International Journal of Reliability and Applications
Vol. 12, No. 2, pp. 103-116, 2011

Joint reliability importance of series-parallel systems

I. Dewan
Stat. Maths Unit, Indian Statistical Institute, New Delhi-110016, INDIA

K. Jain*
Department of Statistics, Panjab University, Chandigarh-160014, INDIA

Abstract. A series-parallel system with independent but non-identical components is considered. The expressions have been derived for the joint reliability importance (JRI) of m ($\geq$ 2) components, chosen from a series-parallel system. JRIs of components of two different series-parallel systems are studied analytically and graphically.

Key Words: *Reliability, coherent system, joint reliability importance, series-parallel system, partition function*

1. INTRODUCTION

Several electronic devices used in real life are examples of series-parallel systems. For example, a combination of series and parallel connections is used in solar batteries for increasing the battery's bank voltage and the amp hour capacity. This combination is also used in the designing of 3-band equalizers in music systems. In a multiple-speaker hookup, any number of speakers can be linked using a series-parallel wiring scheme, maintaining an impedance load that's compatible with the system's amplifier.

In a coherent system with independent components, every component is not equally important for the functioning of the system. Hence, the quantification of the reliability importance of various components of a complex coherent system, is required. Various importance measures/indices viz structural importance, Monte Carlo variance importance, generalised importance and composite importance have been proposed and widely studied [ref. Birnbaum (1969), Barlow and Proschan (1975a, 1975b), Vesley (1970) , Fussell (1975), Levitin, Podofillini and Zio (2003) and Jose, Marquez and Coit (2005)].

Marginal Reliability Importance (MRI) of a component is the rate at which the system reliability improves as the component reliability improves. But it does not give information about the effect on the system reliability as a result of a change in the reliability of two or more components. Improving the reliability of those components , that are critical for the system functioning, is of great significance for design engineers. Hence,

* Corresponding Author.
E-mail address: *jaink14@gmail.com*

we need to study the joint reliability importance (JRI) of two or more components coming from same or different subsystems.

JRI of two or more components measures the rate of change of system reliability as a function of a change in the reliability of these components. The concept of JRI of two components was introduced by Hong and Lie (1993) and discussed further by Armstrong (1995, 1997), Chang and Jan (2006) and Gao, Cui and Li (2007). JRI for a k-out-of-n: G system was studied by Hong, Koo and Lie (2002). Wu (2005) generalized the concept of JRI for multi-state systems .We wish to study the JRI of several components lying within and across different subsystems of a series-parallel system.

In this paper, section 2 describes a series-parallel system.The definition of JRI of m components is given for completeness. In section 3, we drive expressions for the JRI of two or more components for a series–parallel system with independent but non-identically distributed components. JRIs of components of two series-parallel systems are worked out in section 4. Conclusions are reported in section 5.

Notations:

{1, 2, ..., n}: Set of components;

A_i : the i^{th} component;

$\underline{p} = (p_1, p_2, \ldots, p_n)$ where p_i : probability that the i^{th} component functions;

$(1_i, 0_j, \underline{p})$: vector $\underline{p}$ with $p_i = 1$ and $p_j = 0$;

R ($\underline{p}$): system reliability function;

$MRI(A_i) = \frac{\partial R(\underline{p})}{\partial p_i}$: Marginal reliability importance of i^{th} component;

JRI: Joint reliability importance;

r_s: number of series subsystems;

r_p: number of parallel subsystems;

$r = r_p + r_s$: total number of subsystems;

n_i (m_j) : number of components in i^{th} series (j^{th} parallel) subsystem;

For $a = 1,2,\ldots,n_i, i = 1,2,\ldots,r_s$; $b = 1,2,\ldots,m_j$, $j = 1,2,\ldots,r_p$

p_{sia} : probability of functioning of a^{th} component in i^{th} series subsystem,

p_{pjb} :probability of functioning of b^{th} component in j^{th} parallel subsystem,

A_{sia} : the a^{th} component in i^{th} series subsystem;

A_{pjb} : the b^{th} component in j^{th} parallel subsystem;

P(m): partition function for positive integer m.

