International Journal of Reliability and Applications
Vol. 12, No. 1, pp. 1-14, 2011

Estimation of parameters including a quadratic failure rate semi-Markov reliability model

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Abstract. This paper discusses the stochastic analysis and the statistical inference of a quadratic failure rate semi-Markov reliability model. Maximum likelihood procedure will be used to obtain the estimators of the parameters included in this reliability model. Based on the assumption that the lifetime and repair time of the system units are random variables with quadratic failure rate, the reliability function of this system is obtained. Also, the distribution of the first passage time of this system is derived. Many important special cases are discussed.

Key Words: *Quadratic failure rate distribution, maximum likelihood estimators, semi-Markov model, system reliability, operating unit, first passage*

1. INTRODUCTION

The development of stochastic models in any applied setting has great importance since they have many applications in different fields such as reliability systems, social security policy analysis, health care services (El-Gohary and Al-Khedhairi, 2010, El-Gohary, 2005, El-Gohary, 2004, Kastner and Shachtman, 1982).

The severity of run a discrete semi-Markov risk models and iterative convergence of passage time densities in semi-Markov performance models are discussed in Reinhard and Snoussi (2002) and Jeremy et al. (2005).

A Markov chain analysis can be used to describe patterns of deposition and conditional probability of occurrence of different rock types through transition probability matrices (Dacay and Krumbein, 1970; Krumbein and Graybill, 1965). The stochastic analysis of a semi-Markov reliability model is rarely investigated during the last two decades. For a more extensive overview of the reliability theory of repairable systems, see the well-known books (Korolyuk and Swishchuk, 1994; Barlow and Proschan, 1981).

To discuss the stochastic analysis of our reliability model, we present some important

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definitions. A semi Markov process $\{X(t) : t \geq 0\}$ is a stochastic process in which changes of state occur according to a Markov chain the time interval between two successive transitions is a random variable whose distribution depends on the state from which the transition takes place as well as the state to which the next transition takes place (Korolyuk and Swishchuk, 1994). Generally a semi-Markov process with discrete state space can be defined as a Markov renewal process (El-Gohary, 2005).

In this paper, in section *2*, we will display some important definitions and properties of a semi-Markov process and its kernel. In section *3* we use the stochastic analysis and semi-Markov model to estimate the parameters included in some reliability models. The maximum likelihood method is used to derive the point and confidence interval estimates of these parameters. Further, some properties of this reliability model are discussed.

2. BASIC DEFINITIONS AND SEMI-MARKOV KERNEL

In this section we shall throw some light upon the definitions and properties of semi-Markov processes. The semi-Markov kernel and its properties will be discussed. A semi-Markov process is a stochastic process, $\{X(t) : t \geq 0\}$, where an embedded Markov chain governs the state-to-state transitions of the process while a separate probabilistic mechanism determines the time spent in each state. It is assumed that the transition probabilities depend on the current state and the time spent in each state depends upon the current and next state.

Definition 2.1 Assume that the set of nonnegative integers, S = {0, 1, 2, ...}, represents the states of a stochastic process and let the transitions of the process occur at time instants $t_0 = 0,\ t_1, t_2, \ldots (t_n < t_{n+1})$. Suppose that $X_n$denote the transition occurring at time instant t_n. Then the twice $\{X_n, t_n\}$, n = 0, 1, 2, . . . is said to constitute a Markov renewal process if

$$P\{Xn+1 = k,\ tn+1 - tn \_ t|X0 = i0, X1 = i1, \ldots, Xn = in;\ t0,\ t1, \ldots, tn\} =$$
$$P\{Xn+1 = k,\ tn+1 - tn \_ t|Xn = in\}, \quad (2.1)$$

Definition 2.2 The Markov renewal process {Xn, tn}, n = 0, 1, 2, . . . is said to homogeneous if

$$P\{Xn+1 = k,\ tn+1 - tn \_ t|Xn = i\} = Qik(t) \quad (2.2)$$

does not depend on n

Lemma 2.1 Assume that {Xn, n = 0, 1, 2, …} constitutes a Markov chain with state space S = {0, 1, 2, …},and transition probability matrix P = {pij}. The continuous parameter process Y (t) with state space S = {0, 1, 2, …}, defined by

$$Y(t) = Xn,\ tn \_ t < tn+1 \quad (2.3)$$

is called semi-Markov process.

The semi-Markov process is a stochastic process which changes its state according to a Markov chain and the time interval between two successive transitions is a random variable, whose distribution may be depend not only on the present state but also on the state of the next transition.

Definition 2.3 A two-dimensional Markov process $\{\xi_n, \vartheta_n, n \in \mathrm{N}\}$ with values in $\mathrm{S} \times [0, \infty)$ is called a Markov renewal process if and only if

1. $Q_{ij} = P\{\xi_{n+1} = j,\ \vartheta_{n+1} \le \mathrm{t}\ | \xi_n = i,\ \vartheta_n = t_n, \cdots, \xi_0 = i_0, \vartheta_0 = t_0\}$
   $= P\{\xi_{n+1} = j,\ \vartheta_{n+1} \le \mathrm{t}\ | \xi_n = i\}$
2. $P\{\xi_0 = i,\ \vartheta_0 = 0\} = p_{i0}$

In the Markov renewal process, the non-negative random variables $\vartheta_n, n \ge 1$, define the interval between Markov renewal times:

$$T_n = \sum_{k=1}^{n} \vartheta_k, n \ge 1, T_0 = 0$$

Now, let

$$\nu(t) := \sum_{n=1}^{\infty} I_{[0,t]}(T_n) \tag{2.4}$$

where

$$I_{[0,t]}(T_n) = \begin{cases} 1 & if\ T_n \in [0, t] \\ 0 & otherwisw \end{cases} \tag{2.5}$$

The process $\nu(t)$ is called a counting process. It determines the number of renewal times on the segment [0, *t*].

