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Pooling shrinkage estimator of reliability for exponential failure model using the sampling plan (n, C, T)

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Abstract. One of the most important problems in the estimation of the parameter of the failure model, is the cost of experimental sampling units, which can be reduced by using any prior information available about θ, and devising a two-stage pooling shrunken estimation procedure. We have proposed an estimator of the reliability function ($R(t)$) of the exponential model using two-stage time censored data when a prior value about the unknown parameter (θ) is available from the past. To compare the performance of the proposed estimator with the classical estimator, computer intensive calculations for bias, mean squared error, relative efficiency, expected sample size and percentage of the overall sample size saved expressions, were done for varying the constants involved in the proposed estimator ($\tilde{R}(t)$).

Key Words: *Pooling shrinkage, reliability function, exponential failure model, time censored data, Poisson flow of failure, bias ratio, relative efficiency*

1. INTRODUCTION

1.1 The model and $R(t)$

The exponential failure model is one of the most significant and widely used models in life testing and survival problems and has been used very effectively for analyzing data particularly when the data is censored which is very common in most life testing experiments (Gnedenko, Balyayev, & Solovyev, 1969; Sinha, 1986). Reliability theory and reliability engineering make extensive use of the exponential failure model. Another important topic in reliability engineering is the parameter estimation when there are items that have been tested and have not failed (censored data).

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The reliability of a given system (or component) for a given time has been defined as the probability that the system (or component) functions longer than the time of duration t, and given by,

$$\begin{aligned} R(t) &= P(X > t) \\ &= 1 - F_\theta(t) = \exp(-t\theta), \qquad t > 0, \theta > 0, \end{aligned} \tag{1.1}$$

where θ gives the average or mean life of the item under study and $F_\theta(t)$ is the distribution function of failure time T following the exponential model. Reliability is obviously of great importance in life problems. The classical estimator $\hat{\theta}$ of θ and hence of $R(t)$ can easily be obtained without any complicated mathematical aid.

1.2 Incorporating a guess value

Suppose observations are available from a probability distribution function $F_\theta(t)$, where the functional form of $F_\theta(.)$ is known and θ is an unknown parameter. Further, suppose that the experimenter has prior information regarding the value of θ due to past experiences. In certain situations however, the prior information is available only in the form of an initial guess value (natural origin) θ_o of θ, then this guess may be utilized in the new estimation problem. For example; a bulb producer may know that the average life of his product may be close to 1000 hours. Here we may take $\theta_0 = 1000$. According to Thompson (1968) θ_o is a 'natural origin' and such natural origins may arise for any one of number of reasons, e.g., we are estimating θ and:

(i) the prior value (θ_o) of θ in many practical problems exists.

(ii) we believe θ_0 is close to the true value of θ, or

(iii) we are cautious that θ_0 may be near the true value of θ, i.e., something undesirable happens if $\theta_0 \cong \theta$, and we are not aware of it.

In such a situation it is natural to utilize θ_o in the estimation of θ. The method of constructing an estimator of θ that incorporates θ_o leads to what is known as a shrunken estimator to improve the estimation procedure, i.e., reducing the MSE of the new estimator or giving a saving in sample size.

A standard problem in life testing deals with estimation of the parameter θ and $R(t)$ on the basis of less time and minimizing the cost of experimentation. The cost of experimentation can be achieved by using any prior information available about θ and devising a two-stage pooling shrunken estimation procedure.

2. TPSE, BACKGROUND AND THE AIM

As noted earlier the cost of experimentation of getting an estimator of the parameter θ of the failure model can be reduced by using any prior information available about θ and devising a two-stage pooling shrunken procedure in which it is possible to obtain an estimator from a small first stage sample and additional second stage sample is required only if this estimator is not reliable (see Katti (1962)). A two-stage pooling shrunken estimator (TPSE) of θ is defined as follows. Let $T_{1i}, i = 1,2,...,n_1$, be a random sample of size $n_1 < n$ from the exponential distribution and $\hat{\theta}_1$ to be a good estimator of θ based on n_1 observations. Construct a pretest region R in the space of θ, based on the prior value θ_O and an appropriate criterion. If $\hat{\theta}_1 \in R$, use the estimator $k(\hat{\theta}_1 - \theta_o) + \theta_o$, for θ, but if $\hat{\theta}_1 \notin R$, obtain $T_{2i}, i = 1,2,...,n_2$, compute $\hat{\theta}_2$, and then pool $\hat{\theta}_1$ and $\hat{\theta}_2$ to find $\hat{\theta} = (n_1\hat{\theta}_1 + n_2\hat{\theta}_2)/n$. The TPSE of θ is thus given by,

$$\tilde{\theta} = \left\{ [k(\hat{\theta}_1 - \theta_o) + \theta_o]I_R + \hat{\theta}\ I_{\bar{R}} \right\}, \tag{2.1}$$

where $0 \le k \le 1$, I_R and $I_{\bar{R}}$ are respectively the indicator functions of the acceptance region R, and the rejection region $\bar{R}$.

Several authors have studied the TPSE for the parameter θ of the exponential distribution for complete, right censored data by choosing different k and R. (see Al-Hemyari 2010; Rakesh and Vilpa 2007; Kambo et al. 1991a; Gokhale and Adke 1989; Handa et al. 1988; Adke et al. 1987). Katti (1962), Al-Bayyati and Arnold (1972), Waiker et al. (1984), Kambo et al. (1991b), Waiker et al. (2001), and Al-Hemyari (2009a, 2009b) discussed other choices for k and R with different estimation problems.