Assumptions:

(i) The system is coherent;

(ii) The components in the system are independent.

2. PRELIMINARIES

For a coherent system consisting of n components, the idea of JRI of pairs of components was extended to the concept of JRI of m (≥ 2) components by Gao, Cui and Li (2007).They defined the m component JRI as

$$JRI(A_1, A_2,..., A_m) = \frac{\partial^m R(\underline{p})}{\prod_{i=1}^{m} \partial p_i}$$

where p_i is the probability that the i^{th} component functions.
For independent components

$$JRI(A_1, A_2,..., A_m) = R(1_1,1_2,...,1_m,\underline{p}) \pm R(1_1,1_2,...,0_m,\underline{p}) \pm \\ .... \pm R(0_1,0_2,...,1_m,\underline{p}) \pm R(0_1,0_2,...,0_m,\underline{p}) \quad (2.1)$$

The sign before the function R in (1) is determined according to the following rules:

(1) For odd m, the sign is positive if the number of 1's in corresponding R is odd;
(2) For m even, the sign is positive if the number of 1's in corresponding R is even;
(3) The sign of $R(0_1,0_2,...,0_m,\underline{p})$ is negative if m is odd and is positive for even m.

Hong, Koo and Lie (2002) observed that the sign of JRI plays an important role in the analysis of system reliability. Two components are called reliability substitutes if the sign of JRI is non-positive and reliability compliments if the sign of JRI is non-negative.

The following figure illustrates a configuration of the series-parallel system under study.

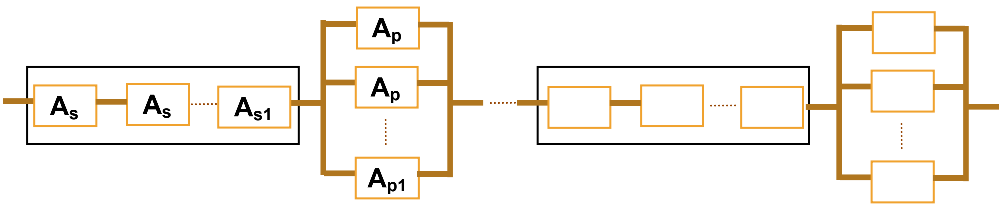

Figure 2.1. A series-parallel system

The reliability of the above system consisting of r_s series subsystems and r_p parallel subsystems is

$$R(\underline{p}) = \left\{ \prod_{i=1}^{r_s} \left(\prod_{a=1}^{n_i} p_{sia} \right) \right\} \left\{ \prod_{j=1}^{r_p} \left(1 - \prod_{b=1}^{m_j} (1 - p_{pjb}) \right) \right\} \quad (2.2)$$

since the reliability of i^{th} series system is $\prod_{a=1}^{n_i} p_{sia}$ and that of j^{th} parallel system is

$$1 - \prod_{b=1}^{m_j} (1 - p_{pjb}).$$

For finding the JRI of m components, the concept of the partition function P(m) is utilised. The partition function gives the number of partitions of m into non-negative integers with sum as m. In our context, P(m) denotes the number of ways in which m components can be chosen from r subsystems such that $2 \le m \le \sum_{j=1}^{r} n_j$. P(0) is defined as 1 and P(1)=1. A recurrence relation for the partition function P (m) is restated below.

Euler's Recurrence Formula (Andrews (1976)):
For m < 0, P(m) =0 and
for $m > 0$,

$$P(m) - P(m-1) - P(m-2) + P(m-5) + P(m-7) + ... + (-1)^k P(m - \frac{k}{2}(3k-1))$$

$$+ (-1)^k P(m - \frac{k}{2}(3k+1)) + ... = 0 .$$

Using the above relations, the values of P(m) for m = 1,2, 3, ... , 10 are listed in the following table.

m	P(m)	m	P(m)
1	1	6	11
2	2	7	15
3	3	8	22
4	5	9	30
5	7	10	42

3. MULTI-COMPONENT JOINT RELIABILITY IMPORTANCE FOR SERIES-PARALLEL SYSTEM

In this section, we generalise the results of Gao, Cui and Li (2007) for a series-parallel system which has independent but non- identical components. In the following theorem, the expressions for the JRI of two components are derived. In case of two components, there are five possibilities viz.