Definition 2.4 A stochastic process $\{\mathrm{X(t) : t} \ge 0\}$ where $\mathrm{X(t)} = \xi_{\nu(\mathrm{t})}$ is called a semi-Markov process that generated by the Markov renewal process with initial distribution $\mathrm{P_i^0} = \mathrm{p}(\xi_0 = \mathrm{i})$ and the kernel $\mathrm{Q(t), t} \ge 0$.

Since the counting process $\nu(t)$ keeps constant values on the half-interval $[t_n, t_{n+1})$ and is continuous from the right, then the semi-Markov process keeps also constant values on the half intervals $[\tau_\mathrm{n}, \tau_{\mathrm{n}+1})$: $X_n(t) = \xi_\mathrm{n}$ for $\mathrm{t} \in [\tau_\mathrm{n}, \tau_{\mathrm{n}+1})$. Moreover the sequence $\{\mathrm{X}(\tau_n) : n \in \mathrm{N}\}$ is a Markov chain with transition probability matrix $\mathrm{P} = \{\mathrm{p}_{ij} = \mathrm{Q}_{ij}(\infty), i, j, \in \mathrm{S}\}$ that is called an embedded Markov chain. The concept of a Markov renewal process is a natural generalization of the concept of the ordinary renewal process given by a sequence of independent identically non-negative random variables $\theta_n, n \ge 1$. The random variables θ_n can be interpreted as lifetimes.

Definition 2.5 The stochastic matrix $Q(t) = [Q_{ij}(t); i, j \in S], t \ge 0$ is said to be a renewal kernel if and only if the following conditions are satisfied:

1. *The functions* $\mathrm{Q}_{ij}(\mathrm{t})$ *are nondecreasing functions in t.*
2. $\sum_{j \in \mathrm{S}} \mathrm{Q}_{ij} = \mathrm{G}_i(\mathrm{t})$ *are distribution functions in t.*
3. $[\mathrm{Q}_{ij}(+\infty) = \mathrm{P}_{ij}, i, j \in \mathrm{S}] = \mathrm{P}$ *is a stochastic matrix.*

Lemma 2.2 Assume that $\{\mathrm{X(t) : t} \ge 0\}$ is a semi-Markov process with renewal kernel

$$\mathrm{Q}(t) = Q_{ij}(t), i, j \in \mathrm{S}, \mathrm{t} \in [0, \infty) \tag{2.6}$$

Then

$$\mathrm{P}\{\xi_0 = i_0, \vartheta_0 = 0, \xi_1 = i_1, \vartheta_1 \le s_1, \ldots, \xi_n = i_n, \vartheta_n \le s_n\} = p_{i_0} \prod_{k=1}^{n} Q_{i_{k-1} i_k}(s_k) \tag{2.7}$$

A main objective of this paper is to use a three state semi-Markov process to describe a reliability system which consists of operating unit, identical spare unit, a switch and repair facility. Also, use the maximum likelihood procedure to obtain the estimators of the unknown parameters included in this reliability system.

3. SEMI-MARKOV PROCESS AND STANDBY MODEL

The semi-Markov process is used to model a reliability system consists of one active unit, an identical spare, a switch and repair facility. This section is devoted to introduce the assumptions of the studying reliability model. Also the semi-Markov kernel of the stochastic process that describe this reliability model will be introduced. Further, the densities corresponding to this kernel will be obtained.

The model of this paper is a slight modification of well a known reliability model introduced by Barlow and Proschan (1965). In order to describe a reliability model of a standby system with a repair facility, the considered reliability system consists of one active unit, an identical spare, a switch and a repair facility and the following assumptions are adopted:

1. As the operating unit fails, the spare is put in motion by the switch immediately.
2. The failed units can be repaired by the repair facility and the repair fully restore the units. This means that the repaired element can be considered as new one.
3. The system fails when the active unit fails and repair has not been finished yet or when the active unit fails and the switch fails .
4. The lifetimes of the active units can be represented by independent and identical nonnegative random variables ξ_1 with probability density function $f_1(t), t \geq 0$.
5. The lengths of repair periods of the units can be represented by independent and identical non-negative random variable ξ_2 with the distribution function $f_2(t) = P\{\xi_2 \leq t\}$.
6. The event E denotes the switch-over as the active unit fails. Then the probability that the switch performs when required is represented by $P(E) = \theta_0$.
7. The whole system can also be repaired, and the failed system is replaced by a new identical one.
8. The replacing time is represented by a non-negative random variable *k* with distribution function $f_3(t) = P\{\xi_3 \leq t\}$.
9. Finally, we assume that all the random variables described above are independent.

