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CRITICAL EXPONENTS FOR A DOUBLY DEGENERATE PARABOLIC SYSTEM COUPLED VIA NONLINEAR BOUNDARY FLUX

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Abstract. The paper deals with the degenerate parabolic system with nonlinear boundary flux. By constructing the self-similar supersolution and subsolution, we obtain the critical global existence curve. The critical Fujita curve is conjectured with the aid of some new results.

1. Introduction

In this paper, we consider the following doubly degenerate parabolic equations

(1.1) $$u_t = (|u_x|^{p_1}(u^{m_1})_x)_x, \quad v_t = (|v_x|^{p_2}(v^{m_2})_x)_x, \quad x > 0,\ 0 < t < T$$

coupled via nonlinear boundary flux

(1.2) $$\begin{cases} -|u_x|^{p_1}(u^{m_1})_x(0,t) = v^{q_1}(0,t), & 0 < t < T, \\ -|v_x|^{p_2}(v^{m_2})_x(0,t) = u^{q_2}(0,t), & 0 < t < T, \end{cases}$$

and initial data

(1.3) $$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x > 0,$$

where parameters $m_i \geq 1, p_i > 0, q_i > 0$ $(i = 1, 2)$, and u_0, v_0 are nonnegative continuous functions with compact support in $\mathbb{R}_+$.

Nonlinear parabolic equations (1.1) appear in population dynamics, chemical reactions, heat transfer, and so on, where $u(x,t)$ and $v(x,t)$ represent the densities of two biological populations during a migration, the thickness of two kinds of chemical reactants in a chemical reaction, or the temperatures of two kinds of porous materials during a propagation.

It is well known that the local existence of the weak solution to the problem (1.1)-(1.3), defined in the usual integral way, as well as, a comparison principle can be easily established (see the survey [13] and books [4, 19, 31]).

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The problems on blow-up and global existence conditions, blow-up rates to nonlinear parabolic equations have been intensively studied (see [1, 2, 3, 5, 8, 11, 10, 13, 16, 17, 18, 21, 23, 26, 30, 31, 28, 29, 32, 34, 35] and references therein). In particular, the critical Fujita exponents are very interesting for various nonlinear parabolic equations of mathematical physics (see [3, 15, 18, 26, 30, 31, 28, 29, 32, 34, 35] and references therein). The concept of critical Fujita exponents was proposed by Fujita in the 1960s during discussion of the heat conduction equation with a nonlinear source (see [6]).

Now we recall some known results. In [8], Galaktionov and Levine studied the single equation case

$$\begin{cases} u_t = (u^m)_{xx}, & x > 0, 0 < t < T, \\ -(u^m)_x(0,t) = u^p(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.4}$$

and the heat conduction equation with gradient diffusion

$$\begin{cases} u_t = (|u_x|^{m-1}u_x)_x, & x > 0, 0 < t < T, \\ -|u_x|^{m-1}u_x(0,t) = u^p(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.5}$$

with $m \geq 1$, $p > 0$ and u_0 has compact support. They proved that for the problem (1.4) the critical global exponent is $p_0 = \frac{1}{2}(m+1)$ and the critical Fujita exponent is $p_c = m+1$, while for the problem (1.5) the critical global exponent is $p_0 = \frac{2m}{m+1}$ and the critical Fujita exponent is $p_c = 2m$. The critical global existence exponent and the critical Fujita exponent of (1.5) were also considered in [8] for the special case $m = 1$.

Wang and Yin [30], Li and Mu [16] studied the following single equation

$$\begin{cases} u_t = (|(u^m)_x|^{p-2}(u^m)_x)_x, & x > 0, 0 < t < T, \\ -|(u^m)_x|^{p-2}(u^m)_x(0,t) = u^q(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.6}$$

where $m > 1, p > 2, q > 0$ and $m > 0, 1 < p < 1 + \frac{1}{m}, q > 0$, respectively, they showed that the critical global existence exponent and critical Fujita exponent are $p_0 = \frac{(m+1)(p-1)}{p}$ and $p_c = (m+1)(p-1)$.

In [28], Wang et al. considered the following problem

$$u_t = u_{xx},\ v_t = v_{xx},\ x > 0,\ 0 < t < T, \tag{1.7}$$

$$-u_x(0,t) = v^p(0,t),\ -v_x(0,t) = u^q(0,t),\ 0 < t < T, \tag{1.8}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x > 0. \tag{1.9}$$

Under some assumptions they established the blow-up estimate near the blow-up time.

In [29], Wang et al. considered the following problem

$$u_t = u_{xx},\ v_t = v_{xx},\ x > 0,\ 0 < t < T, \tag{1.10}$$

$$-u_x(0,t) = u^\alpha v^p(0,t),\ -v_x(0,t) = v^\beta u^q(0,t),\ 0 < t < T, \tag{1.11}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x > 0. \tag{1.12}$$

The global existence and blow-up conditions for solutions of (1.10)-(1.12) are $pq \le (1-\beta)(1-\beta)$ and $pq > (1-\beta)(1-\beta)$, respectively. The blow-up rate of the solution (u,v) is $(O((T-t)^{-\gamma_1}), O((T-t)^{-\gamma_2}))$ as $t \to T$ with $\alpha < 1, \beta < 1$ and $pq \ge (1-\beta)(1-\beta)$, where

$$\gamma_1 = \frac{1}{2}\frac{p+1-\beta}{pq-(1-\alpha)(1-\beta)}, \quad \gamma_2 = \frac{1}{2}\frac{q+1-\alpha}{pq-(1-\alpha)(1-\beta)}.$$