- both components in same series system;
- both components in same parallel system;
- both components in different series systems;
- both components in different parallel systems;
- one component in series and the other component in parallel system.

Theorem 3.1 The JRI of two components u and v in series-parallel system is

(i) $JRI(A_{sju}, A_{sjv}) = \left\{ \prod_{\substack{i=1 \\ i \ne j}}^{r_s} \left(\prod_{a=1}^{n_i} p_{sia} \right) \right\} \left\{ \prod_{h=1}^{r_p} \left(1 - \prod_{b=1}^{m_h} (1 - p_{phb}) \right) \right\} \left\{ \prod_{a \ne u,v}^{n_j} p_{sja} \right\}$

for u and v in the same j^{th} series subsystem;

(ii) $JRI\left(A_{pju}, A_{pjv}\right) = -\left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i}\left(p_{sia}\right)\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{a\neq u,v}^{m_j}\left(1-p_{pja}\right)\right\}$

for u and v in the same j^th^ parallel subsystem;

(iii) $JRI\left(A_{sju}, A_{skv}\right) = \left\{\prod_{\substack{i=1\\i\neq j,k}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\left(\prod_{\substack{a=1\\a\neq u}}^{n_j} p_{sja}\right)\left(\prod_{\substack{a=1\\a\neq v}}^{n_k} p_{ska}\right)\right\}$

for u in j^th^ and v in k^th^ series subsystem;

(iv) $JRI\left(A_{pju}, A_{pkv}\right) = \left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j,k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{b=1\\b\neq u}}^{m_j}\left(1-p_{pjb}\right)\right\}\left\{\prod_{\substack{b=1\\b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\}$

for u in j^th^ and v in k^th^ parallel subsystem;

(v) $JRI\left(A_{sju}, A_{pkv}\right) = \left\{\prod_{\substack{i=1\\i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u}}^{n_j} p_{sja}\right\}\left\{\prod_{b=1}^{m_k}\left(1-p_{pkb}\right)\right\}$

for u in j^th^ series subsystem and v in k^th^ parallel subsystem.

Proof:

(i) Using (2.1) and (2.2), we have

$$R\left(1_{sju}, 1_{sjv}, \underline{p}\right) = \left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{i=1\\i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{a\neq u,v}^{n_j} p_{sja}\right\}.$$

The result follows since all other reliabilities in (2.1) are zero.

(ii) The proof follows on similar lines.

(iii) When both components lie in different series subsystems, the reliability is

$$R\left(1_{sju}, 1_{skv}, \underline{p}\right) = \left\{\prod_{\substack{i=1\\i\neq j,k}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u}}^{n_j} p_{sja}\right\}\left\{\prod_{\substack{a=1\\a\neq v}}^{n_k} p_{ska}\right\}.$$

In this case, all other reliabilities are equal to zero.

(iv) When both components lie in different parallel subsystems

$$R\left(1_{pju}, 1_{pkv}, \underline{p}\right) = \left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j,k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\};$$

$$R\left(1_{pju}, 0_{pkv}, \underline{p}\right) = \left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j,k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{1-\prod_{\substack{b=1\\b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\};$$

$$R\left(0_{pju},1_{pkv},\underline{p}\right)=\left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j,k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{1-\prod_{\substack{b=1\\b\neq u}}^{m_j}\left(1-p_{pjb}\right)\right\};$$

and

$$R\left(0_{pju},0_{pkv},\underline{p}\right)=\left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j,k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{1-\prod_{\substack{b=1\\b\neq u}}^{m_j}\left(1-p_{pjb}\right)\right\}\left\{1-\prod_{\substack{b=1\\b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\}.$$

Using equation (2.1) and the above expressions, (iv) follows.