The reliability model of this paper can be described by a semi-Markov process with three states. Under the model assumptions, the states of the prescribed system can be considered as follows: The system will be in one of the following three states:

state	description
0	System failure
1	Failed unit is repaired and the standby unit is operating
2	Both active and standby units are "Up"

The following random variables and assumptions will be considered for the model:

Random Variable	description
ξ_1	Lifetime of the active unit
ξ_2	Length of the repair period
ξ_3	Replacing time
E	The event that Switch-Over

The random variables ξ_i, (i = 1, 2, 3) assumed to be mutually independent and non-negative. Also $F_i(.)$, (i = 1, 2, 3) are the distribution function of the i-th random variable respectively. These distribution functions are considered to be absolutely continuous and having the probability density functions $f_i(.)$, (i = 1, 2, 3) respectively.

Let $\tau_0^*, \tau_1^*, \tau_2^*, \ldots$ denote the instants which the state of the system changes, where $\tau_0^* = 0$ and let $\{Y(t) : t \geq 0\}$ be a stochastic process with state space $S = \{0, 1, 2\}$. This process keeps constant values on the half intervals $[\tau_n^*, \tau_{n+1}^*)$ and is continuous from the right. Therefore, it is not a semi-Markov process.

Let us define a new stochastic process as follows:

Assuming that $\tau_0 = 0$ and τ_n, n=1, 2, ... represent the instants when the components of the system failed or the whole system renewal. The stochastic process $\{X(t) : t \geq 0\}$ defined by

$$X(0) = 0, X(t) = Y(T_n) for\ t \in [T_n, T_{n+1}) \tag{3.1}$$

is a semi-Markov process and its kernel is given by the following matrix

$$\begin{bmatrix} 0 & 0 & Q_{02} \\ Q_{10} & Q_{11} & 0 \\ Q_{20} & Q_{21} & 0 \end{bmatrix} \tag{3.2}$$

It is well-known that, the semi-Markov process $\{X(t), t \geq 0\}$ is completely specified by its semi-Markov kernel. Let us deduce the elements of the semi-Markov kernel which describe the underlying reliability model as follows:

$$\begin{aligned} Q_{02}(t) &= P\{X(T_{n+1}) = 2, \vartheta_{n+1} \leq t | X(T_n) = 0\} \\ &= P\{\xi_3 \leq t\} = F_3(t), \\ Q_{10}(t) &= P\{X(T_{n+1}) = 0, \vartheta_{n+1} \leq t | X(T_n) = 1\} \\ &= P\{\xi_1 \leq t, \xi_2 > \xi_1\} + P\{\bar{E}, \xi_1 \leq t, \xi_2 < \xi_1\} \\ &= \int_0^t [1 - F_2(t)] dF_1(t) + (1 - \theta_0) \int_0^t F_2(x) dF_1(x) = \\ &= F_1(t) - \theta_0 \int_0^t F_2(x) dF_1(x) \end{aligned} \tag{3.3}$$

$$\begin{aligned} Q_{11}(t) &= P\{X(T_{n+1}) = 1, \vartheta_{n+1} \leq t | X(T_n) = 1\} \\ &= P\{E, \xi_1 \leq t, \xi_2 > \xi_1\} = \theta_0 \int_0^t F_2(x) dF_1(x) \\ &= P\{E, \xi_1 \leq t, \xi_2 < \xi_1\} = \theta_0 \int_0^t F_2(x) dF_1(x) \end{aligned}$$

$$\begin{aligned} Q_{21}(t) &= \mathrm{P}\{X(T_{n+1}) = 1, \vartheta_{n+1} \le t | X(T_n) = 2\} \\ &= \mathrm{P}\{\mathrm{E}, \xi_1 \le \mathrm{t}\} = \theta_0 F_1(t) \\ Q_{20}(t) &= \mathrm{P}\{X(T_{n+1}) = 0, \vartheta_{n+1} \le t | Y(T_n) = 2\} \\ &= \mathrm{P}\{\bar{E}, \xi_1 \le \mathrm{t}\} = (1-\theta_0) F_1(t) \end{aligned}$$

To derive the densities associated to the semi-Markov kernel, we will use the following relations

$$q_{ij}(t|\underline{\Theta}) = \frac{\partial Q_{ij}(t|\underline{\Theta})}{\partial t}, \forall i,j \in \mathrm{S}, \mathrm{t} \in \mathrm{R}+. \tag{3.4}$$

That is

$$\begin{aligned} q_{02}(t|\underline{\Theta}) &= f_3(t), t \ge 0, \\ q_{10}(t|\underline{\Theta}) &= f_1(t) - \theta_0 F_2(t) f_1(t), t \ge 0, \\ q_{11}(t|\underline{\Theta}) &= \theta_0 F_2(t) f_1(t), t \ge 0, \\ q_{20}(t|\underline{\Theta}) &= (1-\theta_0) f_1(t), t \ge 0, \\ q_{21}(t|\underline{\Theta}) &= \theta_0 f_1(t), t \ge 0. \end{aligned} \tag{3.5}$$

Now, we assume that the lifetime of the active units have identically quadratic failure rate distribution with the parameters θ_1, θ_2 and θ_3. Therefore the probability density function of the lifetime of the active units is given by

$$f_1(t) = (\theta_1 + \theta_2 t + \theta_3 t^2) e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)}, \theta_{1,}\theta_3 > 0, \theta_2 > -\sqrt{\theta_1 \theta_3}, t \ge 0 \tag{3.6}$$