In [23], Quiros and Rossi considered the degenerate equation

$$u_t = (u^m)_{xx},\ v_t = (v^n)_{xx}, \quad x > 0\ , 0 < t < T, \tag{1.13}$$

$$-(u^m)_x(0,t) = v^p(0,t),\ -(v^n)_x(0,t) = u^q(0,t),\ 0 < t < T, \tag{1.14}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x > 0 \tag{1.15}$$

with notation

$$\alpha_1 = \frac{2p+n+1}{(m+1)(n+1)-4pq}, \qquad \alpha_2 = \frac{2q+m+1}{(m+1)(n+1)-4pq},$$

$$\beta_1 = \frac{p(m-1-2q)+(n+1)m}{(m+1)(n+1)-4pq}, \qquad \beta_2 = \frac{q(n-1-2p)+(m+1)n}{(m+1)(n+1)-4pq}.$$

They proved that the solutions of (1.13)-(1.15) are global if $pq \le \frac{1}{4}(m+1)(n+1)$, and may blow up in finite time if $pq > \frac{1}{4}(m+1)(n+1)$. In the case of $pq > \frac{1}{4}(m+1)(n+1)$, if $\alpha_1 + \beta_1 \le 0$, or $\alpha_2 + \beta_2 \le 0$, then every non-negative, non-trivial solutions of (1.13)-(1.15) blow up in finite time: if $\alpha_1 + \beta_1 > 0$ and $\alpha_2 + \beta_2 > 0$, then there exist blow-up solutions for large initial and global solutions for small initial data. The critical Fujita exponents to (1.13)-(1.15) are described by $\alpha_i + \beta_i = 0, i = 1, 2$, while the blow-up rate of the positive solution is $O((T-t)^{-\alpha_1})$ for component u and $O((T-t)^{-\alpha_2})$ for v as $t \to T$.

In [7], Galaktionov and Levine studied the following single equation

$$\begin{aligned} &u_t = \nabla(|\nabla u|^\sigma \nabla u^m) + u^p, x \in \mathbb{R}^N, t > 0,\\ &u(x,0) = u_0(x), \qquad\qquad x \in \mathbb{R}^N, \end{aligned}$$

where $\sigma > 0$, $m > 1$, $p > 1$ and $u_0(x)$ is a bounded positive continuous function. They shown that the critical exponent is $p_c = m + \sigma + \frac{\sigma+2}{N}$.

Recently, Jiang and Zheng [10] studied the following single equation:

$$\begin{cases} u_t = (|u_x|^\beta (u^m)_x)_x, & x > 0,\ 0 < t < T,\\ -|u_x|^\beta (u^m)_x(0,t) = u^p(0,t), & 0 < t < T,\\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.16}$$

where $m \ge 1$,$p > 0, \beta > 0$. They obtained the critical global existence exponent $p_0 = \frac{2\beta+m+1}{\beta+2}$ and the critical Fujita exponent $p_c = 2\beta + m + 1$. These results are the extensions of those of Galaktionov and Levine [8].

Motivated by the above mentioned works, the aim of this paper is twofold. On the one hand, we construct the self-similar supersolutions and subsolutions to obtain the critical global existence curve of the system (1.1)-(1.3). On the other hand, the critical curve of Fujita type is conjectured with the aid of some new results. The fact that we are dealing with a system instead of a single equation forces us to develop some new techniques.

To state our results, we need to introduce the following numbers. Let

$$k_1 = \frac{(p_2+1)(p_1+2)q_1 + (p_1+1)(2q_2+m_2+1)}{(p+2)(p+2)q_1q_2 - (2q_1+m_1+1)(2q_2+m_2+1)},$$
$$k_2 = \frac{(p_1+1)(p_2+2)q_2 + (p_2+1)(2q_1+m_1+1)}{(p+2)(p+2)q_1q_2 - (2q_1+m_1+1)(2q_2+m_2+1)},$$
$$l_1 = \frac{k_2q_1 - p_1k_1 - m_1k_1}{p_1+1}, \qquad l_2 = \frac{k_1q_2 - p_2k_2 - m_2k_2}{p_2+1},$$

if $q_1q_2 \neq \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+1)(p_2+2)}$. The values k_1, k_2, l_1, l_2 are the exponents of self-similar solutions to problem (1.1)-(1.3).

Our main results in this paper are stated as follows.

Theorem 1.1. (1) *If* $q_1q_2 \leq \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$, *then every nonnegative solution of the system* (1.1)-(1.3) *is global in time*;

(2) *If* $q_1q_2 > \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$, *then the system* (1.1)-(1.3) *has a solution that blows up in a finite time.*

Theorem 1.2. *Assume* $q_1q_2 > \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$.

(1) *If* $\max\{l_1-k_1, l_2-k_2\} < 0$, *then every nonnegative nontrivial solution of the system* (1.1)-(1.3) *blows up in finite time.*

(2) *If* $\min\{l_1-k_1, l_2-k_2\} > 0$, *then there exists a global solution to the system* (1.1)-(1.3).

Remark 1.1. Theorem 1.1 show that the critical global existence curve of (1.1)-(1.3) is $q_1q_2 = \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+1)(p_2+2)}$, the restriction $\max\{l_1-k_1, l_2-k_2\} < 0$ in the Theorem1.2(2) is rather technical. It comes from the construction of the so-called Zel'dovich-Kompaneetz-Barenblatt profile. We believe that the critical Fujita curve is $\min\{l_1-k_1, l_2-k_2\} = 0$.

Remark 1.2. Unfortunately, we cannot obtain the blow-up rates of the non-global solution.

The rest of this paper is organized as follows. In Section 2, we consider the critical global existence curve and prove Theorem 1.1. The proof of Theorem 1.2 is shown in Section 3.

2. Critical global existence curve

In this section, by constructing self-similar sub- and super-solutions to problem (1.1)-(1.3), we shall prove Theorem 1.1.