(v) When one component lies in a series subsystem and the other is in a parallel subsystem,

$$R\left(1_{sju},1_{pkv},\underline{p}\right)=\left\{\prod_{\substack{i=1\\i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u}}^{n_j}p_{sja}\right\};$$

$$R\left(1_{sju},0_{pkv},\underline{p}\right)=\left\{\prod_{\substack{i=1\\i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u}}^{n_j}p_{sja}\right\}\left\{1-\prod_{\substack{b=1\\b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\};$$

$$R\left(0_{sju},1_{pkv},\underline{p}\right)=R\left(0_{sju},0_{pkv},\underline{p}\right)=0.$$

$$\text{Thus } JRI\left(A_{sju},A_{pkv}\right)=\left\{\prod_{\substack{i=1\\i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{\substack{h=1\\h\neq k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u}}^{n_j}p_{sja}\right\}\left\{\prod_{\substack{b=1\\b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\}.$$

Remark 3.2 It is observed that the JRI of two components is non-negative in all cases except when components belong to the same parallel subsystem.

Theorem 3.3 The JRI of three components u, v and w for a series-parallel system is given as

(i) $$JRI\left(A_{sju},A_{sjv},A_{sjw}\right)=\left\{\prod_{\substack{i=1\\i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u,v,w}}^{n_j}p_{sja}\right\}$$

for u, v and w in the same series subsystem;

(ii) $$JRI\left(A_{pju},A_{pjv},A_{pjw}\right)=\left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i}\left(p_{sia}\right)\right)\right\}\left\{\prod_{\substack{h=1\\h\neq j}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{b=1\\b\neq u,v,w}}^{m_j}\left(1-p_{pjb}\right)\right\}$$

for u, v and w in the same parallel subsystem;

(iii) $$JRI\left(A_{sju},A_{sjv},A_{skw}\right)=\left\{\prod_{\substack{i=1\\i\neq j,k}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\a\neq u,v}}^{n_j}p_{sja}\right\}\left\{\prod_{\substack{a=1\\a\neq w}}^{n_k}p_{ska}\right\}$$

for u, v in the same series subsystem and w in a different series subsystem;

(iv) $JRI\left(A_{pju}, A_{pjv}, A_{pkw}\right) = -\left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1 \\ h\neq j,k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq u,v}}^{m_j}\left(1-p_{pjb}\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq w}}^{m_k}\left(1-p_{pkb}\right)\right\}$

for u, v in the same parallel subsystem and w in a different parallel subsystem;

(v) $JRI\left(A_{sju}, A_{skv}, A_{slw}\right) = \left\{\prod_{\substack{i=1 \\ i\neq j,k,l}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1 \\ a\neq u}}^{n_j} p_{sja}\right\}\left\{\prod_{\substack{a=1 \\ a\neq v}}^{n_k} p_{ska}\right\}\left\{\prod_{\substack{a=1 \\ a\neq w}}^{n_l} p_{sla}\right\}$

for all three in different series subsystems;

(vi) $JRI\left(A_{pju}, A_{pkv}, A_{plw}\right) =$

$$\left\{\prod_{i=1}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1 \\ h\neq j,k,l}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq w}}^{m_l}\left(1-p_{plb}\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq u}}^{m_j}\left(1-p_{pjb}\right)\right\}$$

for all three in different parallel subsystems;

(vii) $JRI\left(A_{sju}, A_{sjv}, A_{pkw}\right) = \left\{\prod_{\substack{i=1 \\ i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1 \\ h\neq k}}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq w}}^{m_k}\left(1-p_{pkb}\right)\right\}\left\{\prod_{\substack{a=1 \\ a\neq u,v}}^{n_j} p_{sja}\right\}$

for u and v in the same series subsystem and w in the parallel subsystem;

(viii) $JRI\left(A_{sjw}, A_{skv}, A_{plw}\right) =$

$$\left\{\prod_{\substack{i=1 \\ i\neq j,k}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1 \\ h\neq l}}^{r_p}\left(1-\prod_{b=1}^{m_h}(1-p_{phb})\right)\right\}\left\{\prod_{\substack{a=1 \\ a\neq u}}^{n_j} p_{sja}\right\}\left\{\prod_{\substack{a=1 \\ a\neq v}}^{n_k}\left(p_{ska}\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq w}}^{m_l}\left(1-p_{plb}\right)\right\}$$

for u, v in the different series subsystems and w in the parallel subsystem;

