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# Three extended geometric process models for modeling reliability deterioration and improvement

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**Abstract.** The geometric process (GP) has been widely used for modeling failure and repair time sequences of repairable systems. The GP is mathematically tractable but restrictive in reliability applications since it actually assumes that the mean function of inter-failure times sequence asymptotically decreases to zero; and the mean function of successive repair times sequence asymptotically increases to infinity. This is generally unrealistic from an engineering perspective. This paper presents three extended GP models for modeling reliability deterioration and improvement (or growth) process. The extensions maintain the advantage of mathematical tractability of GP model. Their usefulness and appropriateness are illustrated with three real-world examples.

**Key Words:** *Repairable system, stochastic process, extended geometric process, reliability, repair time*

## 1. INTRODUCTION

Given a set of failure times of a repairable system, an important reliability problem is to estimate the future failure times of the system. Various stochastic process models have been developed for this issue (e.g., see Calabria and Pulcini (2000) and cited the literature therein). The geometric process (GP) proposed by Lam (1988) has been widely used for representing the stochastic phenomenon that the successive working times of the system after repair stochastically become shorter and shorter (e.g., see Finkelstein (1993), Stanley (1993), Leung and Lee (1998), Wang and Pham (2006), and Lam (2007)). Several authors (e.g., Lai and Yuan (1990), Wang and Pham (2006), and Zhang and Wang (2007)) assume that the successive repair times after failures constitute a non-decreasing GP. Chan et al. (in press) mention that the GP model is very restrictive in applications and develop a Bayesian conditional autoregressive GP model, where the model parameters are time-dependent.

The main problem of the GP is that the mean function of inter-failure times

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asymptotically decreases to zero as the number of failures. Most of repairable systems have an asymptotic MTBF as shown in Figure 1.1. This phenomenon is reproduced by Jiang and Qin (2007) by simulation. They study the failure point process of a series system composed of several components. The system is repaired at failure by replacing the failed component with a new one. As such, the failure process is simulated and the MTBF is evaluated. The simulated MTBF is very similar to the one in Figure 1.1. Using the simulated data, they further examine whether or not the GP model is appropriate for modeling the inter-failure times under this component failure replacement policy and minimal repair policy, and show that the GP is inappropriate for modeling these two stochastic processes.

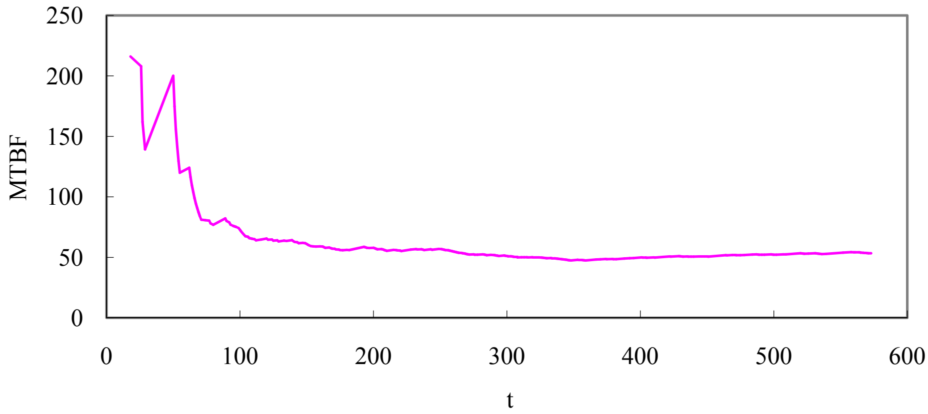


**Figure 1.1** MTBF of an actual repairable system (Note: the time scales of $t$ and MTBF are different)

From an engineering perspective, it is impossible for repair time to increase to infinity. If it is increasing, the rate of increase should gradually decrease with the number of repairs. In other words, the GP is also inappropriate for modeling repair time sequence. In fact, we have not found any published repair time sequence data from field to validate the use of the GP model.