It may be remarked that Chiou (1987) discussed the problem of estimating the reliability function $R(t)$ of the exponential model using single-stage right censored data, and proposed a shrinkage estimator,

$$\bar{R}(t) = \begin{cases} \exp(-t/\theta_0), & if\ \ C_1 \le 2T/\theta_0 \le C_2, \\ \breve{R}(t), & if\ \ C_1 > 2T/\theta_0\ \ or\ \ 2T/\theta_0 > C_2, \end{cases} \tag{2.2}$$

where $T = \sum_{i=1}^{r_j} T_{(i)} + (n-r)T_{(r)}$, C_1 and C_2 respectively are the lower and upper 100(α /2) percentile points of the chi-square distribution of $2r$ degrees of freedom and the minimum variance unbiased estimator $\breve{R}(t)$ (see Basu, 1964) is,

$$\breve{R}(t) = \begin{cases} (1 - t/T)^{r-1}, & if\ \ t < T, \\ 0, & if\ \ t \ge T. \end{cases} \tag{2.3}$$

This estimator has been adapted in various other situations by many authors. For example, Instead of using $exp(-t/\theta_0)$, Chiou (1992) used $k\,exp(-t/\theta_0) + (1-k)\breve{R}(t)$, where k $(0 < k < 1)$ is an arbitrary constant and Chiou (1993) also proposed empirical Bayes shrinkage estimation of $R(t)$. The procedure of Baklizi and Ahmed (2008) was closely related to Chiou (1992). Al-Hemyari (2009c) discussed a two-stage pooling shrinkage estimation procedure for $R(t)$ with a constant shrinkage factor using the complete data. Finally, Al-Hemyari (2010) considered a shrinkage estimator of estimation the scale parameter and reliability function of Weibull distribution in different context.

The purpose of this paper is not to simply extend Al-Hemyari's (2009c) estimator to other estimation problems. Rather, we assume a time censored sample where the aim is to find a estimator of the reliability function which offers some improvement over the classical estimators. The expressions for the bias, mean squared error, expected sample size and relative efficiency are obtained and studied numerically, and numerical results and conclusions drawn from those are presented.

3. TPSE FOR $R(t)$

As mentioned, the exponential model serves as a very useful model in analyzing the life testing and reliability censored data. Among the different censoring schemes, interestingly, the time censored data (plan (n, C, T))) receives a considerable attention, particularly in the reliability analysis. In this section first we define the general proposed estimator, and we obtain the expected value, bias and mean squared error expressions of $\hat{\theta}_j$ and $\hat{R}(t)$, then we describe the choice of the region R, and finally we obtain the necessary expressions of the proposed estimator.

3.1 The proposed estimator using the plan (n, C, T)

The construction of proposed estimator is considered in this section; and we are concerned with the sampling plan (n, C, T) (we use the term ' plan (n, C, T)' which Gnedenko et al. 1969 proposed) where the observations are carried out for a period of time ${}_jT_o$ and each unit that fails is replaced with a new one identical to the old one. For this case we observe the Poisson flow of failures during the period ${}_jT_o$ (Gnedenko et al. 1969; Sinha, 1986; Johnson et al. 2005).

Let $T_{jr_1} \le T_{jr_2} \le \ldots \le T_{j\ rj}, j = 1,2$, be the failure times of the first ${}_jr_0$ items that fail before a specified time ${}_jT_o$. Suppose that the underlying distribution of each T_{ji} is an exponential distribution with $F_\theta(t)$ and θ_o being the prior information about θ. We wish to estimate $R(t)$ by the following PSE: compute $\hat{\theta}_1 = {}_1r_0 / n_1\ {}_1T_0$ and construct a region R in the space of θ based on θ_o. If $\hat{\theta}_1 \in R$, use the estimator $\exp(-t(k(\hat{\theta}_1 - \theta_o) + \theta_o)$ for $R(t)$, but if $\hat{\theta}_1 \notin R$, compute $\hat{\theta}_2 = {}_2r_0 / n_2\ {}_2T_0$ and then pool

$\hat{\theta}_1$ and $\hat{\theta}_2$ to find $\hat{\theta} = (n_1\hat{\theta}_1 + n_2\hat{\theta}_2)/(n_1 + n_2)$ and use the estimator $\exp(-t\hat{\theta})$ for $R(t)$. The TPSE of $R(t)$ is thus given by,

$$\widetilde{R}(t) = \left\{ [e^{-t(k(({}_1r_0 / n_1 {}_1T_0) - \theta_0) + \theta_0)}] I_R + \left[e^{-t(({}_1r_0/{}_1T_0) + ({}_2r_0/{}_2T_0))/(n_1+n_2)} \right] I_{\overline{R}} \right\}, \tag{3.1}$$

where I_R and $I_{\overline{R}}$ are respectively the indicator functions of the acceptance region R, and the rejection region $\overline{R}$, and

$$P(\hat{\theta}_j) = \begin{cases} (n_j \ \theta \ {}_jT_o)^{{}_jr_0} e^{-(n_j \ {}_jT_0 \ \theta)} / \Gamma({}_jr_0 + 1), & {}_jr_0 = 1, 2, \ldots, n_j, \\ 0 \quad , & otherwise \end{cases} \tag{3.2}$$

It may be noted here that the number of failures ${}_jr_0$ has a Poisson distribution with the parameter $(n_j \ \theta \ {}_jT_0)$ and ${}_jr_0$ is sufficient as well as a complete statistic. The measures which are used for studying the behaviour of the proposed pooling shrinkage estimator PSE are the bias ratio, mean squared error, expected sample size required for obtaining $\widetilde{R}(t)$, and the percentage of the overall sample size saved. Whereas the main criterion used for comparison of two pooling estimators is the relative efficiency, which is defined by the ratio of two corresponding mean squared errors.