(ix) $JRI\left(A_{sju}, A_{pkv}, A_{pkw}\right) = -\left\{\prod_{\substack{i=1 \\ i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1 \\ h\neq k}}^{r_p}\left(1-\prod_{b=1}^{m_h}(1-p_{phb})\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq v,w}}^{m_k}\left(1-p_{pkb}\right)\right\}\left\{\prod_{\substack{a=1 \\ a\neq u}}^{n_j} p_{sja}\right\}$

for u in series subsystem and v and w in same parallel subsystem.

(x) $JRI\left(A_{sju}, A_{pkv}, A_{plw}\right) =$

$$\left\{\prod_{\substack{i=1 \\ i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i} p_{sia}\right)\right\}\left\{\prod_{\substack{h=1 \\ h\neq k,l}}^{r_p}\left(1-\prod_{b=1}^{m_h}(1-p_{phb})\right)\right\}\left\{\prod_{a=1}^{n_j} p_{sja}\right\}\left\{\prod_{\substack{b=1 \\ b\neq v}}^{m_k}\left(1-p_{pkb}\right)\right\}\left\{\prod_{\substack{b=1 \\ b\neq w}}^{m_l}\left(1-p_{plb}\right)\right\}$$

for u in series subsystem and v and w in different parallel subsystems.

Proof:

(i) When all three components belong to same series subsystem, then

$$R\left(1_{sju},1_{sjv},1_{sjw},\underline{p}\right)=\left\{\prod_{\substack{i=1\\ i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\ a\neq u,v,w}}^{n_j}p_{sja}\right\}.$$

The other reliabilities are all equal to zero.

Using (2.1) and the above expression, we get

$$JRI\left(A_{sju},A_{sjv},A_{sjw}\right)=\left\{\prod_{\substack{i=1\\ i\neq j}}^{r_s}\left(\prod_{a=1}^{n_i}p_{sia}\right)\right\}\left\{\prod_{h=1}^{r_p}\left(1-\prod_{b=1}^{m_h}\left(1-p_{phb}\right)\right)\right\}\left\{\prod_{\substack{a=1\\ a\neq u,v,w}}^{n_j}p_{sja}\right\}.$$

The expressions in (ii) to (x) can be obtained on similar lines.

In what follows, we find the JRI of m components for a series-parallel system. These m components can be chosen from any one of rs series (rp parallel) subsystems. The following lemma gives an expression for the total number of ways of choosing these m (≥ 2) components from r subsystems.

Lemma 3.4 Suppose that m components are chosen from r subsystems in N(m) ways. Then, for partition function P(.)

$$N(m)=\sum_{j=0}^{m}\left\{P(m-j)P(j)\right\},\ m\geq 2\,.$$

Proof: Suppose that j components are chosen from series subsystems and (m-j) are selected from parallel subsystems. This can be done in P(m-j).P(j) ways. We get the desired result by summing over all j=0,1,…, m.

For example, using Table 1, $N(3)=\sum_{j=0}^{3}\left\{P(3-j)P(j)\right\}=10$.

The following theorem gives expressions for the JRI of m components.

Theorem 3.5 Without loss of generality, suppose that m components are chosen from first q_s series subsystems and first q_p parallel subsystems such that

$$m=\sum_{i=1}^{q_s}m_{si}+\sum_{l=1}^{q_p}m_{pl}$$

where $m_{si}\left(m_{pl}\right)$ is the number of components in ith series (lth parallel) subsystem and $0\leq m_{si}\leq n_i$ and $0\leq m_{pl}\leq m_l$ for $i=1,2,...,q_s$, $1\leq q_s\leq r_s$ and $l=1,2,...,q_p$, $1\leq q_p\leq r_p$.