Substituting from equations (3.6) into (3.5) we get

$$\left.\begin{aligned} q_{10}(t|\underline{\Theta}) &= \left(1 - \theta_0 F_2(t)\right)(\theta_1 + \theta_2 t + \theta_3 t^2) e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)}, \\ q_{11}(t|\underline{\Theta}) &= \theta_0 F_2(t)(\theta_1 + \theta_2 t + \theta_3 t^2) e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)} \\ q_{20}(t|\underline{\Theta}) &= (1-\theta_0)(\theta_1 + \theta_2 t + \theta_3 t^2) e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)}, \\ q_{21}(t|\underline{\Theta}) &= \theta_0(\theta_1 + \theta_2 t + \theta_3 t^2) e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)}, \end{aligned}\right\} \tag{3.7}$$

where $\theta_{0,}\theta_1, \theta_3 > 0$, $\theta_2 > -\sqrt{\theta_1, \theta_3}$, $\mathrm{t}, \ge 0$.

Next, we derive the maximum likelihood estimators of the unknown parameters $\theta_0, \theta_1, \theta_2$ and θ_3 included in the underlying reliability model. The maximum likelihood procedure will be used to derive these estimators.

4. MAXIMUM LIKELIHOOD ESTIMATORS

In this section, we use the maximum likelihood procedure to derive point and interval estimates of the unknown vector parameters $\underline{\Theta} = (\theta_0, \theta_1, \theta_2, \theta_3)$ included in the quadratic failure rate reliability model.

4.1 Maximum likelihood procedure

In this subsection, we use maximum likelihood procedure to derive the point and interval estimates of the parameters. Suppose that *z* denotes the observations

$\{(i_0, t_0), (i_1, t_1), \ldots, (i_n, t_n)\}$ of two dimensional random vector of variables, $\{(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \ldots, (\xi_n, \vartheta_n)\}$ where $i_0, i_1, \ldots, t_n$ and $t_0, t_1, \ldots, t_n \in [0, \infty)$ Further, we assume that this observation is classified as follows:

Let

$$A_{ij} = \{k: i_{k-1} = i, i_k = j, k = 1,2, \ldots, n\} \tag{4.1}$$

be the set of numbers of direct observed transition from the state *i* to the state *j* and n_{ij} is the cardinal number of the set A_{ij} which represents the number of direct transitions from the state *i* to state *j* . In the present case we find that

$$n_{02} + n_{10} + n_{11} + n_{20} + n_{21} = n \tag{4.2}$$

Based on the above observation, the sample likelihood function $L(z; \Theta)$ can be obtained as follows:

Using (3.6) and (3.7) the sample likelihood function $L(z; \Theta)$ takes the form

$$L(z; \underline{\Theta}) = \prod_{i\in A_{02}} q_{02}(t_i|\underline{\Theta}) \prod_{i\in A_{10}} q_{10}(t_i|\underline{\Theta}) \prod_{i\in A_{11}} q_{11}(t_i|\underline{\Theta}) \prod_{i\in A_{20}} q_{20}(t_i|\underline{\Theta}) \prod_{i\in A_{21}} q_{21}(t_i|\underline{\Theta}) \tag{4.3}$$

Substituting the semi-Markov densities from (3.7) into (4.3) we get

$$L(z; \underline{\Theta}) = C\theta_0^{n_{11}+n_{21}}(1-\theta_0)^{n_{20}} W(\theta_0) \prod_{i\in B}(\theta_1 + \theta_2 t_i + \theta_3 t_i^2) e^{-\left(\theta_1 t_i + \frac{1}{2}\theta_2 t_i^2 + \frac{1}{3}\theta_3 t_i^3\right)} \tag{4.4}$$

where

$$\left.\begin{array}{l} W(\theta_0) = \prod_{i\in A_{10}}[1 - \theta_0 F_2(t_i)] \qquad C = \prod_{i\in A_{03}} f_3(t_i) \\ \mathcal{B} = A_{10} \cup A_{11} \cup A_{20} \cup A_{21}, \;\; m = n_{10} + n_{11} + n_{20} + n_{21} \end{array}\right\} \tag{4.5}$$

Finally, the log of the sample likelihood function *L* can be written in the following form

$$\left.\begin{array}{l} \mathcal{L} = (n_{11} + n_{21})\ln\theta_0 + n_{20}\ln(1-\theta_0) + \ln W(\theta_0) + \\ \sum_{i\in\mathcal{B}}(\theta_1 + \theta_2 t_i + \theta_3 t_i^2) - \sum_{i\in\mathcal{B}}\left(\theta_1 t_i + \frac{1}{2}\theta_2 t_i^2 + \frac{1}{3}\theta_3 t_i^3\right) \end{array}\right\} \tag{4.6}$$

The maximum likelihood estimators $\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2$ and $\hat{\theta}_3$ are the values of $\theta_0, \theta_1, \theta_2$ and θ_3, respectively that maximize the sample likelihood $L(z; \underline{\Theta})$. Equivalently $\theta_0, \theta_1, \theta_2$ and θ_3 maximize the log sample likelihood function $L(z; \underline{\Theta})$ since it is a monotone function of $L(z; \underline{\Theta})$.

