Journal of Information Processing Systems, Vol.7, No.1, March 2011 DOI : 10.3745/JIPS.2011.7.1.075

An Embedding of Multiple Edge-Disjoint Hamiltonian Cycles on Enhanced Pyramid Graphs

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Abstract—The enhanced pyramid graph was recently proposed as an interconnection network model in parallel processing for maximizing regularity in pyramid networks. We prove that there are two edge-disjoint Hamiltonian cycles in the enhanced pyramid networks. This investigation demonstrates its superior property in edge fault tolerance. This result is optimal in the sense that the minimum degree of the graph is only four.

Keywords—Enhanced Pyramid Model, Hamiltonian Cycle, Edge-Disjoint Cycle

1. INTRODUCTION

In parallel processing, a common concern is how to map the underlying data structures in the algorithmic aspect into the physical interconnection network topology [1-2]. Under this theoretical background, the graph theoretic problem known as *graph embedding* has been given much attention. An embedding of one guest (or source) graph G into another host (or target) graph H is a one-to-one mapping f from the node (or vertex) set of G to the node set of H such that an edge of G corresponds to a (simple) path of H under the function f [3].

The *pyramid* model (*PM*, for short) as an interconnection network topology is known to be useful for such underlying data structures in hierarchy such as image processing, computer visions, and network computing [4-10]. With *PMs*, many research results have been published on topics such as Hamiltonian and cycle properties [11-13], fault tolerance [14], and so on [15].

For a long time and until quite recently the 2D and 3D *mesh* (or *grid*) graphs have also been studied and implemented as an attractive interconnection network model. To improve the irregular nature of meshes in the border nodes, the *torus* model was proposed by simply adding wraparound edges for the outside nodes [16]. Thus *PM* is naturally extended into the *enhanced pyramid* model (*EPM*, for short) by replacing meshes with tori in each layer of *PM* [17]. This means that *EPM* is a simple graph of the same set in nodes but is super set in edges other than *PM* under the same dimension. The topological properties and the related performances of *EPM* as an interconnection network model are well-known in [18-19].

The growing size of massively parallel processing systems increases the possibility of the situation that there exist failing components such as processors and/or communication channels in the system. To continuously maintain a system's high availability, it is crucial to isolate the faulty components by means of a system-level fault diagnosis mechanism [20]. Once the failing components are identified, the next work is to reconstruct the alternative longest path/cycle possible. In the graph theory representing the underlying topology, the fault-tolerance property

Manuscript received October 20, 2010; accepted February 22, 2011.
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shows how many faulty components such as nodes and/or edges can be tolerated without disturbing normal operation only by using the non-faulty sections of the graph. A graph G is called *k-node-fault Hamiltonian* if there is a cycle that contains all the non-faulty nodes when there are k or less faulty nodes, and a graph G is called *k-edge-fault Hamiltonian* if there is a cycle that contains all the nodes and contains only non-faulty edges when there are k or less faulty nodes and/or edges. Most of the researches associated with Hamiltonian cycle properties are mainly focused on constructing single cycle, but are not concerned with verifying the existence of the multiple disjoint cycles. In general, the existence of multiple disjoint cycles is meaningful enough, but it is not frequent in real models enough to be satisfied due to relatively strict conditions.

In this paper, we are concerned with finding multiple cycles that can be embedded without overlapped edges to each other on the enhanced pyramid graph. This research possibly contributes to improve the graph-theoretic properties in any condition. Under a no fault environment, this guarantees the possibility of concurrent processing in parallel without any resource collision in a massively parallel processing system. On the other hand, this also improves the fault-tolerance property by possibly supplying the alternating path (or cycle) when the underlying graph has fault. Moreover this result is the first trial to find edge-disjoint Hamiltonian cycles in EPM's, and it is the distinguished property that is only possible in EPM's but impossible in PM's because all nodes have degrees of at least 4.

In the remaining sections of this paper, we proceed in such a sequence that the analysis of the basic properties of associated graph models is followed by the proof of two edge-disjoint Hamiltonian cycles.

2. PRELIMINARIES

In graph theory terminology, a Hamiltonian cycle (path) in the given graph G, is defined as the cycle (path) that contains all nodes of the corresponding graph G. The Hamiltonian property of a graph indicates its capability of embedding the largest cycles (paths) when it is adopted as the underlying interconnection network model of a massively parallel processing system. Cycles and paths are two representative data models of the most popular and fundamental structures for such applications as many algebraic and graph problems.