It can be easily shown that the expected value, bias and mean squared error of $\hat{\theta}_j$ are, respectively, given by:

$$E(\hat{\theta}_j \mid \theta) = \theta \ e^{-q_j} G_1(q_j, \ \infty),) \tag{3.3}$$

$$B(\hat{\theta}_j | \theta) = \theta \ [e^{-q_j} G_1(q_j, \ \infty) - 1 \ ], \tag{3.4}$$

$$MSE(\hat{\theta}_j \mid \theta) = \theta^2 [e^{-q_j} [G_2(q_j, \infty) + G_1(q_j, \infty)(e^{-q_j} G_1(q_j, \infty) - 4) + G_0(q_j, \infty)] + 1], \tag{3.5}$$

where for any integer m,

$$G_l(q_j, m) = \sum_{{}_jr_0=0}^{m} {}_jr_0^{\,l} \ q_j^{\,{}_jr_0 - l} / \Gamma({}_jr_0 + 1), \ j = 1, 2, \ l = 0, \ 1, \ 2, \text{and } q_j = (n_j \ \theta \ {}_jT_0). \tag{3.6}$$

3.2 Choices for region R

Estimator (3.1) is completely obtained by specifying the region R. Therefore, it seems reasonable to construct a region denoted by R by the criterion,

$$R = \{\theta : (\theta - \theta_0)^2 \le MSE(\hat{\theta}_1 \mid \theta)\}, \tag{3.7}$$

i.e., the square of differences between the initial estimate θ_o and the actual value of θ should not exceed the $MSE(\hat{\theta}_1 \mid \theta)$. This gives the first choice R_1 by the following interval,

$$R_1 = [q_1 / (1 + \sqrt{d_1}), \ q_1 / (1 - \sqrt{d_1})], \tag{3.8}$$

where $d_1 = [e^{-q_1} [G_2(q_1, \infty) + G_1(q_1, \infty)(e^{-q_1} G_1(q_1, \infty) - 4) + G_0(q_1, \infty)] + 1]$.

Sometimes it may not be possible to express R_1 as an interval, simplified, as $MSE(\hat{\theta}_1 \mid \theta)$ often depends on θ. In this case region R may be approximated by the interval R_2 given by,

$$R_2 = \{\theta : (\theta - \theta_0)^2 \leq MSE(\hat{\theta}_1 \mid \theta_0)\}. \tag{3.9}$$

Simple calculations lead to,

$$R_2 = [Max\ (0, q_1^*(1-\sqrt{d_2}),\ q_1^*(1+\sqrt{d_2})], \tag{3.10}$$

where $d_2 = e^{-q_1^*}[G_2(q_1^*,\infty) + G_1(q_1^*,\infty)(e^{-q_1^*}G_1(q_1^*,\infty) - 4) + G_0(q_1^*,\infty)] + 1,\ \ d_2 \geq 0,$

$q_1^* = (n_1 \quad \theta_0 \quad {}_1T_0)$, and $q_1^* = q_1$, if $\theta_0 = \theta$.

Remarks: Special cases

1) It may be noted here that $d_1 \geq 0$. If $d_i = 0,\ i = 1,2$ or $d_1 \geq 1$, the estimator $\widetilde{R}(t)$ is a single-stage and hence we do not consider it.
2) The estimator $\widetilde{R}(t)$ corresponding to the choice $k = 1$, is Katti's (1962) type estimator and is given by,
$$\widetilde{R}(t) = \left\{ [e^{-t({}_1r_0 / n_{11}T_0)}]I_R + \left[e^{-t((({}_1r_0/{}_1T_0)+({}_2r_0/{}_2T_0))/(n_1+n_2))}\right] I_{\bar{R}} \right\}. \tag{3.11}$$
3) Also, the estimator $\widetilde{R}(t)$ corresponding to the choice $k = 0$, is Kambo et al. (1991) type estimator and is given by,
$$\widetilde{R}(t) = \left\{ [e^{-t(\theta_0)}]I_R + \left[e^{-t((({}_1r_0/{}_1T_0)+({}_2r_0/{}_2T_0))/(n_1+n_2))}\right] I_{\bar{R}} \right\}. \tag{3.12}$$

3.3 Expressions of the classical and proposed estimators

In order to study the behavior of the proposed TPSE, the bias, mean squared error and relative efficiency of the classical and proposed estimator are derived in this section. It can be easily shown that the expected value, bias and mean squared error of the classical estimator $\hat{R}(t)$ are respectively, given by:

$$E(\hat{R}(t)) = G_1(q_1,\infty)G_1(q_2,\infty),) \tag{3.13}$$

$$B(\hat{R}(t)) = G_1(q_1,\infty)G_1(q_2,\infty) - e^{-t\theta}, \tag{3.14}$$

$$MSE(\hat{R}(t)) = e^{-2t\theta}\{e^{2t\theta}[G_3^*(q_1,\infty)G_3^*(q_2,\infty) - 2e^{t\theta}G_1^*(q_1,\infty)G_1^*(q_1,\infty) + 1]\}. \tag{3.15}$$

Let $R_i = [a_i, b_i],\ i = 1,2$ the expressions for the bias ratio, mean squared error and expected sample size of the proposed estimator $\widetilde{R}(t)$ are given respectively by:

$$B(\widetilde{R}(t) \mid R(t); R_i) = E(\widetilde{R}(t) - R(t))/e^{-t\theta}$$
$$= e^{-t\theta((1-k)(\lambda-1))}(G_2^*(q_1, b_i) - G_2^*(q_1, a_i)) + e^{t\theta}G_1^*(q_2,\infty)(G_1^*(q_1,\infty) - G_1^*(q_1, b_i) \tag{3.16}$$
$$- G_1^*(q_1, a_i)) - 1,$$

$$MSE(\widetilde{R}(t) \mid R(t); R_i) = E(\widetilde{R}(t) - R(t))^2$$

$$= e^{-2t\theta}\Big[e^{(-2t\theta((1-k)\lambda-1))}[\, G_4^*(q_1, b_i) - G_4^*(q_1, a_i)\,] - 2e^{(-t\theta((1-k)\lambda-1))}[\, G_2^*(q_2, b_i)$$