Proof of Theorem 1.1 (1). It is enough to construct global supersolutions with initial data as large as needed. To this purpose, we look for a globally defined in time strict supersolution of self-similar form

$$\overline{u}(x,t) = e^{\kappa_1 t}(M + e^{-L_1 x e^{-\kappa_2 t}})^{\frac{1}{m_1}}, \quad x \geq 0, t \geq 0,$$
$$\overline{v}(x,t) = e^{\kappa_3 t}(M + e^{-L_2 x e^{-\kappa_4 t}})^{\frac{1}{m_2}}, \quad x \geq 0, t \geq 0,$$

where $M = \max\{\| u_0 \|_\infty^{m_1} + 1, \| v_0 \|_\infty^{m_2} + 1\}$, the constants $\kappa_i > 0$ $(i = 1,2,3,4)$, and $L_i > 0$ $(i = 1,2)$ are to be determined. Obviously, we have

$$\overline{u}(x,0) \geq u_0(x), \quad \overline{v}(x,0) \geq v_0(x), \quad x \geq 0.$$

After a direct computation, we obtain

$$\overline{u}_t \geq \kappa_1 e^{\kappa_1 t}(M + e^{-L_1 x e^{-\kappa_2 t}})^{\frac{1}{m_1}} \geq \kappa_1 e^{\kappa_1 t} M^{\frac{1}{m_1}},$$

$$\begin{aligned}
&(|\overline{u}_x|^{p_1}(\overline{u}^{m_1})_x \\
= &-\frac{L_1^{p_1+1}}{m_1^{p_1}} e^{p_1(\kappa_1-\kappa_2)t+(m_1\kappa_1-\kappa_2)t} e^{-(L_1 x + p_1 L_1 x)e^{-\kappa_2 t}} (M + e^{-L_1 x e^{-\kappa_2 t}})^{p_1(\frac{1}{m_1}-1)},\\
&(|\overline{u}_x|^{p_1}(\overline{u}^{m_1})_x)_x \\
\leq &(p_1+1)\frac{L_1^{p_1+2}}{m_1^{p_1}} e^{p_1(\kappa_1-\kappa_2)t+(m_1\kappa_1-2\kappa_2)t} M^{p_1(\frac{1}{m_1}-1)},
\end{aligned}$$

and

$$\overline{v}_t \geq \kappa_3 e^{\kappa_3 t}(M + e^{-L_2 x e^{-\kappa_4 t}})^{\frac{1}{m_2}} \geq \kappa_3 e^{\kappa_3 t} M^{\frac{1}{m_2}},$$

$$\begin{aligned}
&(|\overline{v}_x|^{p_2}(\overline{v}^{m_2})_x \\
= &-\frac{L_2^{p_2+1}}{m_2^{p_2}} e^{p_2(\kappa_3-\kappa_4)t+(m_2\kappa_3-\kappa_4)t} e^{-(L_2 x + p_2 L_2 x)e^{-\kappa_4 t}} (M + e^{-L_2 x e^{-\kappa_4 t}})^{p_2(\frac{1}{m_2}-1)},\\
&(|\overline{v}_x|^{p_2}(\overline{v}^{m_2})_x)_x \\
\leq &(p_2+1)\frac{L_2^{p_2+2}}{m_2^{p_2}} e^{p_2(\kappa_3-\kappa_4)t+(m_2\kappa_3-2\kappa_4)t} M^{p_2(\frac{1}{m_2}-1)}
\end{aligned}$$

in $\mathbb{R}_+ \times \mathbb{R}_+$. On the other hand, on the boundary we have

$$\begin{aligned}
-|u_x|^{p_1}(u^{m_1})_x(0,t) &= \frac{L_1^{p_1+1}}{m_1^{p_1}} e^{p_1(\kappa_1-\kappa_2)t+(m_1\kappa_1-\kappa_2)t}(M+1)^{p_1(\frac{1}{m_1}-1)},\\
v^{q_1}(0,t) &= e^{q_1\kappa_3 t}(M+1)^{\frac{q_1}{m_2}},\\
-|v_x|^{p_2}(v^{m_2})_x(0,t) &= \frac{L_2^{p_2+1}}{m_2^{p_2}} e^{p_2(\kappa_3-\kappa_4)t+(m_2\kappa_3-\kappa_4)t}(M+1)^{p_2(\frac{1}{m_2}-1)},\\
u^{q_2}(0,t) &= e^{q_2\kappa_1 t}(M+1)^{\frac{q_2}{m_1}}.
\end{aligned}$$

Therefore, we can see that $(\overline{u},\overline{v})$ is a supersolution of problem (1.1)-(1.3) provided that

$$\kappa_1 e^{\kappa_1 t} M^{\frac{1}{m_1}} \geq (p_1+1)\frac{L_1^{p_1+2}}{m_1^{p_1}} e^{p_1(\kappa_1-\kappa_2)t+(m_1\kappa_1-2\kappa_2)t} M^{p_1(\frac{1}{m_1}-1)},$$
$$\kappa_3 e^{\kappa_3 t} M^{\frac{1}{m_2}} \geq (p_2+1)\frac{L_2^{p_2+2}}{m_2^{p_2}} e^{p_2(\kappa_3-\kappa_4)t+(m_2\kappa_3-2\kappa_4)t} M^{p_2(\frac{1}{m_2}-1)},$$

and

$$\frac{L_1^{p_1+1}}{m_1^{p_1}} e^{p_1(\kappa_1-\kappa_2)t+(m_1\kappa_1-\kappa_2)t}(M+1)^{p_1(\frac{1}{m_1}-1)} \geq e^{q_1\kappa_3 t}(M+1)^{\frac{q_1}{m_2}},$$
$$\frac{L_2^{p_2+1}}{m_2^{p_2}} e^{p_2(\kappa_3-\kappa_4)t+(m_2\kappa_3-\kappa_4)t}(M+1)^{p_2(\frac{1}{m_2}-1)} \geq e^{q_2\kappa_1 t}(M+1)^{\frac{q_2}{m_1}}.$$

