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A Bayesian approach to maintenance strategy for non-renewing free replacement-repair warranty

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Abstract. This paper considers the maintenance model suggested by Jung and Park (2010) to adopt the Bayesian approach and obtain an optimal replacement policy following the expiration of NFRRW. As the criteria to determine the optimal maintenance period, we use the expected cost during the life cycle of the system. When the failure times are assumed to follow a Weibull distribution with unknown parameters, we propose an optimal maintenance policy based on the Bayesian approach. Also, we describe the revision of uncertainty about parameters in the light of data observed. Some numerical examples are presented for illustrative purpose.

Key Words: *Bayesian replacement model, expected cost rate per unit time; replacement-repair warranty*

1. INTRODUCTION

A manufacturer usually provides a certain type of warranty policy to the user at the sale of its system. Thus, a number of maintenance policies following the warranty period have been proposed and discussed in the literature. Sahin and Polatoglu (1996) consider the maintenance strategies following the expiration of replacement warranty and Jung and Park (2003) suggest an optimal preventive maintenance policy after the replacement warranty period is expired. Later, Chen and Chien (2007) deal with a model to study the PM effect carried out by the buyer on items which are sold under a free-replacement renewing warranty. On the other hand, Yeh, Chen and Lin (2007) propose an optimal replacement policy for repairable system under free repair warranty.

Recently, some papers consider a combination of the free-replacement and pro-rata warranty policies. Especially, Chien (2010) develops a method to determine the optimal preventive replacement age for non-repairable products under a fully renewable free replacement with a pro-rata warranty policy and Jung and Park (2010) study the

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maintenance policy following the expiration of non-renewing free replacement-repair warranty (NFRRW).

In most of practical situations, however, the failure distribution of the system or its parameters may not be known and so they must be estimated to obtain the optimal maintenance policy. For such purposes, many authors have adopted the Bayesian approach. Mazzuchi and Soyer (1996) suggest a Bayesian decision theoretic approach to obtain the optimal replacement strategies. Sheu, Yeh, Lin and Juang (1999) extend Mazzuchi and Soyer' model (1996) by taking the randomness of minimal repair cost. Recently, Jung, Han and Park (2010) consider a Bayesian approach to determine the optimal maintenance policy following the expiration of non-renewing replacement warranty.

In this paper, we consider the maintenance model suggested by Jung and Park (2010) to adopt the Bayesian approach and obtain an optimal replacement policy following the expiration of NFRRW. Under NFRRW, the system is replaced with a new one or is minimally repaired by the manufacturer at no charge to the user when the system fails during the replacement warranty period and the repair warranty period, respectively. And if the system failures occur after the warranty is expired, the system is minimally repaired at each failure. As the criteria to determine the optimality of the maintenance policy, we consider the expected cost rate per unit time. All maintenance costs of the system incurred after the warranty is expired are paid by the user. Given the cost structures during the life cycle of the system, we determine the Bayesian optimal maintenance period following the expiration of NFRRW, after which the system is replace by a new one.

In Section 2, we briefly review Jung and Park's (2010) model and Section 3 presents a Bayesian approach to determine the optimal maintenance policy following the expiration of NFRRW. The prior and posterior distributions for the parameters of our interests are presented in the light of data observed and the likelihood function is given. Section 4 gives numerical examples for illustrative purpose

Nomenclature

T	time to failure of a system
$F(t)$, $f(t)$	life distribution and probability density function of T
w	NRRW period
w_R	NRW period
w_M	NMW period
$h(t)$	failure rate function
x	length of maintenance period after NRRW is expired
$c_{f,m}$	unit failure cost during the maintenance period
c_r	unit cost of replacement at the end of the maintenance period
$c_{m,m}$	unit cost of minimal repair during the maintenance period
y	age of the system in use at the end of NRRW period
k	number of replacements during the NRRW period
$ECL(x)$	expected cycle length
$ETC(x)$	expected total cost
$C_C(x)$	expected cost rate per unit time in classical approach
$C_B(x)$	expected cost rate per unit time in Bayesian approach

2. JUNG AND PARK'S MODEL

In this section we briefly review the replacement model following the expiration of NFRRW and its optimization, discussed in Jung and Park (2010). Under NFRRW, the system is replaced with a new one or is minimally repaired by the manufacturer at no charge to the user when the system fails during the replacement warranty period and the repair warranty period, respectively. It also should be pointed out that the replacement period and the repair period are not renewed under NFRRW. However, if the system failure occurs during the maintenance period, it is minimally repaired at the user's own expenses by the user. At the end of maintenance period of a fixed length, the system is replaced by a new one, regardless of its current age. Once the original non-renewing warranty is expired, the user is fully responsible for each failure of the system during the entire maintenance period.