Then JRI of m components is

$$(-1)^{q_p+\sum_{l=1}^{q_p} m_{pl}} \left\{ \prod_{\substack{i=1 \\ \text{excluding } q_s \\ \text{subsystems from} \\ \text{which components} \\ \text{are chosen}}}^{r_s} \left(\prod_{a=1}^{n_i} p_{sia} \right) \right\} \left\{ \prod_{\substack{h=1 \\ \text{excluding } q_p \\ \text{subsystems from} \\ \text{which components} \\ \text{are chosen}}}^{r_p} \left(1 - \prod_{b=1}^{m_h} \left(1 - p_{phb}\right) \right) \right\}$$

$$\left\{ \prod_{i=1}^{q_s} \left(\prod_{\substack{a=1 \\ \text{excluding } m_{si} \\ \text{components of} \\ \text{interest}}}^{n_i} p_{sia} \right) \right\} \left\{ \prod_{l=1}^{q_p} \left(1 - \prod_{\substack{b=1 \\ \text{excluding } m_{pl} \\ \text{components of} \\ \text{interest}}}^{m_l} \left(1 - p_{plb}\right) \right) \right\}.$$

The following result gives the expressions for JRI of m components when components in the same subsystem are identical.

Corollary 3.6
Let $p_{sia} = p_{si}, a = 1,2,...,n_i, i = 1,2,...,r_s; p_{pjb} = p_{pj}, b = 1,2,...,m_j, j = 1,2,...,r_p$.

Then JRI of m components is

$$(-1)^{q_p+\sum_{l=1}^{q_p} m_{pl}} \left\{ \prod_{\substack{i=1 \\ \text{excluding } q_s \\ \text{subsystems from} \\ \text{which components} \\ \text{are chosen}}}^{r_s} \left(p_{si}\right)^{n_i} \right\} \left\{ \prod_{\substack{h=1 \\ \text{excluding } q_p \\ \text{subsystems from} \\ \text{which components} \\ \text{are chosen}}}^{r_p} \left[1 - \left(1 - p_{ph}\right)^{m_h} \right] \right\}$$

$$\left\{ \prod_{i=1}^{q_s} \left(p_{si}^{\left(n_i - m_{si}\right)} \right) \right\} \left\{ \prod_{l=1}^{q_p} \left(1 - p_{p_l}^{\left(m_l - m_{pl}\right)} \right) \right\}.$$

Corollary 3.7: If $p_{sia} = p_{pjb} = p, i = 1,2,...,r_s; j = 1,2,...,r_p$, then JRI of m components is

$$(-1)^{q_p+\sum_{l=1}^{q_p} m_{pl}} \left\{ \prod_{\substack{i=1 \\ \text{excluding } q_s \\ \text{subsystems from} \\ \text{which components} \\ \text{are chosen}}}^{r_s} \left(p\right)^{n_i} \right\} \left\{ \prod_{\substack{h=1 \\ \text{excluding } q_p \\ \text{subsystems from} \\ \text{which components} \\ \text{are chosen}}}^{r_p} \left[1 - \left(1 - p\right)^{m_h} \right] \right\}$$

$$\left\{ \prod_{i=1}^{q_s} \left(p^{\left(n_i - m_{si}\right)} \right) \right\} \left\{ \prod_{l=1}^{q_p} \left(1 - p^{\left(m_l - m_{pl}\right)} \right) \right\}.$$

This expression gives JRI of m components when components in the same system are independent and identical.

4. ILLUSTRATIONS

Example 4.1 We consider the following series-parallel system consisting of seven independent components where p_i is the reliability of the i^{th} component.

