-1)}{2} + k - 1, \\ f_1(x_i) = f_1(y_i) = -1 & \text{if} & \frac{k(q-1)}{2} + k \leq i \leq kq - 1 \end{array}$$

and for $2 \leq j \leq q$ and $0 \leq i \leq kq-1$,

$$f_j(x_i) = f_{j-1}(x_{i+2k}) \text{ and } f_j(y_i) = f_{j-1}(y_{i+2k}),$$

where the sum is taken modulo kq. In addition, if $r > 0$,

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \leq j \leq q \text{ and } kq \leq i \leq n-1.$$

It is easy to see that f_j is a signed total k-dominating function of $K_{n,n}$ for each $1 \leq j \leq q$ and $\{f_1, \ldots, f_q\}$ is a signed total k-dominating family of $K_{n,n}$. Hence $d^t_{kS}(K_{n,n}) \geq q$. Therefore $d^t_{kS}(K_{n,n}) = q$, as desired.

Subcase 1.2 q is even. Then $r+k$ is even. Define the functions $f_1, \ldots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = f_1(y_i) = 1 & \text{if} & 0 \leq i \leq \frac{k(q-2)}{2} + k - 1, \\ f_1(x_i) = f_1(y_i) = -1 & \text{if} & \frac{k(q-2)}{2} + k \leq i \leq k(q-1) - 1 \end{array}$$

and for $2 \leq j \leq q-1$ and $0 \leq i \leq k(q-1)-1$,

$$f_j(x_i) = f_{j-1}(x_{i+2k}) \text{ and } f_j(y_i) = f_{j-1}(y_{i+2k}),$$

where the sum is taken modulo $k(q-1)$. In addition,

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } k(q-1) \le i \le n-1.$$

It is easy to see that f_j is a signed total k-dominating function of $K_{n,n}$ for each $1 \le j \le q-1$ and $\{f_1, \dots, f_{q-1}\}$ is a signed total k-dominating family on $K_{n,n}$. Hence, $d^t_{kS}(K_{n,n}) \ge q-1$ and so $d^t_{kS}(K_{n,n}) = q-1$ as desired.

Case 2. $n \not\equiv k \pmod 2$. Then $n \equiv k+1 \pmod 2$. Suppose that $n = (k+1)q + r$, where q is a positive integer and $0 \le r \le k$. By Theorem 3, $\gamma^t_{kS}(K_{n,n}) = 2(k+1)$. Hence, by Theorems 4 and 8, $d^t_{kS}(K_{n,n}) \le q$ if q is odd and $d^t_{kS}(K_{n,n}) \le q-1$ if q is even.

Subcase 2.1 q is odd. Then r is even. Define the functions $f_1, \dots, f_q$ as follows.

$$\begin{aligned} f_1(x_i) = f_1(y_i) = 1 \quad & \text{if} \quad 0 \le i \le \tfrac{(k+1)(q-1)}{2} + k, \\ f_1(x_i) = f_1(y_i) = -1 \quad & \text{if} \quad \tfrac{(k+1)(q-1)}{2} + k + 1 \le i \le (k+1)q - 1 \end{aligned}$$

and for $2 \le j \le q$ and $0 \le i \le (k+1)q - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}) \text{ and } f_j(y_i) = f_{j-1}(y_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)q$. In addition, if $r > 0$,

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)q \le i \le n-1.$$

It is easy to see that f_j is a signed total k-dominating function of $K_{n,n}$ for each $1 \le j \le q$ and $\{f_1, \dots, f_q\}$ is a signed total k-dominating family of $K_{n,n}$. Hence, $d^t_{kS}(K_{n,n}) \ge q$. Therefore $d^t_{kS}(K_{n,n}) = q$, as desired.

Subcase 2.2 q is even. Then $r + k + 1$ is even. Define the functions $f_1, \dots, f_{q-1}$ as follows.

$$\begin{aligned} f_1(x_i) = f_1(y_i) = 1 \quad & \text{if} \quad 0 \le i \le \tfrac{(k+1)(q-2)}{2} + k, \\ f_1(x_i) = f_1(y_i) = -1 \quad & \text{if} \quad \tfrac{(k+1)(q-2)}{2} + k + 1 \le i \le (k+1)(q-1) - 1 \end{aligned}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+1)(q-1) - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}) \text{ and } f_j(y_i) = f_{j-1}(y_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)(q-1)$. In addition,

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)(q-1) \le i \le n-1.$$

It is easy to see that f_j is a signed total k-dominating function of $K_{n,n}$ for each $1 \le j \le q-1$ and $\{f_1, \dots, f_{q-1}\}$ is a signed total k-dominating family of $K_{n,n}$. Hence, $d^t_{kS}(K_{n,n}) \ge q-1$ and so $d^t_{kS}(K_{n,n}) = q-1$, as desired. □

Now we are ready to prove the main theorem of this section.