The age of the system that is still in use at the end of warranty period could be anywhere between 0 and w in case of non-renewing warranty and this age should be known to the user when the warranty period expires. Let y and k be the age of the system in use at the end of a non-renewing warranty period and the number of replacements during a non-renewing warranty period, respectively. Here, y is assumed to be known. Under the non-renewing warranty policy, the life cycle starts with the original purchase of the system and ends when the system is replaced by a new one at the end of maintenance period, regardless of the number of replacements during the warranty period. Thus, the expected cycle length is always equal to $w + x$ independent of y.

Under the Jung and Park's (2010) maintenance model, the expected cost rate per unit time is given as

$$C(x) = \frac{ETC(x)}{ECL(x)} = \frac{1}{w+\tau}\left\{c_0 + \left(c_{f,m} + c_{m,m}\right)\int_{y+w_M}^{y+w_M+x\tau} h(t)\,dt\right\}, \tag{2.1}$$

where $c_0 = c_{f,w}\left(k + \int_y^{y+w_M} h(t)\,dt\right) + c_r$.

Assume that the failure time of the system follows a Weibull distribution with the scale and shape parameters of α and β, respectively. Then, the probability density function and failure rate are given as

$$f(t) = \alpha\beta t^{\beta-1}\exp\{-\alpha t^{\beta}\},\ t > 0, \alpha > 0, \beta > 1,$$

and

$$h(t) = \alpha\beta t^{\beta-1},\ t > 0, \alpha > 0, \beta > 1, \tag{2.2}$$

respectively. If the Weibull distribution, defined in (2.2), is assumed, the expected cost rate per unit time, given in (2.1), becomes

$$C_C(x) = \frac{1}{w+x}\{c_{f,w}k + c_{f,w}\alpha\left((y+w_M)^{\beta} - y^{\beta}\right) + c_r + \left(c_{f,m} + c_{m,m}\right)\alpha\left((y+w_M+x)^{\beta} - (y+w_M)^{\beta}\right)\}. \tag{2.3}$$

The optimal replacement period can be obtained by minimizing the expected cost rate per unit time, given in (2.3).

3. BAYESIAN MAINTENANCE POLICY AFTER NFRRW

In this section, we extend the Jung and Park's (2010) model to adopt the Bayesian approach and derive the optimal maintenance policy following the expiration of NFRRW. Firstly, we obtain the formulas to compute the expected cost rate per unit time in the Bayesian framework, assuming that the failure times of the system follow a Weibull distribution, as defined in (2.2).

3.1 Optimal maintenance policy

As suggested in Mazzuchi and Soyer (1996), an appropriate prior distribution for the scale parameter α, given in (2.2), is considered to be a gamma distribution, which is defined as

$$f(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}, \alpha > 0 , \tag{3.1}$$

where $a > 0$ and $b > 0$ are hyperparameters. As for the prior distribution of the shape parameter β, given in (2.2), Mazzuchi and Soyer (1996) suggests that the discretized Beta distribution is appropriate since it allows for great flexibility in representing prior uncertainty. Define the Beta distribution as

$$g(\beta) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \frac{(\beta-\beta_L)^{c-1}(\beta_U-\beta)^{d-1}}{(\beta_U-\beta_L)^{c+d-1}}, \; 0 \le \beta_L \le \beta \le \beta_U \tag{3.2}$$

where the hyperparameters c and d are assumed to be positive. Then, the prior distribution for β is assigned as

$$\begin{aligned} P_l &= \Pr(\beta = \beta_l) \\ &= \int_{\beta_l-\delta/2}^{\beta_l+\delta/2} g(\beta)d\beta, \end{aligned} \tag{3.3}$$

where, $\beta_l = \beta_L + \delta(2l-1)/2$ and $\delta = (\beta_U - \beta_L)/m$ for $l = 1,2,\cdots,m$. Here, m is the number of distinct values that β assumes.