In order to verify the above inequalities, we only need impose

$$\kappa_1 \geq (p_1+m_1)\kappa_1-(p_1+2)\kappa_2,\ \kappa_3 \geq (p_2+m_2)\kappa_3-(p_2+2)\kappa_4, \tag{2.1}$$

$$p_1(\kappa_1-\kappa_2)+m_1\kappa_1-\kappa_2 \geq q_1\kappa_3,\ p_2(\kappa_3-\kappa_4)+m_2\kappa_3-\kappa_4 \geq q_2\kappa_1, \tag{2.2}$$

and

$$\kappa_1 M^{\frac{1}{m_1}} \geq (p_1+1)\frac{L_1^{p_1+2}}{m_1^{p_1}} M^{p_1(\frac{1}{m_1}-1)},$$
$$\kappa_3 M^{\frac{1}{m_2}} \geq (p_2+1)\frac{L_2^{p_2+2}}{m_2^{p_2}} M^{p_2(\frac{1}{m_2}-1)}, \tag{2.3}$$
$$\frac{L_1^{p_1+1}}{m_1^{p_1}}(M+1)^{p_1(\frac{1}{m_1}-1)} \geq (M+1)^{\frac{q_1}{m_2}},$$
$$\frac{L_2^{p_2+1}}{m_2^{p_2}}(M+1)^{p_2(\frac{1}{m_2}-1)} \geq (M+1)^{\frac{q_2}{m_1}}. \tag{2.4}$$

Now we show that such choice in (2.1)-(2.4) is valid. Firstly, by taking

$$L_1 = m_1^{\frac{p_1}{p_1+1}}(M+1)^{\frac{q_1}{(p_1+1)m_2}-\frac{p_1-m_1p_1}{m_1(p_1+1)}},\quad L_2 = m_2^{\frac{p_2}{p_2+1}}(M+1)^{\frac{q_2}{(p_2+1)m_1}-\frac{p_2-m_2p_2}{m_2(p_2+1)}},$$

we see that (2.4) holds.

Secondly, to obtain (2.1), we take

$$\kappa_1 = (p_1+m_1)\kappa_1-(p_1+2)\kappa_2,\quad \kappa_3 = (p_3+m_3)\kappa_3-(p_3+2)\kappa_4,$$

which can also written as

$$\kappa_2 = \frac{p_1+m_1-1}{p_1+2}\kappa_1,\qquad \kappa_4 = \frac{p_2+m_2-1}{p_2+2}\kappa_3. \tag{2.5}$$

To obtain the inequalities (2.2), we substitute (2.5) into (2.2) and then only need to confirm

$$\frac{2p_1+m_1+1}{p_1+2}\kappa_1 \geq \frac{q_1(p_2+2)}{p_2+m_2-1}\kappa_4,\qquad \frac{2p_2+m_2+1}{p_2+m_2-1}\kappa_4 \geq q_2\kappa_1. \tag{2.6}$$

It follows from the assumption

$$q_1q_2 \le \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)},$$

that (2.6) is true for suitable κ_1 and κ_4. Finally, we can further choose κ_1 and κ_3 large enough such that inequalities (2.3) hold.

Therefore, we have proved that $(\overline{u},\overline{v})$ is a global supersolution of system (1.1)-(1.3). The global existence of solutions to problem (1.1)-(1.3) follows from the comparison principle.

(2) To prove the non-existence of global solutions, we construct a blow-up self-similar subsolution of the system. Construct

$$\underline{u}(x,t) = (T-t)^{-k_1} f_1(\xi), \qquad \xi = x(T-t)^{-l_1}, \tag{2.7}$$

$$\underline{v}(x,t) = (T-t)^{-k_2} f_2(\eta), \qquad \eta = x(T-t)^{-l_2}, \tag{2.8}$$

where T is a positive constant and f_1, f_2 are two compactly supported functions to be determined.

After some computations, we have

$$\begin{aligned}
\underline{u}_t &= (T-t)^{-(k_1+1)}(k_1f_1(\xi)+l_1\xi {f_1}'(\xi)),\\
|\underline{u}_x|^{p_1}(\underline{u}^{m_1})_x &= (T-t)^{-p_1k_1-p_1l_1-m_1k_1-l_1}|{f_1}'|^{p_1}(f_1^{m_1})'(\xi),\\
(|\underline{u}_x|^{p_1}(\underline{u}^{m_1})_x)_x &= (T-t)^{-p_1k_1-p_1l_1-m_1k_1-2l_1}(|{f_1}'|^{p_1}(f_1^{m_1})'(\xi))',\\
\underline{v}_t &= (T-t)^{-(k_2+1)}(k_2f_2(\eta)+l_2\eta {f_2}'(\eta)),\\
|\underline{v}_x|^{p_2}(\underline{v}^{m_2})_x &= (T-t)^{-p_2k_2-p_2l_2-m_2k_2-l_2}|{f_2}'|^{p_2}(f_2^{m_2})'(\eta),\\
(|\underline{v}_x|^{p_2}(\underline{v}^{m_2})_x)_x &= (T-t)^{-p_2k_2-p_2l_2-m_2k_2-2l_2}(|{f_2}'|^{p_2}(f_2^{m_2})'(\eta))'
\end{aligned}$$

and

$$\begin{aligned}
|\underline{u}_x|^{p_1}(\underline{u}^{m_1})_x(0,t) &= (T-t)^{-p_1k_1-p_1l_1-m_1k_1-l_1}|{f_1}'|^{p_1}(f_1^{m_1})'(0),\\
\underline{v}^{q_1}(0,t) &= (T-t)^{-k_2q_1}{f_2}^{q_1}(0),\\
|\underline{v}_x|^{p_2}(\underline{v}^{m_2})_x(0,t) &= (T-t)^{-p_2k_2-p_2l_2-m_2k_2-l_2}|{f_2}'|^{p_2}(f_2^{m_2})'(0),\\
\underline{u}^{q_2}(0,t) &= (T-t)^{-k_1q_2}{f_1}^{q_2}(0).
\end{aligned}$$