Theorem 19. *Let $1 \le k \le n \le m$ be integers. If $m \equiv n \pmod 2$, then $d^t_{kS}(K_{n,m}) = d^t_{kS}(K_{n,n})$. If $m \not\equiv n \pmod 2$, then*

$$d^t_{kS}(K_{n,m})$$

$$= \begin{cases} \lfloor \frac{n}{k} \rfloor & \textit{if} \quad n \equiv k \pmod 2,\ \lfloor \frac{n}{k} \rfloor \le \lfloor \frac{m}{k+1} \rfloor \textit{ and } \lfloor \frac{n}{k} \rfloor \textit{ is odd,} \\ \lfloor \frac{n}{k} \rfloor - 1 & \textit{if} \quad n \equiv k \pmod 2,\ \lfloor \frac{n}{k} \rfloor \le \lfloor \frac{m}{k+1} \rfloor \textit{ and } \lfloor \frac{n}{k} \rfloor \textit{ is even,} \\ \lfloor \frac{m}{k+1} \rfloor & \textit{if} \quad n \equiv k \pmod 2,\ \lfloor \frac{n}{k} \rfloor \ge \lfloor \frac{m}{k+1} \rfloor \textit{ and } \lfloor \frac{m}{k+1} \rfloor \textit{ is odd,} \\ \lfloor \frac{m}{k+1} \rfloor - 1 & \textit{if} \quad n \equiv k \pmod 2,\ \lfloor \frac{n}{k} \rfloor \ge \lfloor \frac{m}{k+1} \rfloor \textit{ and } \lfloor \frac{m}{k+1} \rfloor \textit{ is even,} \\ \lfloor \frac{n}{k+1} \rfloor & \textit{if} \quad n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \textit{ is odd,} \\ \lfloor \frac{n}{k+1} \rfloor - 1 & \textit{if} \quad n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \textit{ is even.} \end{cases}$$

Proof. First let $m \equiv n \pmod 2$. By Corollary 11, $d^t_{kS}(K_{n,m}) \le \frac{n}{k}$ if $n \equiv k \pmod 2$ and $d^t_{kS}(K_{n,m}) \le \frac{n}{k+1}$ if $n \equiv k+1 \pmod 2$. Hence, by Theorem 18, $d^t_{kS}(K_{n,m}) \le d^t_{kS}(K_{n,n})$. Let $\{f_1, f_2, \dots, f_d\}$ be a STkD family of $K_{n,n}$ and $d = d^t_{kS}(K_{n,n})$. We extend this family to a STkD family of $K_{n,m}$. Add the new vertices $\{w_1, w_2, \dots, w_{m-n}\}$ to a partite set of $K_{n,n}$ and join each w_i, $1 \le i \le m-n$, to every vertex in the other partite set of $K_{n,n}$ to obtain a $K_{n,m}$. Define $f^*_j : K_{n,m} \to \{-1, 1\}$ as follows: $f^*_j(v) = f_j(v)$ if $v \in V(K_{n,n})$ and $f_j(w_i) = (-1)^{i+j}$ for $1 \le i \le m-n$ and $1 \le j \le d$. Since $m \equiv n$ (mod 2), $\{f^*_1, f^*_2, \dots, f^*_d\}$ is a STkD family of $K_{n,m}$, so $d^t_{kS}(K_{n,m}) \ge d$, and hence $d^t_{kS}(K_{n,m}) = d^t_{kS}(K_{n,n})$.

Now assume $m \not\equiv n \pmod 2$. Let $X = \{x_0, x_1, \dots, x_{n-1}\}$ and $Y = \{y_0, y_1, \dots, y_{m-1}\}$ be the partite sets of $K_{n,m}$. We consider two cases.

Case 1. $n \equiv k \pmod 2$. Then $m \equiv k+1 \pmod 2$. Assume $n = kq_1 + r_1$, where $0 \le r_1 \le k-1$, and $m = (k+1)q_2 + r_2$, where $0 \le r_2 \le k$.