A main advantage of the GP model is its mathematical tractability. To avoid the limitations without loss of this advantage, we present three extended GP models in this paper. The basic idea is to transform a non-negative random variable sequence $Z=\{Z_j, j\geq 1\}$ into a GP sequence $X=\{X_j, j\geq 1\}$ so that $Z=\{Z_j, j\geq 1\}$ follows an extended GP. Three simple transformations are introduced and result in three extended GP models. The extended GP models can be used for modeling reliability deterioration process, repair time sequence if the repair times are increasing, and reliability improvement process. The usefulness and appropriateness of the proposed models will be illustrated with three real-world examples.

The paper is organized as follows. Section 2 outlines the GP and its properties. The three extended GP models are presented and illustrated in Sections 3, 4 and 5, respectively. Section 3 also presents extensions of the power-law and log-linear point process models. The paper is concluded in Section 6.

## 2. GEOMETRIC PROCESS AND ITS PROPERTIES

### 2.1 GP model

Let $X = \{X_j, j \ge 1\}$ be a set of non-negative random variables. If $Y = \{Y_j = \rho^{j-1} X_j, j \ge 1\}$ forms a renewal process, then $X$ is called a GP and $\rho$ (a positive real number) is called the ratio of the GP.

Let $\mu$ and $\sigma^2$ denote the mean and variance of the renewal process $Y$. The mean and variance of the GP $X$ are given by

$$E(X_j) = m_j = \mu / \rho^{j-1}, \quad Var(X_j) = s_j^2 = \sigma^2 / \rho^{2(j-1)}. \tag{2.1}$$

Letting $\alpha = 1/\rho$, (2.1) can be written as

$$m_j = \mu\alpha^{j-1}, \quad s_j = \sigma\alpha^{j-1}. \tag{2.2}$$

As such, a GP can be fully specified by (2.2).

### 2.2 Cumulative working time and MTBF

A failure point process can be represented using different measures of failure time. Two important measures are cumulative working time and MTBF. Let $X$ be an inter-failure times sequence, $T = \{T_j, j \ge 1\}$ denote the cumulative working time process, and $M = \{M_j, j \ge 1\}$ denote the MTBF process.

The cumulative working time to the $j$-th failure is a random variable and given by

$$T_j = \sum_{i=1}^{j} X_i. \tag{2.3}$$

From (2.2) and (2.3) we have

$$\mu_j = E(T_j) = \sum_{i=1}^{j} m_i = \mu \sum_{i=1}^{j} \alpha^{i-1} = \mu \frac{1-\alpha^j}{1-\alpha}. \tag{2.4}$$

The MTBF of the first $j$ working cycles is a random variable and given by

$$M_j = T_j / j, \quad j \ge 1. \tag{2.5}$$

From (2.4) and (2.5), we have:

$$\delta_j = E(M_j) = \mu_j / j = \frac{\mu(1-\alpha^j)}{j(1-\alpha)}. \tag{2.6}$$

We call $m_j$, $\mu_j$ and $\delta_j$ the mean functions of $X$, $T$ and $M$, respectively. It is noted that the cumulative working time and MTBF processes contain the information of the history of the failure process. Therefore, one often transforms the inter-failure data into the cumulative working time data for reliability modeling.

### 2.3 Some properties of GP as a failure point process

In reliability and maintenance applications of the GP, it is usually assumed that an inter-failure time sequence follows the GP with $\alpha < 1$. If this assumption holds, from (2.2), (2.4) and (2.6), the failure point process should have the following properties:

$$m_j \to 0\,,\ \ \mu_j \to \frac{\mu}{1-\alpha},\ \ \delta_j \to 0\,. \tag{2.7}$$

Eq. (2.7) implies that the asymptotic MTBF of the failure point process tends to zero and the mean of cumulative working time after many repairs is always smaller than a certain value. Generally, practical engineering systems do not have such features. Therefore, the GP cannot well represent the practical failure behavior of engineering systems, and is generally inappropriate for modeling inter-failure sequence.