Notice that

$$\begin{aligned}
&k_1+1 = p_1k_1+p_1l_1+m_1k_1+2l_1, \quad p_1k_1+p_1l_1+m_1k_1+l_1 = k_2q_1,\\
&k_2+1 = p_2k_2+p_2l_2+m_2k_2+2l_2, \quad p_2k_2+p_2l_2+m_2k_2+l_2 = k_1q_2.
\end{aligned}$$

Thus, $(\underline{u},\underline{v})$ is subsolution of (1.1)-(1.3) provided that

$$\begin{cases} (|{f_1}'|^{p_1}(f_1^{m_1})'(\xi))' \ge k_1f_1(\xi)+l_1f_1'(\xi)\xi,\\ (|{f_2}'|^{p_2}(f_2^{m_2})'(\eta))' \ge k_2f_2(\eta)+l_2f_2'(\eta)\eta, \end{cases} \tag{2.9}$$

$$\begin{cases} -|{f_1}'|^{p_1}(f_1^{m_1})'(0) \le {f_2}^{q_1}(0),\\ -|{f_2}'|^{p_2}(f_2^{m_2})'(0) \le {f_1}^{q_2}(0). \end{cases} \tag{2.10}$$

Set

$$f_1(\xi) = A_1(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}}, \quad f_2(\eta) = A_2(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}}, \tag{2.11}$$

where $A_i, a_i (i = 1, 2)$ are constants to be determined. It is easy to see that

$$f_1'(\xi) = -A_1\frac{p_1+1}{p_1+m_1-1}(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}-1}, \tag{2.12}$$

$$|f_1'|^{p_1}(f_1^{m_1})' = -m_1A_1^{m_1+p_1}(\frac{p_1+1}{p_1+m_1-1})^{p_1+1}(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}}, \tag{2.13}$$

$$(|f_1'|^{p_1}(f_1^{m_1})')' = m_1A_1^{m_1+p_1}(\frac{p_1+1}{p_1+m_1-1})^{p_1+2}(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}-1}, \tag{2.14}$$

and

$$f_2'(\eta) = -A_2\frac{p_2+1}{p_2+m_2-1}(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}-1}, \tag{2.15}$$

$$|f_2'|^{p_2}(f_2^{m_2})' = -m_2A_2^{m_2+p_2}(\frac{p_2+1}{p_2+m_2-1})^{p_2+1}(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}}, \tag{2.16}$$

$$(|f_2'|^{p_2}(f_2^{m_2})')' = m_2A_2^{m_2+p_2}(\frac{p_2+1}{p_2+m_2-1})^{p_2+2}(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}-1}. \tag{2.17}$$

Substituting (2.11)-(2.17) into (2.9), then inequalities (2.9) are valid provided that

$$\begin{aligned} k_1A_1(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}} &- l_1\xi A_1\frac{p_1+1}{p_1+m_1-1}(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}-1} \\ &- m_1A_1^{m_1+p_1}(\frac{p_1+1}{p_1+m_1-1})^{p_1+2}(a_1-\xi)_+^{\frac{p_1+1}{p_1+m_1-1}-1} \le 0 \end{aligned}$$

and

$$\begin{aligned} k_2A_2(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}} &- l_2\eta A_2\frac{p_2+1}{p_2+m_2-1}(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}-1} \\ &- m_2A_2^{m_2+p_2}(\frac{p_2+1}{p_2+m_2-1})^{p_2+2}(a_2-\eta)_+^{\frac{p_2+1}{p_2+m_2-1}-1} \le 0. \end{aligned}$$

To show that the above two inequalities hold, we choose a_1 and a_2 with

$$a_1 = \omega_1A_1^{m_1+p_1-1}, \qquad a_2 = \omega_2A_2^{m_2+p_2-1}, \tag{2.18}$$

where

$$\begin{aligned} \omega_1 &= \frac{m_1(p_1+m_1-1)}{k_1(p_1+m_1-1)+|l_1|(p_1+1)}(\frac{p_1+1}{p_1+m_1-1})^{p_1+2}, \\ \omega_2 &= \frac{m_2(p_2+m_2-1)}{k_2(p_2+m_2-1)+|l_2|(p_2+1)}(\frac{p_2+1}{p_2+m_2-1})^{p_2+2}. \end{aligned}$$

Here, we remark that the assumptions $q_1q_2 > \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$ imply $k_1 > 0, k_2 > 0$, then the inequalities (2.9) hold.

On the other hand, the boundary conditions in (2.10) are satisfied if we have

$$A_1^{m_1+p_1}\rho_1 a_1^{\frac{p_1+1}{p_1+m_1-1}} \le A_2^{q_1} a_2^{\frac{q_1(p_2+1)}{p_2+m_2-1}}, \tag{2.19}$$

$$A_2^{m_2+p_2}\rho_2 a_2^{\frac{p_2+1}{p_2+m_2-1}} \le A_1^{q_2} a_1^{\frac{q_2(p_1+1)}{p_1+m_1-1}}, \tag{2.20}$$

where $\rho_1 = m_1(\frac{p_1+1}{p_1+m_1-1})^{p_1+1} > 0$, $\rho_2 = m_2(\frac{p_2+1}{p_2+m_2-1})^{p_2+1} > 0$.

According to (2.18), we see that (2.19) and (2.20) hold provided that A_1 and A_2 are chosen to satisfy

$$A_1^{m_1+2p_1+1}\rho_1\omega_1^{\frac{p_1+1}{p_1+m_1-1}} \le A_2^{q_1(p_2+2)}\omega_2^{\frac{q_1(p_2+1)}{p_2+m_2-1}}, \tag{2.21}$$

$$A_2^{m_2+2p_2+1}\rho_2\omega_2^{\frac{p_2+1}{p_2+m_2-1}} \le A_1^{q_2(p_1+2)}\omega_1^{\frac{q_2(p_1+1)}{p_1+m_1-1}}. \tag{2.22}$$

The condition $q_1q_2 > \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$ ensures that we can take A_1 and A_2 large enough such that the inequalities (2.21) and (2.22) are valid.