First, we formally define the basic graph models to be referred to in the remaining part of this paper.

DEFINITION 1. Let $M(m,n)$ be a 2D mesh with size $m \times n$. The node set $V(M(m,n))$ and edge set $E(M(m,n))$ of $M(m,n)$ are defined as follows:

$$V(M(m,n)) = \{ (x,y) \mid 0 \le x < m,\ 0 \le y < n \} \tag{1}$$

$$E(M(m,n)) = \{ ((x_1,y_1),(x_2,y_2)) \mid |x_1-x_2|+|y_1-y_2|=1 \} \tag{2}$$

By adding some additional edges called *wraparound edges* into the mesh $M(m,n)$, we can also define the 2D torus, denoted as $T(m,n)$, with the same size as follows:

$$V(T(m,n)) = V(M(m,n)) \tag{3}$$

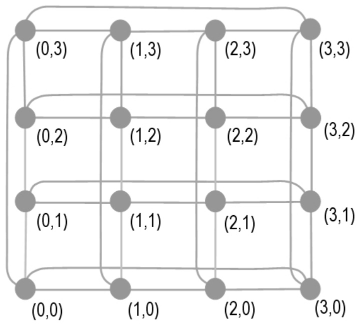


Fig. 1. An example of a regular torus graph ST^2

$$E(T(m,n)) = E(M(m,n)) \cup \{ ((x,n\text{-}1),(x,0)), ((m\text{-}1,y),(0,y)) \mid 0 \le x < m,\ 0 \le y < n \} \tag{4}$$

When $m=n$, $M(m,n)$ and $T(m,n)$ can be shortened as $SM(n)$ (or $SM(m)$) and $ST(n)$ (or $ST(m)$), and are called a *regular square mesh* and *torus*, respectively. Specially, a regular square mesh and torus with size $2^k \times 2^k$ can be denoted by SM^k and ST^k, respectively. Fig. 1 shows an example of a regular square tori.

DEFINITION 2. Let PM^n be the *pyramid graph* with dimension n. Then the node set $V(PM^n)$ and the edge set $E(PM^n)$ in PM^n are defined as follows:

$$V(PM^n) = \{ (l,x,y) \mid 0 \le l < n,\ 0 \le x,y < 2^{n-l-1} \} \tag{5}$$

$$\begin{aligned} E(PM^n) = &\{ ((l_1,x_1,y_1),(l_2,x_2,y_2)) \mid |x_1\text{-}x_2|+|y_1\text{-}y_2|=1,\ l_1=l_2 \} \\ &\cup \{ ((l_1,x_1,y_1),(l_2,x_2,y_2)) \mid |x_1\text{-}x_2|+|y_1\text{-}y_2|=1,\ l_2=l_1+1 \} \end{aligned} \tag{6}$$

Thus, the number of nodes in PM^n is $3(4^n\text{-}1)/4$. The edge set can be classified into two subset groups known as *intra-layer* edges and *inter-layer* edges as shown in its definition, respectively.

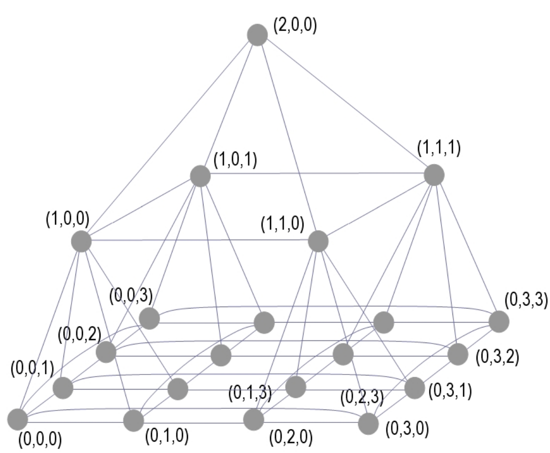


Fig. 2. An example of an enhanced pyramid graph EPM^3

DEFINITION 3. Let EPM^n be the *enhanced pyramid graph* with dimension n. In EPM^n, node set $V(EPM^n)$ and edge set $E(EPM^n)$ are defined as follows:

$$V(EPM^n) = V(PM^n) \tag{7}$$

$$E(EPM^n) = E(PM^n) \cup \{ ((x,n\text{-}1),(x,0)), ((n\text{-}1,y),(0,y)) \mid 0 \le x,y < n \} \tag{8}$$