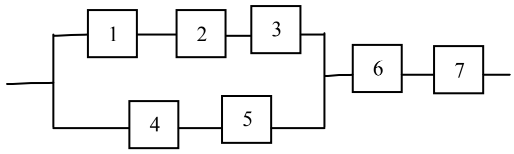

Figure 4.1. A series–parallel system with seven components

The system reliability of the above system is

$$R(\underline{p}) = (p_1p_2p_3 + p_4p_5 - p_1p_2p_3p_4p_5)p_6p_7.$$

It is observed that

$$JRI(1,2) = p_3(1 - p_4p_5)p_6p_7, JRI(1,3) = p_2(1 - p_4p_5)p_6p_7, JRI(1,2,3) = (1 - p_4p_5)p_6p_7$$

Hence

$$JRI(1,2) < JRI(1,2,3) \text{ and } JRI(1,3) < JRI(1,2,3).$$

Also

$$\text{JRI}(2,4) = -p_1p_3p_5p_6p_7, \text{JRI}(2,4,6) = -p_1p_3p_5p_7, \text{JRI}(1,2,4,6) = -p_3p_5p_7,$$
$$\text{JRI}(1,2,4,5,6) = -p_3p_7, \text{JRI}(1,2,3,4,5,6) = -p_7, \text{JRI}(1,2,3,4,5,6,7) = -1.$$

It is noticed that

$$\text{JRI}(2,4) > \text{JRI}(2,4,6) > \text{JRI}(1,2,4,6) > \text{JRI}(1,2,4,5,6)$$
$$> \text{JRI}(1,2,3,4,5,6) > \text{JRI}(1,2,3,4,5,6,7).$$

It can also be concluded that

(i) JRI of components from same subsystem(s) is positive;
(ii) JRI of components from different subsystem(s) is negative;
(iii) JRI of all seven components in the system is -1.

The following figure depicts the behaviour of JRI (1,2) as a function of p_4 and p_7 when $p_3 = .89, p_5 = .92 \; and \; p_6 = .98$.

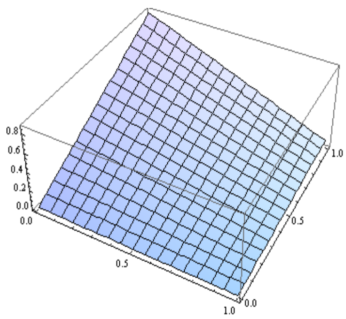

Figure 4.2. JRI (1,2) as a function of p_4 and p_7

It is seen that JRI (1,2) decreases in p_4 and increases in p_7.

In the next example, we study a system where a component is replicated in different subsystems.

Example 4.2 We consider the following configuration

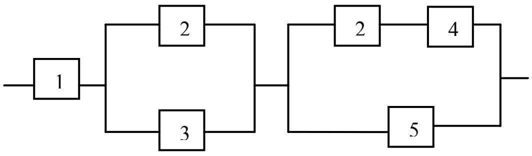

Figure 4.3. A series-parallel system with replicated components

The reliability of the above system (Birolini (2007)) is

$$R(\underline{p}) = p_1 p_2 (p_4 + p_5 - p_4 p_5) + (1 - p_2) p_1 p_3 p_5,$$

$$JRI(1,5) = p_2(1 - p_3 - p_4),$$

$$JRI(2,3) = -p_1 p_5, JRI(2,4) = p_1(1 - p_5), JRI(2,5) = p_1(1 - p_3 - p_4), JRI(3,4) = 0,$$

$$JRI(1,2,3) = -p_5, JRI(1,2,4) = 1, JRI(1,2,5) = 1 - p_3 - p_4, JRI(1,3,5) = 1 - p_2,$$

$$JRI(1,2,3,4) = JRI(1,3,4,5) = JRI(2,3,4,5) = 0,$$

$$JRI(1,2,3,5) = JRI(1,2,4,5) = -1,$$

$$JRI(1,2,3,4,5) = 0.$$

Hence $JRI(1,2,5) > JRI(2,5)$, $JRI(1,2,4) > JRI(2,4)$

and $JRI(1,3,5) > JRI(1,5) > JRI(1,3)$.

It can also be concluded that

(i) JRI of components from same series subsystem is positive;
(ii) JRI of components from same parallel subsystem is negative;
(iii) JRI of all components in the system is zero.

The following figure depicts JRI(2,3).