Therefore, if the initial data u_0, v_0 are large enough so that $u_0(x) \ge \underline{u}(x,0)$ and $v_0(x) \ge \underline{v}(x,0)$, then $(\underline{u},\ \underline{v})$ is a subsolution to (1.1)-(1.3). By the comparison principle, it implies that the solutions of (1.1)-(1.3) with large initial data blow up in a finite time. The proof is complete. □

3. Critical Fujita curve

We devote this section to proof of Theorem 1.2. We borrow some ideas from [8, 10] to construct suitable auxiliary functions, however, the fact that we are dealing with a system instead of a single equation forces us to develop some new techniques.

Proof of Theorem 1.2 (1). We construct the following well-known self-similar solution (the so-called Zel'dovich-Kompaneetz-Barenblatt profile [8, 13, 24]) to (1.1)-(1.3) in the form

$$u_B(x,t) = (\tau+t)^{-\frac{1}{m_1+2p_1+1}}h_1(\xi), \qquad \xi = x(\tau+t)^{-\frac{1}{m_1+2p_1+1}}, \tag{3.1}$$

$$v_B(x,t) = (\tau+t)^{-\frac{1}{m_2+2p_2+1}}h_2(\eta), \qquad \eta = x(\tau+t)^{-\frac{1}{m_2+2p_2+1}}, \tag{3.2}$$

where $\tau > 0$ and

(3.3)

$$h_1(\xi) = C_1(h_1^{\frac{p_1+2}{p_1+1}} - \xi^{\frac{p_1+2}{p_1+1}})_+^{\frac{p_1+1}{p_1+m_1-1}}, \qquad h_2(\eta) = C_2(h_2^{\frac{p_2+2}{p_2+1}} - \eta^{\frac{p_2+2}{p_2+1}})_+^{\frac{p_2+1}{p_2+m_2-1}},$$

with $h_1 > 0$, $h_2 > 0$ and

$$C_1 = (\frac{1}{m_1(m_1+2p_1+1)}(\frac{p_1+m_1-1}{p_1+2})^{p_1+1})^{\frac{1}{p_1+m_1-1}}, \tag{3.4}$$

$$C_2 = (\frac{1}{m_2(m_2+2p_2+1)}(\frac{p_2+m_2-1}{p_2+2})^{p_2+1})^{\frac{1}{p_2+m_2-1}}. \tag{3.5}$$

It is not difficult to check that

$$(|h_1'|^{p_1}(h_1^{m_1})')'(\xi)+\frac{1}{m_1+2p_1+1}\xi h_1'(\xi)+\frac{1}{m_1+2p_1+1}h_1(\xi)=0,\ h_1'(0)=0,$$

$$(|h_2'|^{p_2}(h_2^{m_2})')'(\eta)+\frac{1}{m_2+2p_2+1}\eta h_2'(\eta)+\frac{1}{m_2+2p_2+1}h_2(\eta)=0,\ h_2'(0)=0.$$

Since $u(x,t)$ and $v(x,t)$ are nontrivial and nonnegative, we see that $u(0,t_0)>0$ and $v(0,t_0)>0$ for some $t_0>0$ (compare with a Barenblatt solution of the corresponding equations). Noticing that $u(x,t_0>0)$, $v(x,t_0)>0$ are continuous (see [9, 31]), there exists $\tau>0$ large enough and h_1,h_2 small enough such that

$$u(x,t_0)>u_B(x,t_0),\qquad v(x,t_0)>v_B(x,t_0)\ \text{ for }\ x>0.$$

A direct calculation shows that $(u_B(x,t),v_B(x,t))$ is a weak subsolution of (1.1)-(1.3) in $(0,+\infty)\times(t_0,+\infty)$. By the comparison principle, we obtain that

$$u(x,t)>u_B(x,t),\qquad v(x,t)>v_B(x,t)\ \text{ for }\ x>0,\ t>t_0.$$

Since $\max\{l_1-k_1,l_2-k_2\}<0$, we get $T^{l_1}\ll T^{k_1}$ and $T^{l_2}\ll T^{k_2}$ for large T. Furthermore, there exists $t*\geq t_0$ satisfying

$$T^{l_1}\ll(\tau+t^*)^{\frac{1}{m_1+2p_1+1}}\ll T^{k_1},\qquad T^{l_2}\ll(\tau+t^*)^{\frac{1}{m_2+2p_2+1}}\ll T^{k_2}. \tag{3.6}$$

Let $\underline{u},\underline{v}$ be the functions defined as in the proof of Theorem 1.1(2). Then for any $x>0$,

$$\underline{u}(x,0)\leq u_B(x,t^*)\leq u(x,t^*),\qquad \underline{v}(x,0)\leq v_B(x,t^*)\leq v(x,t^*).$$

It follows from the comparison principle that

$$\underline{u}(x,t)\leq u(x,t+t^*),\quad \underline{v}(x,t)\leq v(x,t+t^*)\ \text{ for }\ x>0,\ t>0.$$

As the proof of Theorem 1.1(2), we see that $(\underline{u},\underline{v})$ blows up in a finite time T. Therefore, (u_1,u_2) blows up in a finite time which is not larger than $T+t^*$. Observing that (3.6) holds for general nontrivial (u_0,v_0), and we know that every nonnegative, nontrivial solution of (1.1)-(1.3) blows up in finite time.

(2) Set

$$\overline{u}(x,t)=(\tau+t)^{-k_1}F_1(\xi),\qquad \xi=x(\tau+t)^{-l_1}, \tag{3.7}$$

$$\overline{v}(x,t)=(\tau+t)^{-k_2}F_2(\eta),\qquad \eta=x(\tau+t)^{-l_2}, \tag{3.8}$$

where $k_i,l_i(i=1,2)$ were defined as before, T is a positive constant and F_1,F_2 are two compactly supported functions to be determined.