An edge $((l_1,x_1,y_1),(l_2,x_2,y_2))$ in EPM^n satisfies one of the following statements ($l_1 \le l_2$):

(1) $l_2 = l_1, \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid = 1$

(2) $l_2 = l_1, \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid = 2^{n-l1-1} - 1$, where $x_1 = x_2$ or $y_1 = y_2$

(3) $l_2 = l_1 + 1, x_2 = \lfloor x_1 / 2 \rfloor, y_2 = \lfloor y_1 / 2 \rfloor$

We call the edge satisfying condition (1), (2), and (3) a *mesh* edge, a *wraparound* edge, and an *inter-layer* edge respectively. Moreover mesh edges are classified into two parts as follows: a mesh edge (u,v) is classified into the *shared-parent edge* (*SP*-edge, for short) if the two distinct end nodes u and v share a common parent, or the *neighbor-parent edge* (*NP*-edge, for short) if its two end nodes have different parents.

In this paper, we only focus on such specially-shaped tori as regular square tori with a size of a power of 2 because those are the bases of *EPM*'s. Thus we only focus on the properties of those tori.

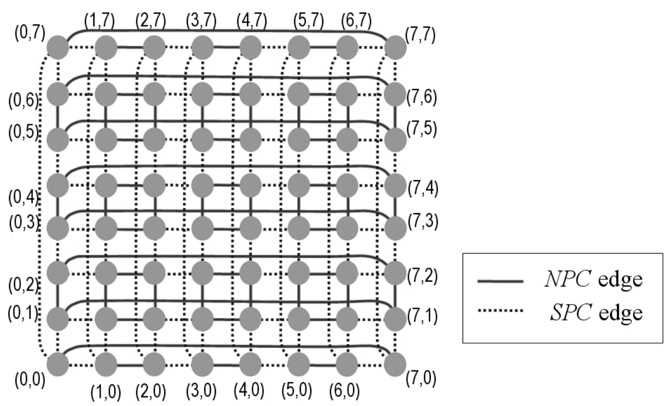
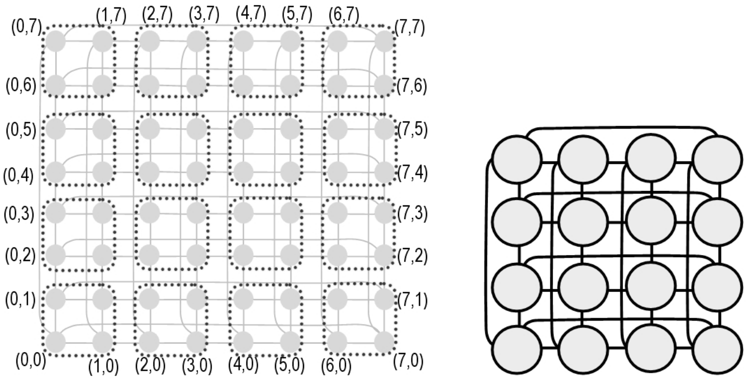


Fig. 3. *NPC*-edges and *SPC*-edges in ST^3

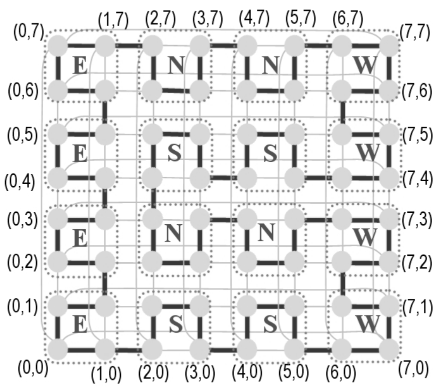
DEFINITION 4. Given a regular square torus ST^n, where $n > 1$, the edges for *NP candidate* (*NPC*-edges, for short), denoted by $NPC(ST^n)$, in ST^n are defined as a subset of edge set $E(ST^n)$ as follows:

$$NPC(ST^n) = \{ ((2i+1, y), (2i+2)\%2^n, y)) \mid 0 \le i < 2^{n-1} \;\&\&\; 0 \le y < 2^n \} \\ \cup \{ ((x, 2j+1), (x, (2j+2)\%2^n)) \mid 0 \le j < 2^{n-1} \;\&\&\; 0 \le x < 2^n \} \tag{9}$$