The maximum likelihood equations are given by :

$$\frac{\partial\mathcal{L}}{\partial\theta_0} = 0, \frac{\partial\mathcal{L}}{\partial\theta_1} = 0, \frac{\partial\mathcal{L}}{\partial\theta_2} = 0, \frac{\partial\mathcal{L}}{\partial\theta_3} = 0 \tag{4.7}$$

Using (4.6) and (4.7) the maximum likelihood equations are

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\theta_0} &= \frac{n_{11}+n_{21}}{\theta_0} - \frac{n_{20}}{1-\theta_0} + \frac{1}{W(\theta_0)}\frac{\partial W(\theta_0)}{\partial\theta_0} = 0, \\ \frac{\partial\mathcal{L}}{\partial\theta_1} &= \sum_{i\in\mathcal{B}} \frac{1}{\theta_0 + \theta_2 t_i + \theta_3 t_i^2} - \sum_{i\in\mathcal{B}} t_i = 0, \\ \frac{\partial\mathcal{L}}{\partial\theta_2} &= \sum_{i\in\mathcal{B}} \frac{t_i}{\theta_0 + \theta_2 t_i + \theta_3 t_i^2} - \frac{1}{2}\sum_{i\in\mathcal{B}} t_i^2 = 0, \\ \frac{\partial\mathcal{L}}{\partial\theta_3} &= \sum_{i\in\mathcal{B}} \frac{t_i^2}{\theta_0 + \theta_2 t_i + \theta_3 t_i^2} - \frac{1}{3}\sum_{i\in\mathcal{B}} t_i^3 = 0, \end{aligned} \tag{4.8}$$

The maximum likelihood estimators $\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2$ and $\hat{\theta}_3$ for the unknown parameters $\theta_0, \theta_1, \theta_2$ and θ_3 are the solution of the non-linear system (4.8). As it seems, the general solution of this system is very difficult to find in a closed form. The general solution is intractable and numerical procedures are required (Dacay and Krumbein, 1970;

Krumbein and Graybill, 1965; El-Gohary and Sarhan, 2004).

Next, we discuss some important special cases of both the time lengths of the repair periods of the units and the lifetimes of the active units.

4.2 Numerical simulation study

In this subsection, we will discuss illustrative numerical example for the maximum likelihood estimators of the unknown parameters $\theta_0, \theta_1, \theta_2$ and θ_3 included in the semi-Markov reliability model. The following table displays the mean square errors (MSE) of the parameters against the different values of the sample size *n*.

n	MSE(θ_1)	MSE(θ_2)	MSE(θ_3)	n	MSE(θ_1)	MSE(θ_2)	MSE(θ_3)
40	2.32	1.40	3.34	360	0.35	0.21	0.31
80	1.16	0.71	1.24	400	0.31	0.20	0.30
120	0.72	0.66	0.88	440	0.30	0.19	0.29
160	0.65	0.54	0.85	500	0.24	0.14	0.25
200	0.49	0.37	0.79	540	0.23	0.13	0.23
250	0.42	0.34	0.48	600	0.21	0.12	0.22
300	0.41	0.33	0.45	700	0.17	0.11	0.21

where the assumed values of the parameters are θ_0 = 0.5, θ_1 = 2.0, θ_2 = 2.5 and the partial of the sample size are such that $n_{11} = n_{21} = n_{10} = n_{20}$. Further the distribution of the length of the repair time is such that $H(t_l) = 1, \forall l \in A_{10}$. Note that the mean squareerror of the parameter θ_0 is zero for all different values of the sample size.

4.3 Important special cases

This subsection is devoted to study some important special cases. Such cases occur when, both the time lengths of the repair periods of the units and the lifetimes of the active units are exponentially, linear failure rate and Rayleigh random variables. In order to obtain the first special case, the following assumptions are needed:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - \theta_0 F_2(t_i) = 1 - \theta_0$ for every $i \in A_{10}$.
2. The lifetimes of the active units can be represented by identically exponential random variables with parameter θ_1. That is, $\theta_2 = \theta_3 = 0$

In this case, the maximum likelihood estimators are given by:

$$\hat{\theta}_0 = \frac{n_{22}+n_{12}}{m}, \hat{\theta}_1 = \frac{m}{\tau}, \tau = \sum_{i \in \mathcal{B}} t_i \tag{4.9}$$

The second special case can be obtained by considering the following assumptions:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - \theta_0 F_2(t_i) = 1 - \theta_0$ for every $i \in A_{10}$.
2. The lifetimes of the active units can be represented by identically linear failure rate random variables with two parameters θ_1 and θ_2. That is, $\theta_3 = 0$

In this case, the maximum likelihood estimators are given by:

$$\hat{\theta}_0 = \frac{n_{22}+n_{12}}{m}, \hat{\theta}_1 = \frac{2m+\hat{\theta}_2 \sum_{i\in\mathcal{B}} t_i^2}{2\tau} \tag{4.10}$$

where the estimator $\hat{\theta}_2$ is the solution of the nonlinear equation

$$2\sum_{i\in\mathcal{L}}\left(\frac{\sum_{s\in\mathcal{B}} t_s}{2m-\hat{\theta}_2\sum_{s\in\mathcal{B}} t_s^2+2\hat{\theta}_2\sum_{i\in\mathcal{B}}\sum_{s\in\mathcal{B}} t_i t_s}\right)-\tau=0 \tag{4.11}$$