Assuming that the prior distributions of α and β are independent, the joint prior distribution of α and β can be expressed as

$$\begin{aligned} P(\alpha, \beta_l) &= f(\alpha)\Pr(\beta = \beta_l) \\ &= \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha} P_l. \end{aligned} \tag{3.4}$$

Now, using the joint prior distribution of α and β, given in (3.4), when the failure time is Weibull distribution with uncertain parameters, the expected cost rate per unit time during the life cycle of the system under non-renewing warranty can be obtained as

$$C_B(x) = \sum_{l=1}^{m} \frac{c_{f,w}k + c_r + c_{f,w}\left(\frac{a}{b}\right)\left((y+w_M)^{\beta_l} - y^{\beta_l}\right) + \left(c_{f,m} + c_{m,m}\right)\left(\frac{a}{b}\right)\left((y+w_M+x)^{\beta_l} - (y+w_M)^{\beta_l}\right)}{w+x} P_l . \tag{3.5}$$

Therefore, when the failure time is Weibull distribution, the optimal maintenance period in Bayesian approach is the value such that the expected cost rate per unit time, given in (3.5), is minimized.

3.2 Adaptive maintenance policy

Now, we consider the update of uncertainty about parameters in the light of data observed. When the failure data are collected at the end of each replacement, we may update the uncertainty about the parameters α and β of a Weibull distribution as considered in Mazzuchi and Soyer (1996). Suppose that there are n failures recorded at times $t_1, t_2, \cdots, t_n$ before the system is replaced by a new one at $y + w_M$ by the user. Given the failure data$t_1, t_2, \cdots, t_n$, the likelihood function is given by

$$L(\alpha, \beta|t) = \left\{\prod_{i=1}^{n} \alpha\beta t_i^{\beta-1}\right\} \exp\{-\alpha(y + w_M + x)^{\beta}\}, \tag{3.8}$$

where $\prod_{i=1}^{n}\{\cdot\} \equiv 1$ accounts for the case of no failures. Using the joint prior distribution of α and β, given in (3.4), and the likelihood function, given in (3.8), the joint posterior distribution of α and β can be expressed as

$$f(\alpha, \beta_l|t) \propto \left\{\prod_{i=1}^{n} \alpha\beta_l t_i^{\beta_l-1}\right\} \exp\{-\alpha(y + w_M + x)^{\beta_l}\}\alpha^{a-1}\exp\{-b\alpha\}P_l. \tag{3.9}$$

Rearranging terms in the equation (3.9), we can write

$$\begin{aligned} f(\alpha, \beta_l|t) &= \frac{\alpha^{n+a-1}\beta_l^n\{\prod_{i=1}^{n} t_i\}^{\beta_l-1}\exp\left\{-\alpha\left(b+\left((y+w_M+x)\right)^{\beta_l}\right)\right\}P_l}{\sum_{j=1}^{m} P_j\beta_j^n\{\prod_{i=1}^{n} t_i\}^{\beta_j-1}\int \alpha^{n+a-1}\exp\left\{-\alpha\left(b+\left((y+w_M+x)\right)^{\beta_l}\right)\right\}d\alpha} \\ &= \frac{\alpha^{n+a-1}\beta_l^n\{\prod_{i=1}^{n} t_i\}^{\beta_l-1}\exp\left\{-\alpha\left(b+\left((y+w_M+x)\right)^{\beta_l}\right)\right\}P_l}{\sum_{j=1}^{m} P_j\beta_j^n\{\prod_{i=1}^{n} t_i\}^{\beta_j-1}/\Gamma(a+n)/\left(b+(y+w_M+x)^{\beta_j}\right)^{a+n}}. \end{aligned} \tag{3.10}$$

Since $f(\alpha|\beta_l, t) = f(\alpha, \beta_l, t)/P(\beta_l, t)$, the conditional posterior distribution of α is given by

$$f(\alpha|\beta_l, t) = \frac{\left(b+(y+w_M+x)^{\beta_l}\right)^{a+n}}{\Gamma(a+n)}\alpha^{(a+n)-1}\exp\left\{-\alpha\left(b + (y + w_M + x)^{\beta_l}\right)\right\}. \tag{3.11}$$

Note that the conditional posterior distribution of α, given in (3.11), follows a gamma distribution with parameters $a^* = a + n$ and $b^* = b + (y + w_M + x)^{\beta_l}$. From the joint posterior distribution of α and β and the conditional posterior distribution of α, given in (3.10) and (3.11), respectively, the marginal posterior distribution of β can be written as follows.

$$\begin{aligned} P(\beta = \beta_l|t) &= P_l^* \\ &= \frac{\beta_l^n\{\prod_{i=1}^{n} t_i\}^{\beta_l-1}/\left(b+(y+w_M+x)^{\beta_l}\right)^{a+n}}{\sum_{j=1}^{m} P_j\beta_j^n\{\prod_{i=1}^{n} t_i\}^{\beta_j-1}/\left(b+(y+w_M+x)^{\beta_j}\right)^{a+n}}P_l. \end{aligned} \tag{3.12}$$

Thus, the posterior distributions of α and β are no longer independent. Therefore, the update of the optimal maintenance policy following the expiration of NFRRW as well as the probability distribution for the expected cost rate per unit time can be achieved by replacing the prior quantities a, b and P_l by the posterior quantities a^*, b^* and P_l^* in the formulas (3.5).