## 3. EXTENDED GEOMETRIC PROCESS MODEL 1

### 3.1 Extended GP-1

Let $Z = \{Z_j, j \ge 1\}$ denote an inter-failure time sequence, which is stochastically decreasing and asymptotically tends to a positive constant $a$. The stochastic process $X = Z - a$ will asymptotically tend to zero. We call $Z$ is an extended GP (or EGP-1 for short) if

$$X = Z - a \tag{3.1}$$

is a GP. In this case, the stochastic process $Y = \{(Z_j - a)/\alpha^{j-1}, j \ge 1\}$ is a renewal process with mean $\mu$ and variance $\sigma^2$.

Let $E(Z_j) = m_{z,j}$ denote the mean function of $Z$. From (3.1), we have

$$E(X_j) = m_j = m_{z,j} - a\,. \tag{3.2}$$

Using (2.2) to (3.2) we have:

$$m_{z,j} = a + \mu\alpha^{j-1},\ \ 0 < \alpha < 1\,. \tag{3.3}$$

Clearly, $m_{z,j} \to a$, implying that the stochastic process $Z$ meets the desired asymptotic property. Also, it is noted that $m_{z,j+1}/m_{z,j} = \alpha + a(1-\alpha)/(a+\mu\alpha^{j-1})$ increases and tends to 1 as $j$ increases, implying that the rate of decrease gets gradually slow. This feature is similar to the one shown in Figure 1.1 and hence desired.

The mean function of the stochastic process $T = \{T_j = \sum_{i=1}^{j} Z_i\}$ is given by:

$$\mu_j = aj + \mu\frac{1-\alpha^j}{1-\alpha}. \tag{3.4}$$

The mean function of the stochastic process $M = \{M_j = T_j / j\}$ is given by:

$$\delta_j = a + \frac{\mu(1-\alpha^j)}{j(1-\alpha)}. \tag{3.5}$$

Clearly, $\delta_j \to a$, implying that the MTBF process meets the desired asymptotic property.

### 3.2 Discussion

Eq. (3.4) gives an important observation: *the mean of cumulative working time* $\mu_j$ *is an asymptotically linear function of* $j$ *if the repairable system has an asymptotic positive MTBF.*

Two well-known failure point process models are the power-law model (e.g., see Crow (2006)) given by

$$E[J(t)] = \lambda t^{\beta} \quad \text{or} \quad \mu_j = (j/\lambda)^{1/\beta} \tag{3.6}$$

and the log-linear model (e.g., see Lawless and Thiagarajah (1996)) given by

$$E[J(t)] = a(e^{t/b} - 1) \quad \text{or} \quad \mu_j = b\ln(1 + j/a) \tag{3.7}$$

where $J(t)$ is the number of failures in $(0, t)$. For the power-law model, $\mu_j - \mu_{j-1} \to j^{1/\beta-1}/(\beta\lambda^{1/\beta})$ so that $\mu_j - \mu_{j-1} \to 0$ for $\beta > 1$ (for modeling the reliability deterioration process) and $\mu_j - \mu_{j-1} \to \infty$ for $\beta < 1$ (for modeling the reliability growth process). For the log-linear model, $\mu_j - \mu_{j-1} \to b\ln[1 + 1/(j + a - 1)] \to 0$. As such, both the models do not meet the above-mentioned asymptotic property.

To improve, we extend the power-law model given by (3.6) as below:

$$\mu_j = aj + bj^c \quad , \quad a, b > 0, c \in (0,1) \tag{3.8}$$

so that $\mu_{j+1} - \mu_j \to a$ as $j$ increases. Similarly, the log-linear model given by (3.7) can be extended as

$$\mu_j = aj + b\ln(1 + j/c) , \quad a, b, c > 0 . \tag{3.9}$$

### 3.3 Parameter estimation

Consider a failure point process:

$$\{T\} = \{t_0 = 0 \le t_1 \le t_2 \le ... \le t_n \le t_s\} \tag{3.10}$$

where $t_s$ is the observation stopping time. Let $\mu_j = \varphi(j;\theta)$ denote the mean function of $T$, where $\theta$ is the model parameter set. For EGP-1 model, $\varphi(j;\theta)$ is given by (3.4) and $\theta = (\mu, \alpha, a)$. We estimate the model parameters by fitting $\mu_j = \varphi(j;\theta)$ to the data set (3.10) using least squares method.