Subcase 1.1 $q_1 \le q_2$. Then $d^t_{kS}(K_{n,m}) \le q_1$ if q_1 is odd by Corollary 11, and $d^t_{kS}(K_{n,m}) \le q_1 - 1$ if q_1 is even by Corollary 11 and Theorem 8. Let $q_2 = q_1 + s$ for some $s \ge 0$. Then

$$m = (k+1)(q_1 + s) + r_2 = (k+1)q_1 + s(k+1) + r_2.$$

If q_1 is odd, by assumptions, r_1 and $s(k+1) + r_2$ are both even.

Define the functions $f_1, \dots, f_{q_1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \le i \le \frac{k(q_1+1)}{2} - 1, \\ f_1(x_i) = -1 & \text{if} & \frac{k(q_1+1)}{2} \le i \le kq_1 - 1, \\ f_1(y_j) = 1 & \text{if} & 0 \le j \le \frac{(k+1)(q_1+1)}{2} - 1, \\ f_1(y_j) = -1 & \text{if} & \frac{(k+1)(q_1+1)}{2} \le j \le (k+1)q_1 - 1 \end{array}$$

and for $2 \le \ell \le q_1$, $0 \le i \le kq_1 - 1$ and $0 \le j \le (k+1)q_1 - 1$,

$$f_\ell(x_i) = f_{\ell-1}(x_{i+2k}) \text{ and } f_\ell(y_j) = f_{\ell-1}(y_{j+2(k+1)}),$$

where the sum is taken modulo kq_1 if i is involved, and modulo $(k+1)q_1$ if j is involved. In addition, for $1 \le \ell \le q_1$, $kq_1 \le i \le n-1$ and $(k+1)q_1 \le j \le m-1$,

$$f_\ell(x_i) = (-1)^{i+\ell} \text{ and } f_\ell(y_j) = (-1)^{j+\ell}.$$

It is straightforward to see that $\{f_1, \dots, f_{q_1}\}$ is a signed total k-dominating family of $K_{n,m}$. Hence $d^t_{kS}(K_{n,m}) \ge q_1$. Therefore $d^t_{kS}(K_{n,m}) = q_1$, as desired.

Now let q_1 be even. We have $n = k(q_1 - 1) + k + r_1$ and

$$m = (k+1)(q_1 + s) + r_2 = (k+1)(q_1 - 1) + (s+1)(k+1) + r_2.$$

By assumptions, $k+r_1$ and $(s+1)(k+1)+r_2$ are both even. Define the functions $f_1,\dots,f_{q_1-1}$ as follows.

$$\begin{array}{lll} f_1(x_i)=1 & \text{if} & 0\le i\le \frac{kq_1}{2}-1,\\ f_1(x_i)=-1 & \text{if} & \frac{kq_1}{2}\le i\le k(q_1-1)-1,\\ f_1(y_j)=1 & \text{if} & 0\le j\le \frac{(k+1)q_1}{2}-1,\\ f_1(y_j)=-1 & \text{if} & \frac{(k+1)q_1}{2}\le j\le (k+1)(q_1-1)-1 \end{array}$$

and for $2\le \ell\le q_1-1$, $0\le i\le k(q_1-1)-1$ and $0\le j\le (k+1)(q_1-1)-1$,

$$f_\ell(x_i)=f_{\ell-1}(x_{i+2k}) \text{ and } f_\ell(y_j)=f_{\ell-1}(y_{j+2(k+1)}),$$

where the sum is taken modulo $k(q_1-1)$ if i is involved, and modulo $(k+1)(q_1-1)$ if j is involved. In addition, for $1\le \ell\le q_1-1$, $k(q_1-1)\le i\le n-1$ and $(k+1)(q_1-1)\le j\le m-1$,

$$f_\ell(x_i)=(-1)^{i+\ell} \text{ and } f_\ell(y_j)=(-1)^{j+\ell}.$$

It is straightforward to see that $\{f_1,\dots,f_{q_1-1}\}$ is a signed total k-dominating family of $K_{n,m}$. Hence $d^t_{kS}(K_{n,m})\ge q_1-1$. Therefore $d^t_{kS}(K_{n,m})=q_1-1$, as desired.