4.4 Numerical simulation study

This subsection is devoted to study the behavior of the mean square errors of the maximum likelihood estimators of the unknown parameters θ_0, θ_1 and θ_2 against the sample size. The following table displays the mean square errors (MSE) of the parameters against the different values of the sample size *n*.

n	MSE(θ_1)	MSE(θ_2)	n	MSE(θ_1)	MSE(θ_2)
40	2.42	1.50	440	0.38	0.18
80	1.17	0.91	480	0.35	0.17
120	0.73	0.86	520	0.34	0.15
160	0.67	0.74	600	0.31	0.13
200	0.51	0.57	680	0.28	0.11
250	0.48	0.45	700	0.27	0.07
300	0.43	0.33	780	0.19	0.05
360	0.41	0.21	860	0.17	0.03
400	0.40	0.20	900	0.12	0.02

where the assumed values of the parameters are $\theta_0 = 0.5$, $\theta_1 = 2.1$, $\theta_2 = 3.5$ and the partial of the sample size are such that $n_{11} = n_{21} = n_{10} = n_{20}$. Further the distribution of the length of the repair time is such that $H(t_l) = 1, \forall l \in A_{10}$. Note that the mean square error of the parameter θ_0 is zero for all different values of the sample size. The numerical study shows that for sufficiently large sample size the mean square error closes to zero.
The third special case can be obtained by considering the following assumptions:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1-\theta_0 F_2(t_i) = 1-\theta_0$ for every $i \in A_{10}$.
2. The lifetimes of the active units can be represented by identically Rayleigh random variables with one parameter θ_2.

In this case, the maximum likelihood estimators are given by:

$$\hat{\theta}_0 = \frac{n_{22}+n_{12}}{m}, \ \hat{\theta}_2 = \frac{2m}{\sum_{s\in\mathcal{B}} t_s^2}. \tag{4.12}$$

In the next section, we will derive the confidence intervals for the unknown parameters included in the quadratic failure rate semi-Markov reliability model.

5. ASYMPTOTIC CONFIDENCE BOUNDS

Since the maximum likelihood estimators $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ of the unknown parameters $(\theta_0, \theta_1, \theta_2, \theta_3)$ cannot be derived in closed forms, we cannot get the exact confidence bounds of the parameters. In this section we will use some of the most widely used methods to construct approximately confidence intervals for the unknown parameters. The idea is to use the large sample approximation. The maximum likelihood estimators of $\underline{\Theta}$ can be treated as being approximately multi-normal with mean $\underline{\Theta} = (\theta_0, \theta_1, \theta_2, \theta_3)$ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\left((\hat{\theta}_0-\theta_0),(\hat{\theta}_1-\theta_1),(\hat{\theta}_2-\theta_2),(\hat{\theta}_3-\theta_3)\right) \to N_4\left(0, I^{-1}(\hat{\underline{\Theta}})\right), \tag{5.1}$$

where $I^{-1}(\hat{\underline{\Theta}})$ is the variance-covariance matrix of the unknown parameters vector $\underline{\Theta}$. The element $I_{ij}(\hat{\underline{\Theta}})$, $i, j = 0, 1, 2, 3$, of the 4×4 matrix I^{-1} is given by

$$I_{ij}(\hat{\underline{\Theta}}) = -\mathcal{L}_{\theta_i\theta_j}\Big|_{\underline{\Theta}=\hat{\underline{\Theta}}} \tag{5.2}$$

From expression (4.8), the second partial derivatives of the log-likelihood function are found to be

$$\frac{\partial^2\mathcal{L}}{\partial\theta_0^2} = -\frac{n_{11}+n_{21}}{\theta_0^2} - \frac{n_{20}}{(1-\theta_0)^2} - \frac{1}{W^2(\theta_0)}\left(\frac{\partial W(\theta_0)}{\partial\theta_0}\right)^2 + \frac{1}{W(\theta_0)}\frac{\partial^2 W(\theta_0)}{\partial\theta_0^2},$$
$$\frac{\partial^2 W(\theta_0)}{\partial\theta_0\theta_1} = 0, \frac{\partial^2 W(\theta_0)}{\partial\theta_0\theta_2} = 0, \frac{\partial^2 W(\theta_0)}{\partial\theta_0\theta_3} = 0$$
$$\frac{\partial^2\mathcal{L}}{\partial\theta_1^2} = -\sum_{i\in\mathcal{B}}\frac{1}{(\theta_1+\theta_2 t_i+\theta_3 t_i^2)^2}, \qquad \frac{\partial^2\mathcal{L}}{\partial\theta_1\partial\theta_2} = -\sum_{i\in\mathcal{B}}\frac{t_i}{(\theta_1+\theta_2 t_i+\theta_3 t_i^2)^2}$$
$$\frac{\partial^2\mathcal{L}}{\partial\theta_1\theta_3} = -\sum_{i\in\mathcal{B}}\frac{t_i^2}{(\theta_1+\theta_2 t_i+\theta_3 t_i^2)^2}, \qquad \frac{\partial^2\mathcal{L}}{\partial\theta_2^2} = -\sum_{i\in\mathcal{B}}\frac{t_i^2}{(\theta_1+\theta_2 t_i+\theta_3 t_i^2)^2}$$
$$\frac{\partial^2\mathcal{L}}{\partial\theta_2\theta_3} = -\sum_{i\in\mathcal{B}}\frac{t_i^3}{(\theta_1+\theta_2 t_i+\theta_3 t_i^2)^2}, \qquad \frac{\partial^2\mathcal{L}}{\partial\theta_3^2} = -\sum_{i\in\mathcal{B}}\frac{t_i^4}{(\theta_1+\theta_2 t_i+\theta_3 t_i^2)^2} \tag{5.3}$$