In the failure point process (3.10), $t_s$ contains more life information than $t_j$ for $j \le n$. Therefore, we introduce the following constraint:

$$\varphi(n;\theta) = t_s . \tag{3.11}$$

The parameter set $\theta$ is estimated by minimizing the following sum of squared errors:

$$SSE = \sum_{j=1}^{n} [t_j - \varphi(j;\theta)]^2 . \tag{3.12}$$

subject to the constraints: the one given by (3.11), $a \ge 0$ and

$$\min(x_j = z_j - a = t_j - t_{j-1} - a, 1 \le j \le n) \ge 0 . \tag{3.13}$$

The constraint given by (3.13) ensures that the GP $X$ is non-negative. The Solver of Microsoft Excel can be used to directly find the parameters that minimize $SSE$.

Once the estimates of $a$ and $\alpha$ are obtained, the estimate of $\sigma^2$ can be obtained by

$$\sigma^2 = Var(x_j / \alpha^{j-1}, 1 \le j \le n). \tag{3.14}$$

### 3.4 Example 1: Automobile data

The data shown in Table 3.1 deal with the failure times on the cumulative time scale with $n = 21$ and $t_s = t_n$, and can be found in Calabria and Pulcini (2000). Using the approach outlined in Section 3.3 to fit the EPG-1 given by (3.4) to the data, we obtained the estimates of the model parameters: $(a, \mu, \alpha) = (0.1160, 5.9473, 0.9326)$. From (3.14), we have $\sigma = 3.2433$.

Figure 3.1 shows the plots of the observed data and fitted mean function. As seen, they are fairly close to each other, particularly for those points with large $j$.

**Table 3.1.** Cumulative failure times (in thousand miles)

| | | | | | | |
|---|---|---|---|---|---|---|
| 11.016 | 16.336 | 24.435 | 26.231 | 26.347 | 30.701 | 34.967 |
| 38.517 | 42.594 | 43.350 | 45.082 | 46.686 | 51.225 | 55.321 |
| 59.344 | 60.671 | 63.523 | 66.505 | 67.659 | 69.110 | 70.271 |

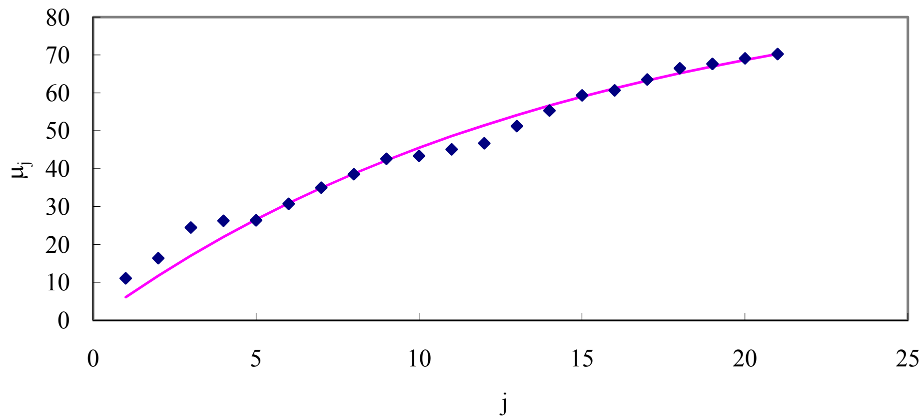


**Figure 3.1.** Plots of the fitted mean function and observed data

## 4. EXTENDED GEOMETRIC PROCESS MODEL 2

### 4.1 Extended GP-2

Let $Z = \{Z_j, j \ge 1\}$ denote a stochastic process (e.g., a reliability improvement process or a repair time process), which is stochastically increasing and asymptotically tends to a positive constant $a$. The stochastic process $X = \{ X_j = a - Z_j, j \ge 1 \}$ is stochastically decreasing and tends to zero. We call the stochastic process