4. NUMERICAL EXAMPLES

To describe the optimal Bayesian maintenance policy following the expiration of NFRRW, we select the gamma and discretized beta prior for α and β, respectively, as defined in (3.1) and (3.3) with $a = 2.1$, $b = 3$, $c = 2$, $d = 2$, $\beta_L = 1$, $\beta_U = 3$, and $m = 20$.

In addition, we assume that $w = 0.5$, $w_R = 0.2$, $w_M = 0.3$, $c_r = 20$, $c_{fw} = 1$, $c_{fm} = 1$, and $c_{m,m} = 3$. All the parameter values are chosen arbitrarily for illustrative purpose of the proposed Bayesian method.

Table 4.1 shows the values of x^* and its corresponding expected cost rate $C_B(x^*)$ for various choice of y following the expiration of NFRRW. For instance, when $y = \frac{2}{4} w_R$, the expected cost rate is minimized at $x^* = 1.933$ and its expected cost rate equals $C_B(x^*) = 15.203616$. It suggests that the optimal maintenance period equals 1.933 time units after NFRRW is expired. In Table 4.2, we demonstrate the update of uncertainty about parameters in the light of data observed under NFRRW. At cycle 0, the optimal maintenance period is derived based on the prior distributions without the data. At the end of cycle 0, the failure data is simulated to determine the new optimal maintenance period using the updated parameters, which makes the cycle 1. We repeat the same process to update the uncertainty of parameters and move on to the next cycle. Table 2 describes the adaptive nature of our approach by considering three cycles, where column 2 gives the simulated failure data in each cycle and k is the number of replacements occurred during the warranty period. Columns 5 and 6 show the optimal maintenance period x^* and its corresponding expected cost rate $C_B(x^*)$ for each cycle, respectively.

Table 4.1. Optimal Bayesian maintenance policies with NFRRW

Optimal policy	y		
	$\frac{1}{4} w_R$	$\frac{2}{4} w_R$	$\frac{3}{4} w_R$
x^*	1.949	1.933	1.917
$C_B(x^*)$	14.930251	15.203616	15.479242

Table 4.2. Adaptive maintenance policy with NFRRW when $y = 0.15$

Cycle	Failure Data	k	y	x^*	$C_B(x^*)$
0	based on the prior	1	0.15	1.917	15.479242
1	0.23886 0.34063 0.61730 0.75832 0.81644 0.89211 1.28697 1.32275 1.59933 1.89400	0	0.2	1.901	15.757169
2	0.06746* 0.54753 0.74183 0.77638 0.89433 0.95937 0.96309 0.98799 1.04741 1.39670	1	0.13254	1.923	15.382733
3	0.14949* 0.28799 0.36531 0.38413 0.61981 0.66464 0.97217 1.01994 1.05152 1.09623	1	0.05051	1.950	14.933028

* denotes the time of failure before the NPRW period is expired.

5. CONCLUDING REMARKS

This paper considers the maintenance model suggested by Jung and Park (2010) to adopt the Bayesian approach and obtain an optimal replacement policy following the expiration of NFRRW. Under NFRRW, the system is replaced with a new one or is minimally repaired by the manufacturer at no charge to the user when the system fails during the replacement warranty period and the repair warranty period, respectively. And if the system failures occur after the warranty is expired, the system is minimally repaired at each failure.

When the failure times follow a Weibull distribution with uncertain parameters, a useful Bayesian approach is established to express and update the uncertain parameters for deriving an optimal maintenance policy. The criterion used to determine the optimal maintenance period following the expiration of NFRRW is the expected cost rate per unit time. In the numerical examples, it is observed that the older the system is when the non-renewing replacement warranty period is expired, the shorter is the optimal maintenance period, which is as expected. Also, we describe the revision of uncertainty about parameters in the light of data observed.

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