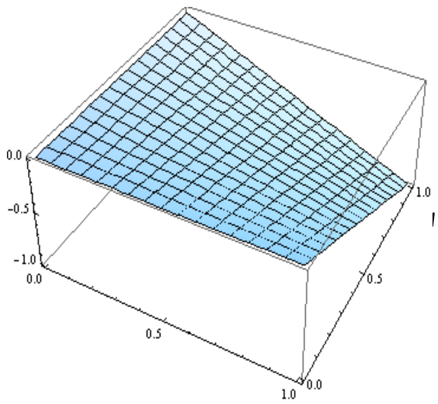

Figure 4.4. JRI(2,3)

It is noticed that as p_1 and p_5 increase from 0 to 1, JRI(2,3) decreases from its maximum value zero and equals -1 for $p_1 = p_5 = 1$.

From the next figure, we conclude the following

(a) As p_1 increases from 0 to 1 and p_5 decreases from 1 to 0, JRI (2,4) increases and finally equals 1 for $p_1 = 1$ and $p_5 = 0$;

(b) As p_1 decreases from 1 to 0 and p_5 increases from 0 to 1, JRI (2,4) decreases and finally equals 0 for $p_1 = 0$ or 1 and $p_5 = 1$ and for $p_1 = 0$ and $p_5 = 0$.

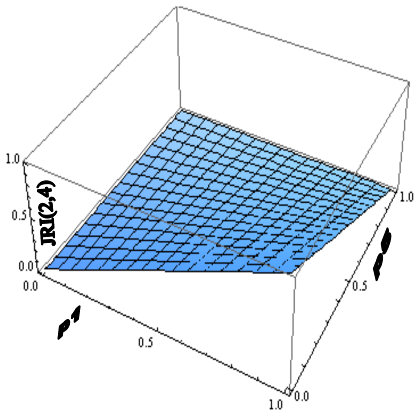

Figure 4.5. JRI(2,4)

In the following figure, JRI (1,2,3) is observed to be a decreasing function of p_5.

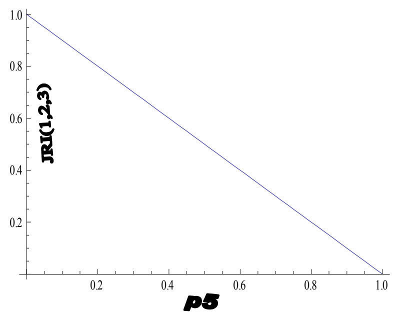

Figure 4.6. JRI(1,2,3)

From Figure 4.7, it is concluded that

(a) JRI (1,2,5)=1 for $p_3 = p_4 = 0$;

(b) JRI (1,2,5)=0 for $p_3=0$, $p_4=1$ or $p_3=1$, $p_4=0$ or $p_3=.5$, $p_4=.5$;

(c) As p_3 increases from 0 to 1, JRI (1,2,5) decreases as a function of p_4;

(d) As p_4 increases from 0 to 1, JRI (1,2,5) decreases as a function of p_3.

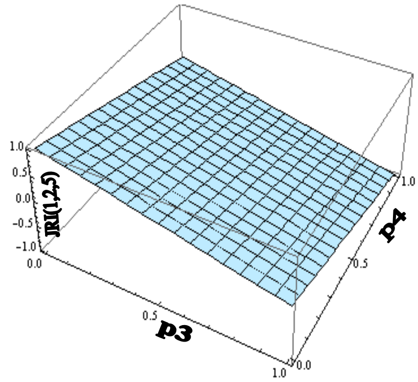

Figure 4.7. JRI (1,2,5)

5. CONCLUSIONS

Gao, Cui and Li (2007) found the JRI of two components of a series-parallel system when the components are identical and independently distributed. We have generalized their results to derive the JRI of $m \geq 2$ components when components are independent but not necessarily identically distributed. These m components can lie in different subsystems of the main system. The results have been illustrated with two series-parallel systems having independent components. In system 1, none of the components is replicated but in system 2, the components can be replicated. These results can be utilized for studying the joint effect of reliability of several components on the reliability of the system. This can further help in designing systems that function more effectively and for a longer period.

ACKNOWLEDGEMENT

The authors are grateful to the referees for their suggestions that have helped in better presentation of the manuscript.

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