$$- G_2^*(q_2, a_i)\,] + 2e^{2t\theta} G_3^*(q_2, \infty)[\, G_3^*(q_1, \infty) - G_3^*(q_1, b_1) + G_3^*(q_1, a_i)\,] - 2e^{t\theta} G_1^*(q_2, \infty) \times$$

$$\times [\, G_1^*(q_1, \infty) - G_1^*(q_1, b_i) + G_1^*(q_1, a_i)\,] + 1\,\Big] \quad (3.17)$$

$$E(n \mid \widetilde{R}(t), R_i) = n - (n - n_1) e^{-q_1} (G_0^*(q_1, b_i) - G_0^*(q_1, a_i)), 6) \quad (3.18)$$

where, for any integer m:

$$G_0^*(q_j, m) = \sum_{{}_j r_o = 0}^{m} e^{(-(n_j\ {}_jT_0\ \theta))} (n_j\ \theta\ {}_jT_0)^{{}_j r_0} / \Gamma({}_j r_0),\ j = 1,2, \quad (3.19)$$

$$G_1^*(q_j, m) = \sum_{{}_j r_o = 0}^{m} e^{(-((t\ {}_j r_0 / n_j\ {}_jT_0) + n_j\ {}_jT_0\ \theta))} (n_j\ \theta\ {}_jT_0)^{{}_j r_0} / \Gamma({}_j r_0)\ , j = 1,2, \quad (3.20)$$

$$G_2^*(q_j, m) = \sum_{{}_j r_o = 0}^{m} e^{(-((t\ K\ {}_j r_0 / n_j\ {}_jT_0) + n_j\ {}_jT_0\ \theta))} (n_j\ \theta\ {}_jT_0)^{{}_j r_0} / \Gamma({}_j r_0)\ , j = 1,2, \quad (3.21)$$

$$G_3^*(q_j, m) = \sum_{{}_j r_o = 0}^{m} e^{(-((2\ t\ {}_j r_0 / n_j\ {}_jT_0) + n_j\ {}_jT_0\ \theta))} (n_j\ \theta\ {}_jT_0)^{{}_j r_0} / \Gamma({}_j r_0)\ , j = 1,2, \quad (3.22)$$

$$G_4^*(q_j, m) = \sum_{{}_j r_o = 0}^{m} e^{(-((2\ t\ K {}_j r_0 / n_j\ {}_jT_0) + n_j\ {}_jT_0\ \theta))} (n_j\ \theta\ {}_jT_0)^{{}_j r_0} / \Gamma({}_j r_0),\ \ j = 1,2, \quad (3.23)$$

$a_1 = [q_1/(1+\sqrt{d_1})]$, $a_2 = [\ Max(0, q_0(1-\sqrt{d_2}))]$, $b_1 = q_1/(1-\sqrt{d_1})]$, $b_1 = [q_0(1+\sqrt{d_2})]$ and $\lambda = (\theta_0/\theta)$. The efficiency of $\widetilde{R}(t)$ relative to $\hat{R}(t)$ is easily obtained as follows,

$$RE(\widetilde{R}(t) \mid \hat{R}(t)) = MSE(\hat{R}(t)) / MSE(\widetilde{R}(t) \mid R(t); R_i)$$

$$= [G_3^*(q_1, \infty) G_3^*(q_2, \infty) - 2e^{-t\theta} G_1^*(q_1, \infty) G_1^*(q_1, \infty) + 1] / \{ e^{-2t\theta((1-k)\lambda} [G_4^*(q_1, b_i)$$

$$- G_4^*(q_1, a_i)] - 2e^{-2t\theta((1-k)\lambda)} [G_2^*(q_1, b_i) - G_2^*(q_2, a_i)] + 2G_3^*(q_2, \infty)[G_3^*(q_1, \infty)$$

$$- G_3^*(q_1, b_i) + G_3^*(q_1, a_i)] - 2e^{-t\theta} G_1^*(q_2, \infty)[G_1^*(q_1, \infty) - G_1^*(q_1, b_i) + G_1^*(q_1, a_i)] + 1\} \quad (3.24)$$

4. SIMULATION AND NUMERICAL RESULTS

An exact analytical study of the performance of the proposed estimator $\tilde{R}(t)$ is not possible because of the expressions of the bias ratio, expected sample size, and relative efficiency appear to be complicated. Therefore, we are left with no other better choice than an empirical study.

To observe the performance of the proposed estimator $\tilde{R}(t)$, and to give useful comparison between the proposed and classical estimator of $R(t)$, we perform computer calculations for bias ratio, mean squared error, relative efficiency, expected sample size expressions and percentage of the overall sample size saved $100(n_2/n)\times \Pr(\hat{\theta}_1 \in R_i)$, were done for each of the two types of region R_i, $i=1,2,$ for different values of the first and second stage samples, and for varying the constants involved in the proposed estimator $\tilde{R}(t)$. Specifically, numerical computation were performed by taking $n_1 = 4(2)12$, $n_2 = 4(2)12$, $q_j = 0.5(0.5)2$, $q_1^* = \lambda q_1$, $t\theta = 0.3(0.1)3.0$, $k = 0.1(0.1)1$, and $\lambda = 0(0.1)10$. Some of these results are presented in Figures 4.1-4.9.

We make the following observations from tables presented in this paper as numerical results, comparisons, and limitations.

4.1 Numerical results:

1. Both regions give highest relative efficiency when θ is close to θ_0, i.e., $\lambda \cong 1$ and decreases as $|\lambda - 1|$ increases.
2. The regions R_1 and R_2 yield mean squared error, bias and expected sample size of approximately the same order. Region R_1 is useful when it can be expressed as an interval. Otherwise to avoid computational difficulties R_2 may be preferred over R_1, and hence the computations of the bias ratio, expected sample size and relative efficiency of $\tilde{R}(t)$ with R_1 are not reported here to save space.
3. Relative efficiency of $\tilde{R}(t)$ is a concave function of λ, i.e., the proposed estimator has maximum efficiency in the neighborhood of $\lambda \cong 1$.
4. The relative efficiency of $\tilde{R}(t)$ is a decreasing function of k, n_1, q_1, and q_2.