The first part of $NPC(ST^n)$ is called *column edges* which connect two nodes by column direction (same as the y-direction), and the second part as *row edges* layered in a row direction (same



(a) Super-nodes and super-edges



(b) A Hamiltonian cycle construction

Fig. 4. An example of a $SG_1(ST^3)$ in a 1-shrink graph

as the *x*-direction) in ST^n. The remaining edges except *NPC*-edges in $E(ST^n)$ are automatically classified as the *SP candidate* edges (*SPC*-edges, for short), denoted by $SPC(ST^n)$ as follows:

$$SPC(ST^n) = E(ST^n) - NPC(ST^n) \tag{10}$$

This classification of edges in ST^n is demonstrated in Fig. 3. The concepts of *NP*-edge and *NPC*-edge are basically the same in the sense that an *NPC*-edge in ST^m is also said to be an *NP*-edge EPM^n because EPM^n contains ST^m as an underlying layer structure when $n>m$.

DEFINITION 5. Given a regular square torus $ST^n = (V, E)$, where $n > 1$, the *k*-shrink graph $SG_k(ST^n) = (SV_k, SE_k)$ is defined as a super-graph of super-node set SV_k and super-edge set SE_k as follows:

$$SV_k = \{ (i, j) \mid 0 \le i < 2^{k-1} \;\&\&\; 0 \le y < 2^n \} \tag{11}$$

$$SE_k = \{ ((x_1, y_1),(x_2,y_2)) \mid 0 \le i < 2^{n-1} \;\&\&\; 0 \le y < 2^n \} \tag{12}$$

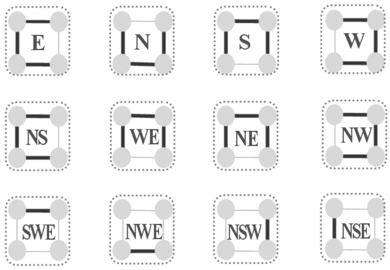


Fig. 5. Possible patterns of super-nodes on 1-shrink graph SG_1

Each $2^k \times 2^k$ sub-mesh in ST^n is shrunk into one super-node in $SG_k(ST^n)$, and 2^k edges between two adjacent $2^k \times 2^k$ sub-meshes are mapped into one super-edge in $SG_k(ST^n)$ as shown in Fig. 4. Thus a node (x, y) in ST^n is mapped into the node $(\lfloor x/2^k \rfloor, \lfloor y/2^k \rfloor)$ in k-shrink graph of ST^n.

In this paper we especially concerned with 1-shrink graphs of tori. Each super-node in SG_1 can be classified into several types according to the directional patterns of internal edges not to be used for connection in constructing a path or cycle as shown in Fig. 5.

3. Basic Property on Tori

Lemma 1. The number of *NPC*-edges in a Hamiltonian cycle constructed in ST^n is at least 2^{2n-2}.

Proof. By using the symmetric property of tori, it is possible to construct multiple isomorphic sub-graphs together. In constructing a Hamiltonian cycle on the corresponding 1-shrink graph to torus, it is strongly recommended to adopt the strategy that focuses on minimizing as many super-edges as possible because the super-edge on 1-shrink graph represents *NPC*-edges on original torus graphs. Fig. 6 shows one of these cases when n=3. Thus it is also possible to analyze the number of *NPC*-edges under this special case.