After some computations, we have

$$\begin{aligned}\overline{u}_t&=(\tau+t)^{-(k_1+1)}(-k_1F_1(\xi)-l_1\xi F_1{}'(\xi)),\\ |\overline{u}_x|^{p_1}(\overline{u}^{m_1})_x&=(\tau+t)^{-p_1k_1-p_1l_1-m_1k_1-l_1}|F_1{}'|^{p_1}(F_1^{m_1})'(\xi),\\ (|\overline{u}_x|^{p_1}(\overline{u}^{m_1})_x)_x&=(\tau+t)^{-p_1k_1-p_1l_1-m_1k_1-2l_1}(|F_1{}'|^{p_1}(F_1^{m_1})'(\xi))',\\ \overline{v}_t&=(\tau+t)^{-(k_2+1)}(-k_2F_2(\eta)-l_2\eta F_2{}'(\eta)),\end{aligned}$$

$$|\overline{v}_x|^{p_2}(\overline{v}^{m_2})_x = (\tau+t)^{-p_2k_2-p_2l_2-m_2k_2-l_2}|F_2{}'|^{p_2}(F_2^{m_2})'(\eta),$$
$$(|\overline{v}_x|^{p_2}(\overline{v}^{m_2})_x)_x = (\tau+t)^{-p_2k_2-p_2l_2-m_2k_2-2l_2}(|F_2{}'|^{p_2}(F_2^{m_2})'(\eta))'.$$

and

$$|\overline{u}_x|^{p_1}(\overline{u}^{m_1})_x(0,t) = (\tau+t)^{-p_1k_1-p_1l_1-m_1k_1-l_1}|F_1{}'|^{p_1}(F_1^{m_1})'(0),$$
$$|\overline{v}_x|^{p_2}(\overline{v}^{m_2})_x(0,t) = (\tau+t)^{-p_2k_2-p_2l_2-m_2k_2-l_2}|F_2{}'|^{p_2}(F_2^{m_2})'(0),$$
$$\overline{v}^{q_1}(0,t) = (\tau+t)^{-k_2q_1}F_2{}^{q_1}(0), \quad \overline{u}^{q_2}(0,t) = (\tau+t)^{-k_1q_2}F_1{}^{q_2}(0).$$

Notice that

$$k_1+1 = p_1k_1+p_1l_1+m_1k_1+2l_1, \quad p_1k_1+p_1l_1+m_1k_1+l_1 = k_2q_1,$$
$$k_2+1 = p_2k_2+p_2l_2+m_2k_2+2l_2, \quad p_2k_2+p_2l_2+m_2k_2+l_2 = k_1q_2.$$

Thus, $(\overline{u},\overline{v})$ is supsolution of (1.1)-(1.3) provided that

$$\left\{\begin{array}{l} (|F_1{}'|^{p_1}(F_1^{m_1})'(\xi))' + k_1F_1(\xi) + l_1F_1'(\xi)\xi \le 0, \\ (|F_2{}'|^{p_2}(F_2^{m_2})'(\eta))' + k_2F_2(\eta) + l_2F_2'(\eta)\eta \le 0, \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{l} -|F_1{}'|^{p_1}(F_1^{m_1})'(0) \ge F_2{}^{q_1}(0), \\ -|F_2{}'|^{p_2}(F_2^{m_2})'(0) \ge F_1{}^{q_2}(0), \end{array}\right. \tag{3.10}$$

we choose

$$F_1(\xi) = A_1C_1((a_1b_1)^{\frac{p_1+2}{p_1+1}} - (\xi+a_1)^{\frac{p_1+2}{p_1+1}})_+^{\frac{p_1+1}{p_1+m_1-1}} = A_1h_1(\xi+a_1), \tag{3.11}$$

$$F_2(\eta) = A_2C_2((a_2b_2)^{\frac{p_2+2}{p_2+1}} - (\eta+a_2)^{\frac{p_2+2}{p_2+1}})_+^{\frac{p_2+1}{p_2+m_2-1}} = A_2h_2(\eta+a_2), \tag{3.12}$$

where C_1 and C_2 were defined by (3.4) and (3.5), h_1 and h_2 were defined by (3.3), $a_i > 0, b_i > 1, A_i > 0$ $(i = 1,2)$. We claim that exist A_i, b_i, a_i $(i = 1,2)$ such that the inequalities (3.9) are valid for F_1, F_2 defined by (3.10) and (3.11), then $h_1(\xi+a_1)$ and $h_2(\eta+a_2)$ satisfy the following equations

$$(|h_1'|^{p_1}(h_1^{m_1})')' = -\frac{1}{m_1+p_1+1}(\xi+a_1)h_1' - \frac{1}{m_1+p_1+1}h_1, \tag{3.13}$$

$$(|h_2'|^{p_2}(h_2^{m_2})')' = -\frac{1}{m_2+p_2+1}(\eta+a_2)h_2' - \frac{1}{m_2+p_2+1}h_2 \tag{3.14}$$

and

(3.15) $h_1'(\xi+a_1)$

$$= -C_1\frac{p_1+2}{p_1+m_1-1}((a_1b_1)^{\frac{p_1+2}{p_1+1}} - (\xi+a_1)^{\frac{p_1+2}{p_1+1}})_+^{\frac{p_1+1}{p_1+m_1-1}-1}(\xi+a_1)^{\frac{1}{p_1+1}},$$

(3.16) $h_2'(\eta+a_2)$

$$= -C_2\frac{p_2+2}{p_2+m_2-1}((a_2b_2)^{\frac{p_2+2}{p_2+1}} - (\eta+a_2)^{\frac{p_2+2}{p_2+1}})_+^{\frac{p_2+1}{p_2+m_2-1}-1}(\eta+a_2)^{\frac{1}{p_2+1}}.$$