   $RE(\tilde{R}(t) \mid \hat{R}(t))$ is also an increasing function of $t\theta$ and n_2, i.e., $n_1 \cong 4$, $n_2 \cong 12$, $\lambda \cong 1$, $k = 0.1$, $q_0 = q_1 = q_2 = 0.5$, and $t\theta = 3$ yield the highest efficiency. Thus the choice $n_1 \cong 4, n_2 \cong 12$, $\lambda \cong 1$, k =0.1, $q_0 = q_1 = q_2 = 0.5$ and $t\theta = 3$ is recommended. The computations for other values have not been presented, to save space.

5. Expected sample size is close to n_1 for small values of n_1 and increasing slowly with increase of n_1. $E(n \mid \widetilde{R}\ (t), R_i)$ is generally smaller than n.
6. As expected the percentage of the overall sample saved of $\widetilde{R}(t)$, is an increasing function of n_2, i.e. this percentage is from 20% to 8% (when $n_2 = 2$); whereas the same is 43% to 30% (when $n_2 = 12$).
7. For fixed n_2 the percentage of the overall sample size saved is a decreasing function of n_1, i.e. the saving is from 20% to 45% (when $n_1 = 4$); whereas the same is 8% to 30% (when $n_1 = 12$).

4.2 Comparisons:

1. From Figures 4.1-4.6, we note that the estimator $\widetilde{R}(t)$ for $0 \le \lambda \le 10$ has smaller mean squared error than the pooled classical estimator $\hat{R}(t)$.
2. The relative efficiency of the proposed TPSE is much greater than the classical estimator (as much as 230 times) for small n_1 in the neighborhood of $\lambda \cong 1$.
3. The estimator $\widetilde{R}\ (t)$ is biased. From Figure 7 it is observed that the bias ratio is reasonably small when θ is sufficiently close to θ_0.

5. LIMITATION AND CONCLUSIONS

In this section, we study the limitation, and the conclusions of the proposed estimator.

5.1 Limitation

As was noted earlier, the TSPE perform better than classical estimators in the neighborhood of $\lambda \cong 1$, i.e., they have higher relative efficiency when $\theta \cong \theta_0$ but lower efficiency otherwise. It is not necessary that the prior value θ_0 always be close to the true value θ, thus, our proposed estimator $\widetilde{R}\ (t)$ has the same limitations, but are relatively much better than the classical estimator $\hat{R}(t)$ in terms of higher relative efficiency and broader range of λ for which efficiency is greater than unity.

5.2 Conclusions:

1. As the main objective of TPSE is to reduce the cost of the sampling units of the estimator, i.e., to cut down the sample size without reducing efficiency, we prefer to study empirically the relationship between the relative efficiency, λ and the ratio

(n_2 / n_1). Indeed the value of n_1 is dictated by the availability of the experimental data and the second sample n_2 can be produced whenever necessary, by performing a new experiment. It is observed from our computation (Figures 4.1-4.6), that (for $0.8 > \lambda > 1.2$) the increment of the maximum increase in relative efficiency decreases with (n_2 / n_1), and is between 15% (when $(n_2 / n_1) = 0.25$) to 3% (when $(n_2 / n_1) = 3$) approximately. The corresponding increment of increase in (n_2 / n_1) (or in n) is almost fixed, and is (n_2 / n_1)%. Thus the choice $(1/4)\ \ n_1 \le n_2 \le 3n_1$ (which is depends upon the availability of experimental data) is recommended to get relative efficiency more than 1.

2. We have considered TPSE for $R(t)$ when the data is time censored. It is clear that if we have a proper prior value of the unknown parameter θ, then TPSE of $R(t)$ has clear advantage over the classical estimator $\hat{R}(t)$. It is observed that the expected sample size is close to n_1, and increases very slowly with increases of n_1 also for $0 \le \lambda \le 10$, and the TPSE $\widetilde{R}\ (t)$ of R (t) is better than the classical estimator $\hat{R}(t)$ both in terms of higher relative efficiency, and broader range of λ for which efficiency is greater than unity. Accordingly, even if the experimenter has less confidence in the guessed value θ_0 (if $\hat{\theta}_1 \notin R$), the relative efficiency is greater than the classical estimator. We have also reported the optimal choice for the constants involved in the proposed estimator. Thus $\widetilde{R}\ (t)$ may be used to improve the efficiency and to decrease the sample size of experimentation if λ belongs to the region $0 \le \lambda \le 10$.

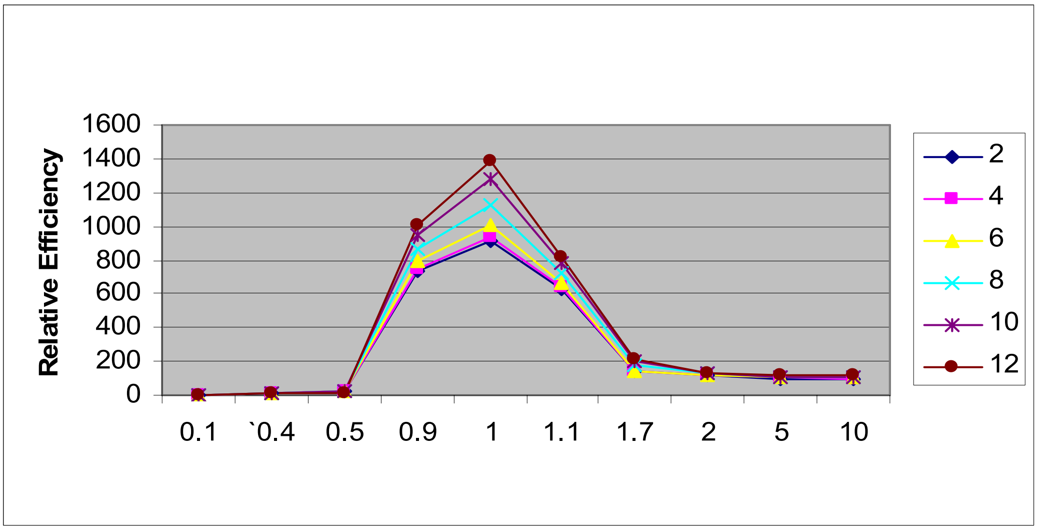

Figure 4.1. $q_i = 0.5$, $t\theta = 3$, $n_1 = 4, n_2 = 2(2)12$, $\lambda = 0.1(0.1)10$, and $k = 0.1$.