Therefore, the approximate $100(1-\alpha)\%$ two sided confidence intervals for $(\theta_0, \theta_1, \theta_2, \theta_3)$ are respectively, given by

$$\hat{\theta}_0 \pm Z_{\alpha/2}\sqrt{I_{00}^{-1}(\hat{\theta}_0)}, \hat{\theta}_1 \pm Z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{\theta}_1)}, \hat{\theta}_2 \pm Z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{\theta}_2)}, \hat{\theta}_3 \pm Z_{\alpha/2}\sqrt{I_{33}^{-1}(\hat{\theta}_3)} \tag{5.4}$$

Here $Z_{\alpha/2}$ is the upper $\alpha/2$th percentile of the standard normal distribution.

From above results, we can deduce the following special cases:

Exponential case: setting $\theta_1 = \theta_2 = 0$, from (5.2) and (5.3), we get the approximate $100(1-\alpha)\%$ two sided confidence intervals for θ_0 and θ_1 respectively

$$\hat{\theta}_0 \pm \frac{\hat{\theta}_0(1-\hat{\theta}_0)Z_{\alpha/2}}{\sqrt{(n_{11}+n_{22})(1-\hat{\theta}_0)^2+(n_{10}+n_{20})\hat{\theta}_0^2}}, \quad \hat{\theta}_1 \pm \frac{\hat{\theta}_1 Z_{\alpha/2}}{\sqrt{m}} \tag{5.5}$$

Linear failure rate case: setting $\theta_3 = 0$, from (5.2) and (5.3), we get the approximate $100(1-\alpha)\%$ two sided confidence intervals for θ_0 and θ_1 respectively.

$$\hat{\theta}_0 \pm \frac{\hat{\theta}_0(1-\hat{\theta}_0)}{\sqrt{(n_{11}+n_{22})(1-\hat{\theta}_0)^2+(n_{10}+n_{20})\hat{\theta}_0^2}} Z_{\alpha/2},$$

$$\hat{\theta}_1 \pm \left[\frac{\sum_{i\in\mathcal{B}}\frac{t_i^2}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2}}{\sum_{i\in\mathcal{B}}\frac{t_i}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}\sum_{i\in\mathcal{B}}\frac{t_s}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}-\sum_{i\in\mathcal{B}}\frac{1}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}\sum_{i\in\mathcal{B}}\frac{t_i^2}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}}\right]^{\frac{1}{2}} Z_{\alpha/2}, \tag{5.6}$$

$$\hat{\theta}_2 \pm \left[\frac{\sum_{i\in\mathcal{B}}\frac{1}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2}}{\sum_{i\in\mathcal{B}}\frac{1}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}\sum_{i\in\mathcal{B}}\frac{t_i^2}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2}-\sum_{i\in\mathcal{B}}\frac{t_i}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}\sum_{i\in\mathcal{B}}\frac{t_s}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}}\right]^{\frac{1}{2}} Z_{\alpha/2},$$

Next, we discuss in details the reliability of our semi-Markov model that consists of one active unit, an identical spare, a switch, and a repair facility.

6. FIRST PASSAGE AND SYSTEM RELIABILITY

In this section, we will discuss the system reliability of the semi-Markov reliability model. The reliability function of the system will be derived. The distribution of the first passage time will be obtained.

6.1 The distribution of the first passage

Now, we will define the first passage time. In order to define the first passage time, we should find an accurate answer for the question "how many transitions will the process take to reach state j for the first time if the system is in state i at time zero". The first passage time of the continuous-time semi-Markov process can be measured in time or in terms of the number of transitions. We will obtain the distribution $\overline{\Phi}_{iA}(t)$ of the first passage time from the state i to a state in a subset $A \subset S$ given that state i was entered at time zero and zeroth transition.

Assuming that $A \subset S = \{0, 1, 2\}$ and $\overline{A} = S - A$, we introduce the following notations

$$\Delta_A = \inf\{n \in N: X(\tau_n) \in A\}, \tag{6.1}$$

and

$$f_{iA}(n) = P\{\Delta_A = n|X(0) = i\}, T_A = \tau_{\Delta A}, \tag{6.2}$$

Therefore, the function $\overline{\Phi}_{iA}(t)$ is given by

$$\Phi_{iA}(t) = P\{T_A \leq= t|X(0) = i\}, i \in \overline{A}, \tag{6.3}$$

which represents the distribution of the first passage time of the semi-Markov process $\{X(t) : t \geq 0\}$, from the state $i \in A$ to state in the subset A.