From the special cases of n=3 as shown in Fig. 6, we can generalize the property in an arbitrary case. Moreover it is also possible to deduce the general case from this special case of n as

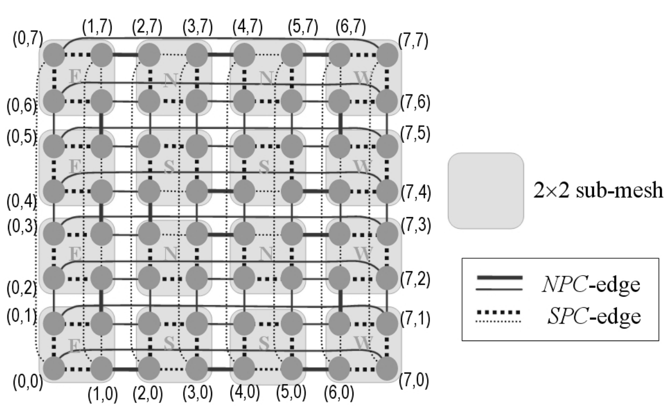


Fig. 6. A construction of a Hamiltonian cycle with minimum *NPC*-edges in ST^3

shown in Fig. 6.

First, the column edges are categorized into three classes as follows:

① $E_{C1} = \{((1,2j+1),(1,2j+2)) \mid 0 \le j < 2^{n-1}-1\}$
② $E_{C2} = \{((2,4j+3),(2,4j+4)) \mid 0 \le j < 2^{n-2}-1\}$
③ $E_{C3} = \{((2^n-2,4j+1),(2^n-2,4j+2)) \mid 0 \le j < 2^{n-2}\}$

Second, the row edges are similarly divided into four groups as follows:

① $E_{R1} = \{((2i+1,0),(2i+2,0)) \mid 0 \le i < 2^{n-1}-1\}$
② $E_{R2} = \{((2i+1,4j),(2i+2,4j)) \mid 1 \le i < 2^{n-1}-1 \,\&\&\, 0 \le j < 2^{n-2}-1\}$
③ $E_{R3} = \{((2i+1,4j+3),(2i+2,4j+3)) \mid 1 \le i < 2^{n-1}-1 \,\&\&\, 0 \le j < 2^{n-2}-1\}$
④ $E_{R4} = \{((2i+1,2^n-1),(2i+2,2^n-1)) \mid 0 \le i < 2^{n-1}-1\}$

The total number of column edges and row edges can be computed by simply summarizing its internal elements as follows:

$S_C = |E_{C1}|+|E_{C2}|+|E_{C3}| = (2^{n-1}-1) + (2^{n-2}-1) + (2^{n-2}) = 2^n-2$
$S_R = |E_{R1}|+|E_{R2}|+|E_{R3}|+|E_{R4}| = 2 * (2^{n-1}-1) + 2 * (2^{n-2}-1) * (2^{n-1}-2) = 2^{2n-2}-2^n+2$

Thus, the sum of above two elements is just the total number of *NPC*-edges included in the given Hamiltonian cycle. This implies that the lemma is correct.

This lemma implies that there are at least eight *NPC*-edges in ST^n when $n \ge 3$.

4. EDGE-DISJOINT HAMILTONIAN CYCLE PROPERTY

THEOREM 1. Given an EPM^n for any $n > 2$, it is always possible to construct two edge-disjoint Hamiltonian cycles in EPM^n.

PROOF. In the case of $n \le 2$, the degree of EPM^n is less than 4, which does not satisfy the minimum requirement for two edge-disjoint Hamiltonian cycles in it. So we only need to consider such dimensions greater than or equal to 3. Note that the two *EPM's*, namely EMP^m and EPM^n ($n>m$), are different from each other only in the structures of the lower *n-m* layers but the same in the top *m* layers. This implies that the greedy approach can be applied to construct a Hamiltonian cycle in *EMP*'s. In the *k* step, we want to accomplish a partial Hamiltonian cycle which contains all nodes on the top *k* layers from layer *n-k* to layer *n*-1.

As a basic case, we consider that the cases of k=3. Fig. 7 shows an example of constructing a partial Hamiltonian cycle using only the nodes in layer *l*, where $n-1 \le l \le n-3$.

Now, we show that the theorem is also satisfied in the successive case of k=4 by such expansion giving special consideration to the newly introduced nodes on layer *n*-4 based on the already constructed cycle when k=3. This processing only means an incremental step as shown in Fig. 8 if the number of *NP*-edges is sufficient.

In this style of approach, for each processing on the intermediate layer *l* ($0 \le l \le$ n-2), two *NP*-edges are needed for connections to the upper layer. By Lemma 1, it is evident that there are at least four numbers of *NP*-edges because the underlying induced sub-graph in layer *n*-3 in EPM^n is isomorphic to ST^2.