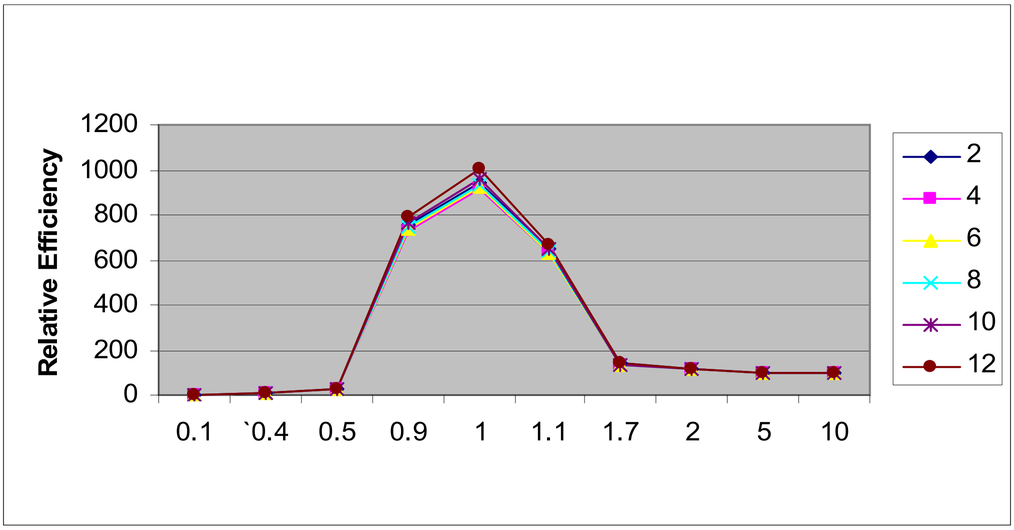

Figure 4.2. $q_i = 0.5$, $t\theta = 3$, $n_1 = 8, n_2 = 2(2)12$, $\lambda = 0.1(0.1)10$, and $k = 0.1$.

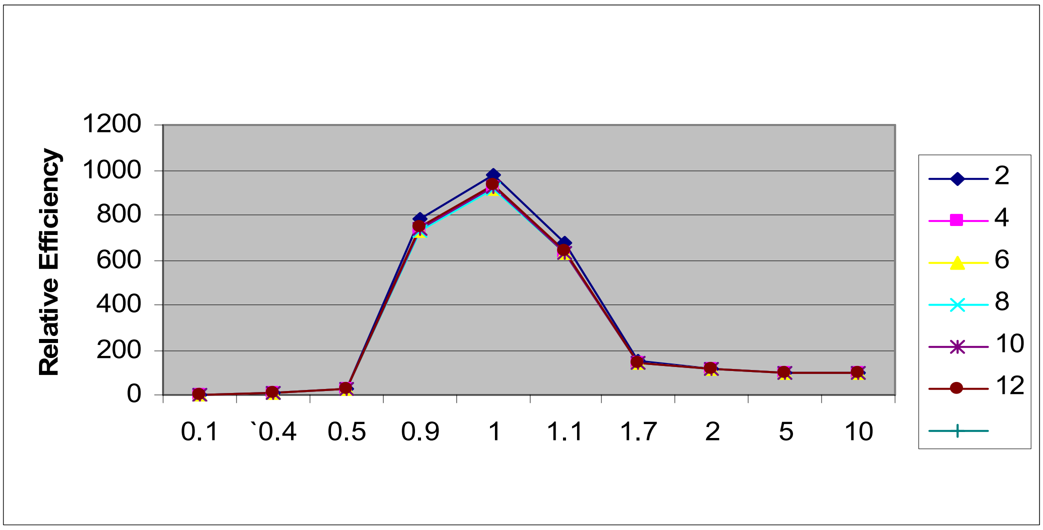

Figure 4.3. $q_i = 0.5$, $t\theta = 3$, $n_1 = 12, n_2 = 2(2)12$, $\lambda = 0.1(0.1)8$, and $k = 0.1$.