Subcase 1.2 $q_1>q_2$. Then $d^t_{kS}(K_{n,m})\le q_2$ if q_2 is odd, by Corollary 11, and $d^t_{kS}(K_{n,m})\le q_2-1$ if q_2 is even, by Corollary 11 and Theorem 8. Let $q_1=q_2+t$ for some $t\ge 0$. Then $n=k(q_2+t)+r_1=kq_2+kt+r_1$. If q_2 is odd, by assumptions, r_2 and $kt+r_1$ are both even. If q_2 is even, then $m=(k+1)(q_2-1)+k+1+r_2$ and $n=k(q_2+t)+r_1=k(q_2-1)+k(t+1)+r_1$. By assumptions, $k+1+r_2$ and $k(t+1)+r_1$ are both even. Hence, by an argument similar to that described in Subcase 1.1 we see that if q_2 is odd, $d^t_{kS}(K_{n,m})=q_2$ and if q_2 is even, $d^t_{kS}(K_{n,m})=q_2-1$.

Case 2. $n\equiv k+1 \pmod 2$. Then $m\equiv k \pmod 2$. Let $n=(k+1)q+r$, where $0\le r\le k$. Then $d^t_{kS}(K_{n,m})\le q$ if q is odd by Corollary 11, and $d^t_{kS}(K_{n,m})\le q-1$ if q is even by Corollary 11 and Theorem 8. Let $m=n+s$, where $s\ge 0$. Then $m=kq+q+r+s$. If q is odd, then r and $q+r+s$ are both even. Define the functions $f_1,\dots,f_q$ as follows.

$$\begin{array}{lll} f_1(x_i)=1 & \text{if} & 0\le i\le \frac{(k+1)(q+1)}{2}-1,\\ f_1(x_i)=-1 & \text{if} & \frac{(k+1)(q+1)}{2}\le i\le (k+1)q-1,\\ f_1(y_j)=1 & \text{if} & 0\le j\le \frac{k(q+1)}{2}-1,\\ f_1(y_j)=-1 & \text{if} & \frac{k(q+1)}{2}\le j\le kq-1 \end{array}$$

and for $2\le \ell\le q$, $0\le i\le (k+1)q-1$ and $0\le j\le kq-1$,

$$f_\ell(x_i)=f_{\ell-1}(x_{i+2(k+1)}) \text{ and } f_\ell(y_j)=f_{\ell-1}(y_{j+2k}),$$

where the sum is taken modulo $(k+1)q$ if i is involved, and modulo kq if j is involved . In addition, for $1\le \ell\le q$, $(k+1)q\le i\le n-1$ and $kq\le j\le m-1$,

$$f_\ell(x_i)=(-1)^{i+\ell} \text{ and } f_\ell(y_j)=(-1)^{j+\ell}.$$

It is straightforward to see that $\{f_1, f_2, \ldots, f_q\}$ is a signed total k-dominating family of $K_{n,m}$. Hence $d^t_{kS}(K_{n,m}) \geq q$. Therefore $d^t_{kS}(K_{n,m}) = q$, as desired.

If q is even, we write $n = (k+1)(q-1)+k+1+r$ and $m = k(q-1)+k+q+r+s$. By assumptions, $k+1+r$ and $k+q+r+s$ are both even. Define the functions $f_1, \ldots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \leq i \leq \frac{(k+1)q}{2} - 1, \\ f_1(x_i) = -1 & \text{if} & \frac{(k+1)q}{2} \leq i \leq (k+1)(q-1) - 1, \\ f_1(y_j) = 1 & \text{if} & 0 \leq j \leq \frac{kq}{2} - 1, \\ f_1(y_j) = -1 & \text{if} & \frac{kq}{2} \leq j \leq k(q-1) - 1 \end{array}$$

and for $2 \leq \ell \leq q-1$, $0 \leq i \leq (k+1)(q-1) - 1$ and $0 \leq j \leq k(q-1) - 1$,

$$f_\ell(x_i) = f_{\ell-1}(x_{i+2(k+1)}) \text{ and } f_\ell(y_j) = f_{\ell-1}(y_{j+2k}),$$

where the sum is taken modulo $(k+1)(q-1)$ if i is involved, and modulo $k(q-1)$ if j is involved. In addition, for $1 \leq \ell \leq q-1$, $(k+1)(q-1) \leq i \leq n-1$ and $k(q-1) \leq j \leq m-1$,

$$f_\ell(x_i) = (-1)^{i+\ell} \text{ and } f_\ell(y_j) = (-1)^{j+\ell}.$$

It is straightforward to see that $\{f_1, f_2, \ldots, f_{q-1}\}$ is a signed total k-dominating family of $K_{n,m}$. Hence $d^t_{kS}(K_{n,m}) \geq q-1$. Therefore $d^t_{kS}(K_{n,m}) = q-1$, as desired. □

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