$$Z = a - X \tag{4.1}$$

is an extended GP (EGP-2 for short) when $X$ is a GP. In this case, the stochastic process $Y = \{(a - Z_j) / \alpha^{j-1}, j \ge 1\}$ is a renewal process with mean $\mu$ and variance $\sigma^2$.

The mean function of the stochastic process $Z$ is given by

$$a - m_{z,j} = \mu\alpha^{j-1} \text{ or } m_{z,j} = a - \mu\alpha^{j-1}, \; 0 < \alpha < 1. \tag{4.2}$$

The sequence $\{ m_{z,j}, j \ge 1 \}$ is increasing and tends to $a$. It is easy to show that $m_{z,j+1} / m_{z,j}$ decreases and tends to 1 as $j$ increases, implying that the rate of reliability improvement gets gradually slow.

The mean function of the stochastic process $T = \{T_j\}$ is given by:

$$\mu_j = aj - \mu \frac{1 - \alpha^j}{1 - \alpha}. \tag{4.3}$$

Clearly, $\mu_j$ is asymptotically linear with $j$.

The mean function of the stochastic process $M = T / j$ is given by:

$$\delta_j = a - \frac{\mu(1 - \alpha^j)}{j(1 - \alpha)}. \tag{4.4}$$

The sequence $\{ \delta_j, j \ge 1\}$ is also increasing and tends to $a$.

### 4.2 Parameter estimation

The parameter estimation method is the same as that outlined in Section 3.3 but (3.13) is replaced by

$$\min(x_j = a - z_j = a - t_j + t_{j-1}, 1 \le j \le n) \ge 0. \tag{4.5}$$

### 4.3 Example 2: Test-fix-test data

The data shown in Table 4.1 is a set of reliability growth test data with $n = 40$ and $t_s = t_n$, and can be found in Crow (2006). Crow fits the data to the power-law model given by (3.6), which has the intensity function given by

$$r(t) = \frac{dE[J(t)]}{dt} = \lambda\beta t^{\beta-1}. \tag{4.6}$$

The maximum likelihood estimates of the parameters are: ($\beta, \lambda$) = (0.4898, 0.7615). The achieved failure intensity and MTBF are estimated by

$$r(t_s) = 0.0060, MTBF = 1 / r(t_s) = 166.22. \tag{4.7}$$

It is noted that the MTBF can be estimated by $\mu_{n+1} - \mu_n$. Using the Crow's estimates to (3.6) we have $\mu_{41} - \mu_{40} = 168.26$. This estimate is larger than but close to the one given by (4.7) with a relative error of 1.2%.

**Table 4.1.** Test-fix-test data (in hours)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.7 | 2.7 | 13.2 | 17.6 | 54.5 | 99.2 | 112.2 |
| 120.9 | 151.0 | 163.0 | 174.5 | 191.6 | 282.8 | 355.2 |
| 486.3 | 490.5 | 513.3 | 558.4 | 678.1 | 699.0 | 785.9 |
| 887.0 | 1010.7 | 1029.1 | 1034.4 | 1136.1 | 1178.9 | 1259.7 |
| 1297.9 | 1419.7 | 1571.7 | 1629.8 | 1702.3 | 1928.9 | 2072.3 |
| 2525.2 | 2928.5 | 3016.4 | 3181.0 | 3256.3 | | |

Fitting the EGP-2 given by (4.3) to the data, we obtained the estimates of the model parameters: $(a,\mu,\alpha)=(8438.45,\ 8438.45,\ 0.9995)$. From (3.14) with $x_j = a - t_j + t_{j-1}$, we have $\sigma = 79.46$. The achieved reliability can be represented by $Z_{41} = T_{41} - T_{40}$ with

$$E(Z_{41}) = \mu_{41} - \mu_{40} = 166.42\,,\quad Var(a - Z_{41}) = Var(Z_{41}) = 77.89^2\,.$$

It is noted that the estimate of achieved MTBF (i.e., 166.42) is almost the same as the estimate of Crow (i.e., 166.22).