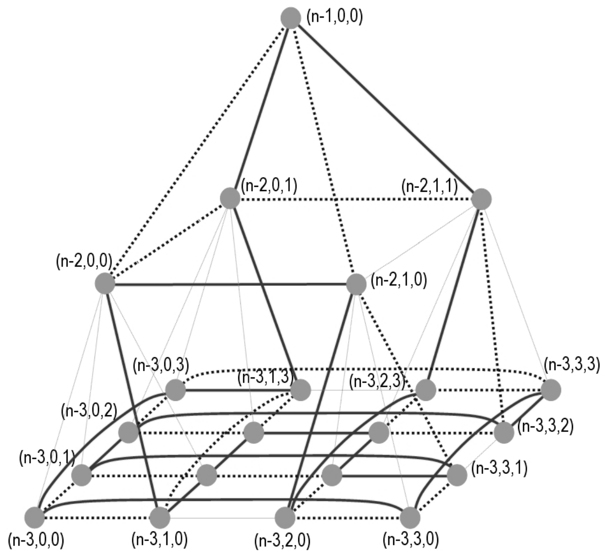


Fig. 7. Two Hamiltonian cycles constructed in EPM^3

In the intermediate step for processing an arbitrary layer $k = l$ such that $4 \leq l < n$, we assume that there has already been constructed two sets of partial edge-disjoint Hamiltonian cycles including all nodes in top l layers between layer n-l and layer n-1 of EPM^n in the previous step. Moreover it is also possible to construct two sets of partial Hamiltonian cycles on the underlying induced sub-graph constituted of the nodes in only the lower layer l+1 by the symmetric property of edges as shown in Fig.8.

Now, we focus only on the extension of connection between two cycles from layer l to l+1 to satisfy the condition that the joined sub-graph results in one larger partial Hamiltonian cycle including all nodes on the top l layer too. This is always possible because there are sufficient *NP*-edges on the induced sub-graph ST^l. By repeatedly applying these steps to reach the last

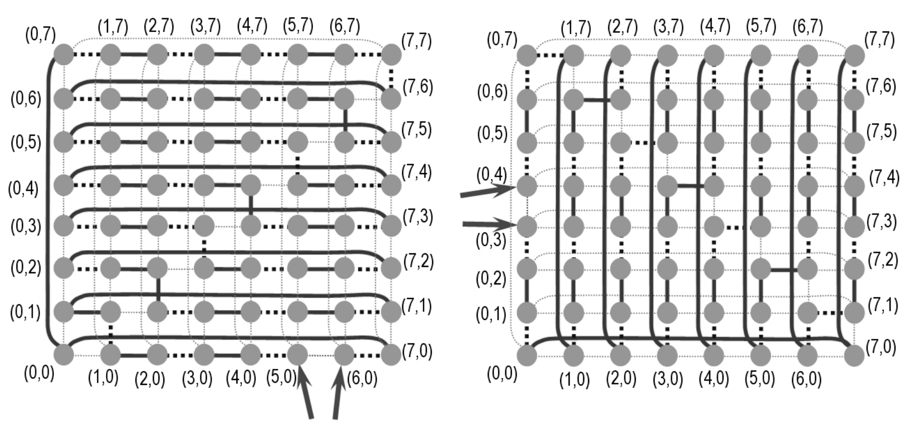


Fig. 8. A connection example of Hamiltonian cycles extension in k-step from (k-1)-step construction

layer 0, we eventually construct two edge-disjoint Hamiltonian cycles on the original graph EPM^n.

The description of the above steps implies that it is always possible to construct two sets of edge-disjoint Hamiltonian cycles by applying these steps to them until k=0. Thus the theorem is proved.

This result is optimal in the sense that EPM^n is an irregular graph that has some nodes of minimum degree 4 in the top-most two layers (layer n-1 and layer n-2).

5. CONCLUDING REMARKS

In this paper we showed that it is always possible to construct two edge-disjoint Hamiltonian cycles in an enhanced pyramid interconnection network model. In diverse applications using the Hamiltonian cycle as underlying topology, the existence of multiple sets implies the improvement of their fault-tolerance property in the sense that edge faults on one Hamiltonian cycle can be tolerated by adopting another one under such operation environment as the active-standby concept.

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