In fact, when $a_1 \le \xi + a_1 \le b_1 a_1$ and $a_2 \le \xi + a_2 \le b_2 a_2$, substituting (3.11)-(3.16) into (3.9), denote by $y = \xi + a_1, z = \eta + a_2$, then (3.9) can be transformed into the following inequality with respect y, z

$$G_1(y) = -e_1 y^{\frac{p_1+2}{p_1+1}} + e_2 a_1 y^{\frac{1}{p_1+1}} - e_3 (a_1 b_1)^{\frac{p_1+2}{p_1+1}} \le 0, \tag{3.17}$$

$$G_2(z) = -\theta_1 y^{\frac{p_2+2}{p_2+1}} + \theta_2 a_2 y^{\frac{1}{p_2+1}} - \theta_3 (a_2 b_2)^{\frac{p_2+2}{p_2+1}} \le 0, \tag{3.18}$$

where

$$\begin{aligned}
e_1 &= (k_1 - \frac{A_1^{m_1+p_1-1}}{m_1+p_1-1}) + \frac{p_1+2}{p_1+m_1-1}(l_1 - \frac{A_1^{m_1+p_1-1}}{m_1+p_1-1}),\\
e_2 &= \frac{l_1(p_1+2)}{p_1+m_1-1},\\
e_3 &= \frac{A_1^{m_1+p_1-1}}{m_1+p_1-1} - k_1,\\
\theta_1 &= (k_2 - \frac{A_2^{m_2+p_2-1}}{m_2+p_2-1}) - \frac{p_2+2}{p_2+m_2-1}(l_2 - \frac{A_2^{m_2+p_2-1}}{m_2+p_1-1}),\\
\theta_2 &= \frac{l_2(p_2+2)}{p_2+m_2-1},\\
\theta_3 &= \frac{A_2^{m_2+p_2-1}}{m_2+p_2-1} - k_2.
\end{aligned}$$

Since $\min\{l_1 - k_1, l_2 - k_2\} > 0$, we can choose a suitable constant $A_1 > 0$ such that $l_1 > \frac{A_1^{m_1+p_1-1}}{m_1+p_1-1} > k_1$ and $(k_1 - \frac{A_1^{m_1+p_1-1}}{m_1+p_1-1}) + \frac{p_1+2}{p_1+m_1-1}(l_1 - \frac{A_1^{m_1+p_1-1}}{m_1+p_1-1}) > 0$ for such A_1, it is easy to verify that $e_i > 0$ $(i = 1, 2, 3)$ and $G_1(y)$ is a concave function with respect to $y^{\frac{1}{p_1+1}}$, then $G_1(y)$ attains its maximum at $z_* = \frac{e_2 a_1}{(p_1+2)e_1}$. Therefore, (3.16) is valid provided that

$$G_1(z_*) = a_1^{\frac{p_1+2}{p_1+1}} \left\{ \frac{p_1+1}{p_1+2} \left(\frac{1}{e_1(p_1+2)} \right)^{\frac{1}{p_1+1}} e_2^{\frac{p_1+2}{p_1+1}} - e_3 b_1^{\frac{p_1+2}{p_1+1}} \right\} \le 0. \tag{3.19}$$

So, we only need to choose b_1 sufficiently large such that

$$b_1 \ge \max\left\{ \left(\frac{(p_1+1)e_2}{(p+2)e_3} \right)^{\frac{p_1+1}{p_1+2}} \left(\frac{e_2}{(p+2)e_1} \right)^{\frac{1}{p_1+2}}, 1 \right\}.$$

Similarly, there exist $A_2 > 0, b_2 > 0$ such that the inequality (3.17) holds. Consequently, we have proved that inequalities (3.9) are true.

Now we consider the boundary condition (3.10), we only need to show that

$$\begin{aligned}
&(A_1 C_1)^{m_1+p_1} m_1 (\frac{p_2+2}{p_1+m_1-1})^{p_1+1} (b_1^{\frac{p_1+2}{p_1+1}} - 1)^{\frac{p_1+1}{p_1+m_1-1}} a_1^{\frac{2p_1+m_1+1}{p_1+m_1-1}}\\
\ge\ &(A_2 C_2)^{q_1} (b_2^{\frac{p_2+2}{p_2+1}} - 1)^{\frac{p_2+1}{p_2+m_2-1}} a_2^{\frac{(p_2+2)q_1}{p_2+m_2-1}},
\end{aligned}$$

$$(A_2C_2)^{m_2+p_2}m_2(\frac{p_2+2}{p_2+m_2-1})^{p_2+1}(b_2^{\frac{p_2+2}{p_2+1}}-1)^{\frac{(p_2+1)}{p_2+m_2-1}}a_2^{\frac{2p_2+m_2+1}{p_2+m_2-1}}$$
$$\geq (A_1C_1)^{q_2}(b_1^{\frac{p_1+2}{p_1+1}}-1)^{\frac{p_1+1}{p_1+m_1-1}}a_1^{\frac{(p_1+2)q_2}{p_1+m_1-1}},$$

where C_1 and C_2 were defined by (3.4) and (3.5). For above choosed A_i, d_i $(i = 1,2)$, the assumption $q_1q_2 > \frac{(2p_1+m_1+1)(2p_2+m_2+1)}{(p_1+2)(p_2+2)}$ ensures that there exist a_2 and a_1 small enough such that the above inequalities hold.

Therefore, it follows from the comparison principle that $(\overline{u}, \overline{v})$ given by (3.7) and (3.8) is a supersolution of the system (1.1)-(1.3) with $\overline{u}(x,0) \geq u_0(x)$, $\overline{v}(x,0) \geq v_0(x)$, which means that the solutions of (1.1)-(1.3) with small initial data have global existence. The proof of Theorem 1.2 is complete. □

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