Figure 4.1 shows the plots of the observed data and fitted mean functions (obtained from the power-law model and EGP-2). As seen, the two fitted curves almost overlap together, and they are fairly close to the observed data.

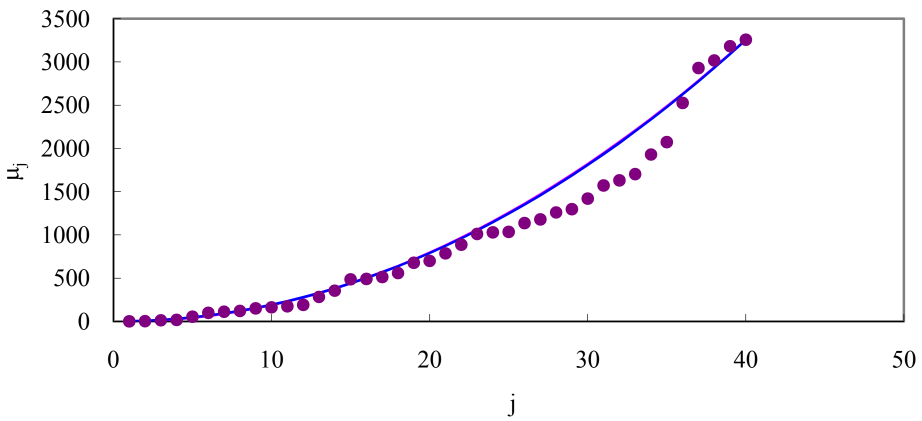


**Figure 4.1.** Plots of the fitted mean functions and observed data

## 5. EXTENDED GEOMETRIC PROCESS MODEL 3

### 5.1 EGP-3

Let $Z=\{Z_j, j\ge 1\}$ denote a stochastic process (e.g., a reliability growth process), which is stochastically increasing. The stochastic process $X = \{\, X_j = Z_j + a, j \ge 1, a > 0 \,\}$ is stochastically increasing and tends to infinity. We call the stochastic process

$$Z = X - a \tag{5.1}$$

is an extended GP (EGP-3 for short) when $X$ is a GP. In this case, the stochastic process

$Y = \{(Z_j + a)/\alpha^{j-1}, \alpha > 1, j \geq 1\}$ is a renewal process with mean $\mu$ and variance $\sigma^2$.

The mean function of the stochastic process $Z$ is given by

$$m_{z,j} + a = \mu\alpha^{j-1} \text{ or } m_{z,j} = \mu\alpha^{j-1} - a. \tag{5.2}$$

To make $m_{z,1} > 0$, it is required:

$$\mu > a. \tag{5.3}$$

The mean function of the stochastic process $T$ is given by:

$$\mu_j = \mu\frac{\alpha^j - 1}{\alpha - 1} - aj. \tag{5.4}$$

The mean function of the stochastic process $M = T/j$ is given by:

$$\delta_j = \frac{\mu(\alpha^j - 1)}{j(\alpha - 1)} - a. \tag{5.5}$$

Different from the previous extensions, this extension has $\alpha > 1$ and, $m_{z,j}$ and $\delta_j$ asymptotically tend to infinite. From (5.2), we have

$$\frac{m_{z,j+1}}{m_{z,j}} = \alpha + \frac{a(\alpha - 1)}{\mu\alpha^{j-1} - a}. \tag{5.6}$$

The RHS of (5.6) is a decreasing function of $j$. This implies that the rate of increase of the stochastic process $Z$ gets gradually slow.