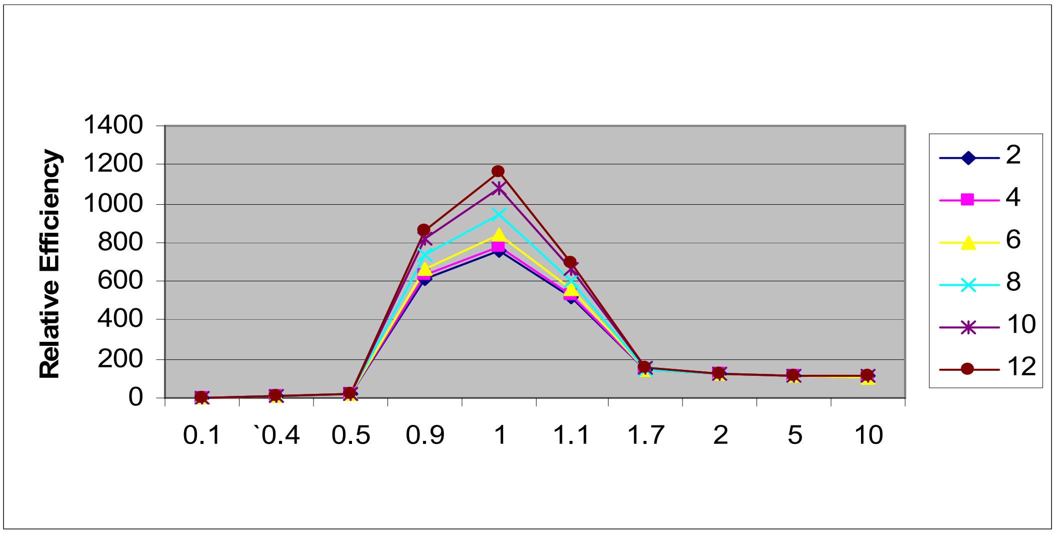

Figure 4.4. $q_i = 0.5$, $t\theta = 3$, $n_1 = 4, n_2 = 2(2)12$, $\lambda = 0.1(0.1)10$, and $k = 0.1$.

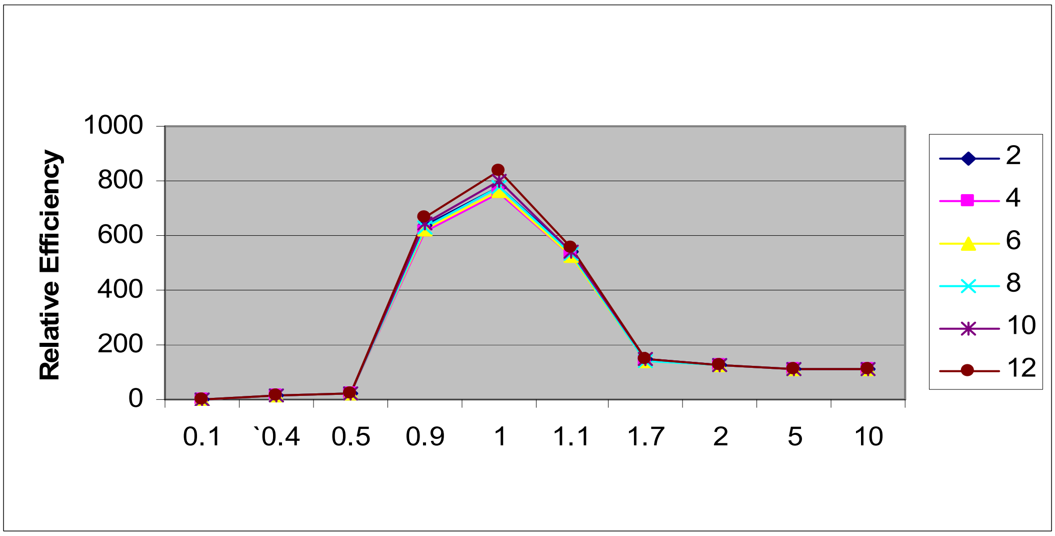

Figure 4.5. $q_i = 0.5$, $t\theta = 2.9$, $n_1 = 8, n_2 = 2(2)12$, $\lambda = 0.1(0.1)10$, and $k = 0.1$.

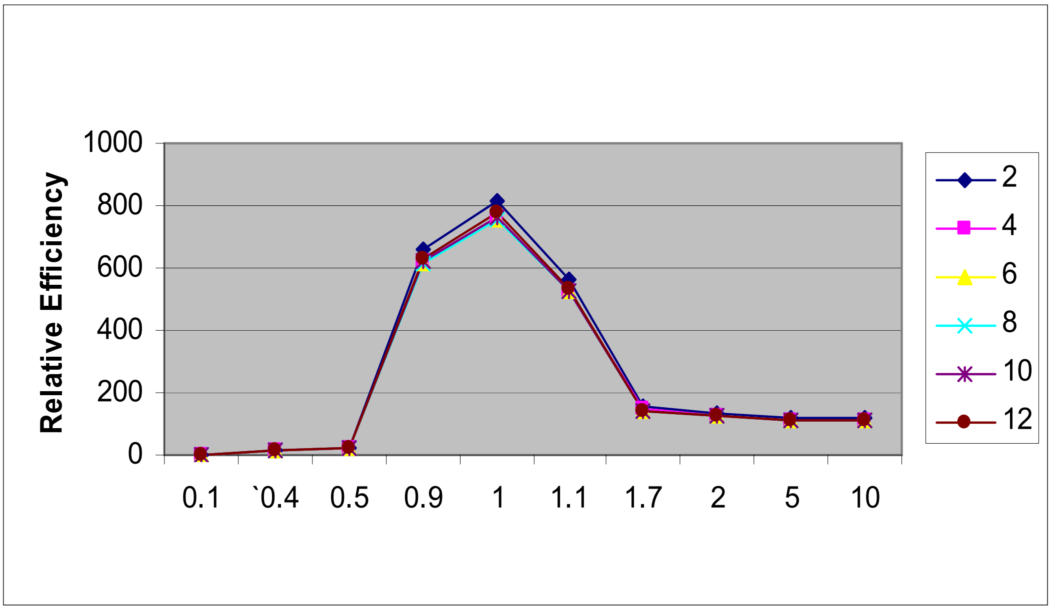

Figure 4.6. $q_i = 0.5,\ t\theta = 2.9,\ \lambda = 0.1(0.1)10,\ n_1 = 12, n_2 = 2(2)12,$ and $k = 0.2.$