6.2 The system reliability function

Now, we will define, the mean and the second moment of the first passage time distribution as follows

$$\overline{\Phi}_{iA} = \int_0^{\infty} t d\Phi_{iA}(t), \text{ and } \overline{\Phi}_{iA}^2 = \int_0^{\infty} t^2 d\Phi_{iA}(t), \tag{6.4}$$

If A denotes the subset of the failed states of the model and $i \in \overline{A}$ is an initial operating state such that $P\{X(0) = i\} = 1$, then the random variable T_A represents the lifetime or the time to failure of our system. That is, the reliability of the system is

$$R(t) = 1 - \Phi_{iA}(t), \ t \geq 0, \tag{6.5}$$

Using [3, 7, 9], some of the reliability characteristics of the system can be defined as follows:

$$\overline{q}_{ik} = \int_0^{\infty} t q_{ik}(t) dt, \text{ and } \overline{q}_{ik}^2 = \int_0^{\infty} t^2 q_{ik}(t) dt, \tag{6.6}$$

To derive the reliability of the system, we will establish the following theorem.

Theorem 4.1 If the following conditions

$$1.\ f_{iA} = 1\ \forall\ i \in \overline{A}, \tag{6.7}$$

$$2.\ \forall\ i,\ j \in S\ \exists\ d > 0\ s.t.\ \overline{q}_{ik}^2 < d \tag{6.8}$$

$$3.\ \sum_{k=1}^{\infty} k^2 f_{iA} < \infty\ \forall\ i \in \overline{A}, \tag{6.9}$$

are satisfied.

Then the functions $\Phi_{iA}(t)$, the mean $\overline{\Phi}_{iA}$ and the second moments $\overline{\Phi}_{iA}^2, i \in \overline{A}$ are only the solution of the following system:

$$1.\ \Phi_{iA}(t) = \sum_{j \in A} Q_{ij}(t) + \sum_{k \in \overline{A}} \int_0^t \Phi_{kA}(t-u) dQ_{ik}(u), i \in \overline{A}, \tag{6.10}$$

$$2.\ \overline{\Phi}_{iA} = \overline{g}_i + \sum_{k \in \overline{A}} p_{ik} \overline{\Phi}_{ik}, i \in \overline{A}, \tag{6.11}$$

$$3.\ \overline{\Phi}_{iA}^2 = \overline{g}_i^2 + 2\sum_{k \in \overline{A}} \overline{q}_{ik} \overline{\Phi}_{kA} + \sum_{k \in \overline{A}} p_{ik} \overline{\Phi}_{ik}^2, i \in \overline{A}, \tag{6.12}$$

which consist of a system of integral equations (6.9) and two linear algebraic systems of equations (6.10) and (6.11).

The system of integral equations (6.9) is equivalent to its Laplace-Stieltjes system

$$\tilde{\varphi}_{iA}(s) = \sum_{j \in A} \tilde{q}_{ij}(s) + \sum_{k \in \overline{A}} \tilde{q}_{ik}(s) \tilde{\varphi}_{kA}(s), i \in \overline{A}, \tag{6.13}$$

where

$$\tilde{\varphi}_{iA}(s) = \int_0^{\infty} e^{-st} d\Phi_{iA}(t), \tilde{q}_{ij}(s) = \int_0^{\infty} e^{-st} dQ_{ij}(t), \tag{6.14}$$

In the underling system, we find that $A = \{0\}$ and $\overline{A} = \{1, 2\}$. From the solution of the system (6.4), we have

$$\tilde{\varphi}_{10}(s) = \frac{\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)}, \tilde{\varphi}_{20}(s) = \tilde{q}_{20}(s) + \frac{\tilde{q}_{21}\tilde{q}_{10}}{1-\tilde{q}_{11}(s)} \tag{6.15}$$

Using the Laplace transformation, the system reliability function (6.12) of the underling reliability model is given by

$$\tilde{R}(s) = \frac{1-\tilde{\varphi}_{20}(s)}{s}, \tag{6.16}$$

From the system of equations (6.10), we can get

$$\overline{\Phi}_{20} = \overline{g}_2 + \frac{p_{21} g_1}{1-p_{11}}, \tag{6.17}$$

For the present model we have:

$$\bar{g}_1 = \bar{g}_2 = E(\xi_1) = \int_0^\infty t(\theta_1 + \theta_2 t + \theta_3 t^2) e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)} dt$$
$$= \int_0^\infty e^{-\left(\theta_1 t + \frac{1}{2}\theta_2 t^2 + \frac{1}{3}\theta_3 t^3\right)} dt \quad (6.18)$$

For the exponential distribution the lifetimes of the active units , we find that:

$$\bar{g}_1 = \bar{g}_2 = E(\xi_1) = \frac{1}{\theta_1}, p_{21} = \theta_0 \quad (6.19)$$

Substituting from (6.18) into (6.16) we obtain a simple form of the mean lifetime of the underling reliability system

$$E(T_A|X(0) = 2) = \varphi_{20} = \frac{1}{\theta_1} + \frac{\theta_0}{\theta_1(1-p_{11})} \quad (6.20)$$

where

$$p_{11} = \theta_0\theta_1 \int_0^\infty F_2(u) e^{-(-\theta_1 u)} du. \quad (6.21)$$

After observation of a piece of the considered semi-Markov process realization, we can substitute $\theta_0 = \hat{\theta}_0$ and $\theta_1 = \hat{\theta}_1$.

7. CONCLUSION

In this paper we have used the stochastic analysis to discuss an important semi-Markov reliability model. Also the likelihood procedure is employed to obtain estimators of the parameters include in this semi-Markov reliability model. The distribution of the first passage time of this reliability model is obtained. The reliability function of this model is derived. Important special cases are discussed.

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