### 5.2 Parameter estimation

The parameter estimation method is the same as that outlined in Section 3.3 but there is an additional constraint given by (5.3).

### 5.3 Example 3: Aircraft generator data

The data shown in Table 5.1 deal with the aircraft generator data with $n = 14$ and $t_s = t_n$, and can be found in Calabria and Pulcini (2000). The data show that the reliability improves with operating time.

**Table 5.1.** Aircraft generator failure time data

| | | | | | | |
|---|---|---|---|---|---|---|
| 10 | 55 | 166 | 205 | 341 | 488 | 567 |
| 731 | 1308 | 2050 | 2453 | 3115 | 4017 | 4596 |

Fitting the EGP-3 given by (5.4) to the data, we obtained the estimates of the model parameters: $(a, \mu, \alpha) = (225.43,\ 225.43,\ 1.1281)$. From (3.14), we have $\sigma = 53.05$. The time to the next failure can be represented by $Z_{n+1} = T_{n+1} - T_n$ if the improvement continues in a similar way. From the fitted model, we have:

$$E(Z_{n+1}) = \mu_{n+1} - \mu_n = 992.84, \quad Var(Z_{n+1}) = 286.70^2.$$

Figure 5.1 shows the plots of the observed data and fitted mean function. As seen, the fitted curve and observed data points are fairly close to each other.

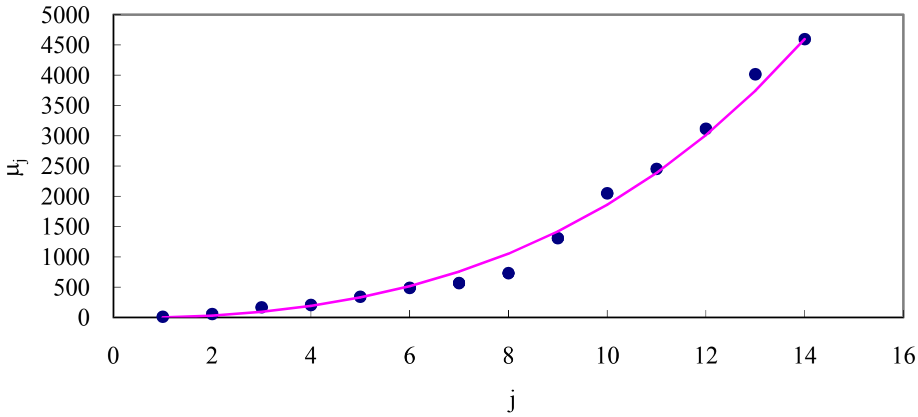


**Figure 5.1.** Plots of the fitted mean functions and observed data

## 6. DISCUSSIONS AND CONCLUSIONS

In this paper, we have identified the limitations of the GP in reliability and maintenance applications and proposed three extended GPs. The extended GPs are mathematically tractable, and their appropriateness and usefulness have been illustrated with three real-world examples. In addition, we have extended the power-law and log-linear models to make them meet the asymptotic property that MTBF tends to a positive constant.

The main property and applicability of these extended models are summarized in Table 6.1. Specifically, EGP-1 model is suitable for modeling reliability deterioration; the extended power-law and log-linear models provide two options fore this case; EDP-2 model is suitable for modeling a reliability growth process or a repair time sequence; and EDP-3 model provides another option for modeling reliability improvement. As a result, this contribution is useful for reliability and maintenance modeling of repairable systems.

**Table 6.1.** Summary of the proposed models

| Model | $\alpha$ | Properties | Applicability |
|---|---|---|---|
| EPG-1 | < 1 | Asymptotic MTBF | Reliability deterioration process |
| EPG-2 | < 1 | Asymptotic MTBF or repair time | Reliability improvement process or repair time process |
| EPG-3 | > 1 | Rate of increase gradually decreases | Reliability growth process |
| Extended power-law model | | Asymptotic MTBF | Reliability deterioration process |
| Extended log-linear model | | Asymptotic MTBF | Reliability deterioration process |

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