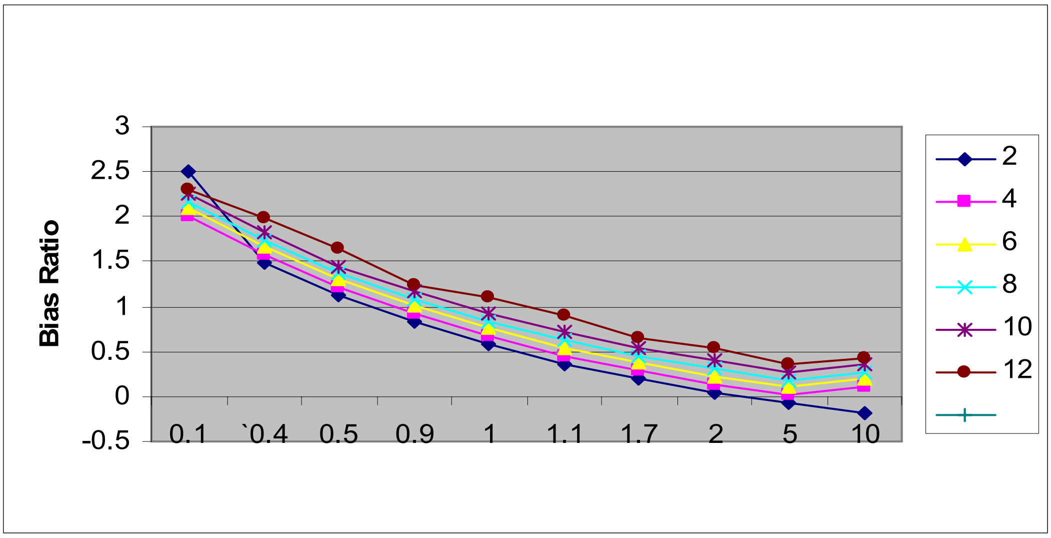

Figure 4.7. $q_i = 0.5,\ n_1 = 4, n_2 = 2(2)12\ \ t\theta = 3,\ \lambda = 0.1(0.1)10,$ and $k = 0.1.$

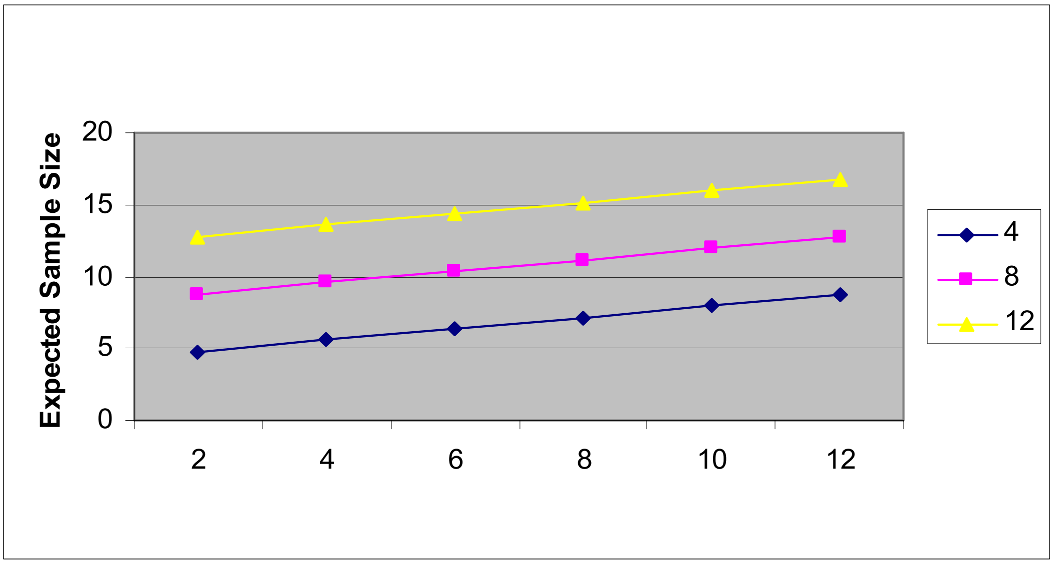

Figure 4.8. $q_i = 0.5,\ \ n_1 = 4{,}8{,}12, n_2 = 2(2)12,\ \ \lambda = 0.1(0.1)10,\ t\theta = 3,$ and $k = 0.1$.

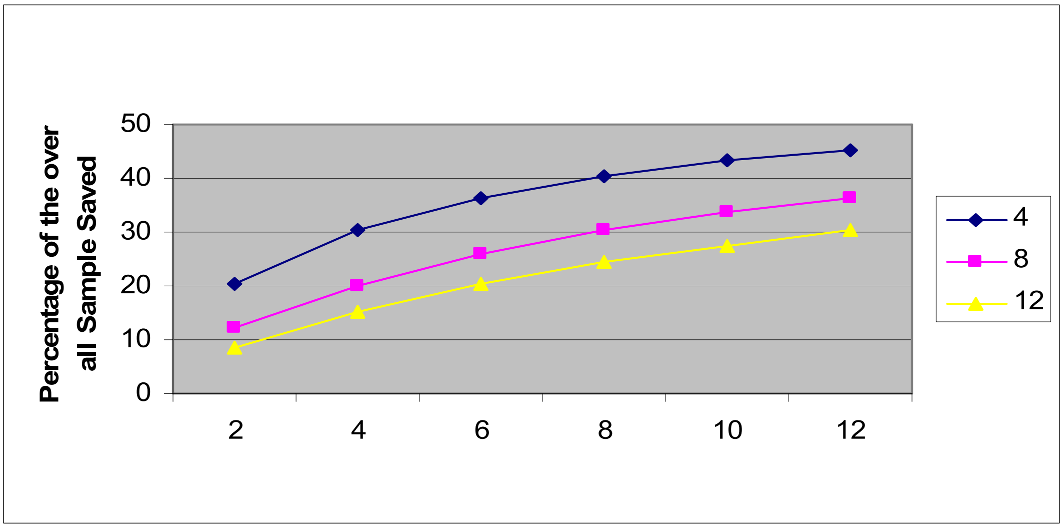

Figure 4.9. $q_i = 0.5,\ \ n_1 = 4{,}8{,}12, n_2 = 2(2)12,\ \ \lambda = 0.1(0.1)10,\ t\theta = 3,$ and $k = 0.1$.

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