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Uniform Ergodicity and Exponential α-Mixing for Continuous Time Stochastic Volatility Model

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Abstract

A continuous time stochastic volatility model for financial assets suggested by Barndorff-Nielsen and Shephard (2001) is considered, where the volatility process is modelled as an Ornstein-Uhlenbeck type process driven by a general Lévy process and the price process is then obtained by using an independent Brownian motion as the driving noise. The uniform ergodicity of the volatility process and exponential α-mixing properties of the log price processes of given continuous time stochastic volatility models are obtained.

Keywords: Continuous time stochastic volatility model, Ornstein-Uhlenbeck type process, stationarity, uniform ergodicity, α-mixing.

1. Introduction

A stochastic volatility model has been treated mostly in continuous time to handle irregularly spaced or ultra high frequency data. Continuous time stochastic volatility(SV) models mainly concern asset-price modelling and have recently been the object of growing interest because of their applications in econometry and finance. Barndorff-Nielsen and Shephard (2001) introduce a continuous time stochastic volatility model for financial assets, where the volatility process is modelled as an Ornstein-Uhlenbeck(OU) type process driven by a subordinator and the price process is then obtained by using a standard Brownian motion independent of the Lévy process.

We consider the following two important continuous time stochastic volatility processes:

$$dG_t = \left\{\mu + \beta\sigma_t^2\right\} dt + \sigma_t dW_t, \quad t \geq 0,\ G_0 = 0 \tag{1.1}$$

and

$$d\tilde{G}_t = \left\{\mu + \beta\sigma_t^2\right\} dt + \sigma_t dW_t + \rho d\bar{Z}_{\lambda t}, \quad t \geq 0,\ \tilde{G}_0 = 0, \tag{1.2}$$

where $(\sigma_t^2)_{t\geq 0}$ is the unobserved instantaneous volatility, $(W_t)_{t\geq 0}$ standard Brownian motion, $(Z_t)_{t\geq 0}$ a nondecreasing Lévy process with $\bar{Z}_t = Z_t - E(Z_t)$ and μ, β and ρ are some constants. As a model for $(\sigma_t^2)_{t\geq 0}$, following simple Lévy driven OU type process is considered:

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_{\lambda t}, \quad t \geq 0,\ \lambda > 0. \tag{1.3}$$

The Equation (1.1) represents a continuous time SV model without leverage effect whereas the Equation (1.2) with leverage effect.

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Classical volatility models driven by Brownian motion such as GARCH diffusion of Nelson (1990) can model heavy tails, but obviously they are not able to model volatility jumps. Such phenomena can be modelled by a Levy driven volatility processes. It is examined that OU type process $(\sigma_t^2)_{t\geq 0}$ of the Equation (1.3) can capture heavy tails, volatility jumps and volatility clusters on high levels, provided the driving process has regularly varying tails (see Barndorff-Nielsen and Shephard, 2001).

Various probabilistic and statistical properties of $(\sigma_t^2)_{t\geq 0}, (G_t)_{t\geq 0}$ or $(\tilde{G}_t)_{t\geq 0}$ are studied by many authors, for example, Sato and Yamazato (1984), Kusuoka and Yoshida (2000), Barndorff-Nielsen and Shephard (2001), Klüppelberg *et al.* (2006), Fasen (2009) and references therein.

In this paper, two important continuous time log asset price $(G_t)_{t\geq 0}$ and $(\tilde{G}_t)_{t\geq 0}$ in (1.1) and (1.2) together with the volatility process $(\sigma_t^2)_{t\geq 0}$ in (1.3) are considered. We first obtain the uniform ergodicity and β-mixing property of volatility process $(\sigma_t^2)_{t\geq 0}$ and then the exponential α-mixing properties of the log price processes are obtained.

2. Uniform Ergodicity of σ_t^2

Let $Z = (Z_t)_{t\geq 0}$ be a time homogeneous càdlàg Lévy process defined on $(\Omega, \mathcal{F}, P)$ to R starting from the origin. Denote by (b, τ^2, ν) the characteristic triple of Z. The Levy measure ν is a nontrivial σ-finite measure on R satisfying $\nu(\{0\}) = 0$ and $\int_R \min(1, |z|^2)\nu(dz) < \infty$. Z has the characteristic function of the form $E(e^{iuZ_t}) = \exp\{t\psi(u)\}$ with

$$\psi(u) = ibu - \frac{1}{2}\tau^2 u^2 + \int_R \left(e^{iuz} - 1 - iuzI_{\{|z|<1\}}\right)\nu(dz), \quad u \in R,\ t \geq 0,$$

where $b \in R$ and nonnegative τ^2 is the variance of the Brownian motion component of Z. I_A denotes the indicator function of A. A Lévy process with nondecreasing sample paths called a subordinator. Subordinators have no Gaussian part, finite variation with nonnegative drift and Lévy measure ν concentrated on $(0, \infty)$. We assume, throughout this paper that a background driving Lévy process Z is a subordinator whose characteristic triple is $(b, 0, \nu)$. Note that the characteristic triple of $(Z_{\lambda t})_{t\geq 0}$ is $(\lambda b, 0, \lambda\nu)$.

OU type process σ_t^2 driven by Z is defined by

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}, \quad t \geq 0,\ \lambda > 0, \tag{2.1}$$

where σ_0^2 is assumed to be independent of Z. Equation (2.1) is equivalently defined as the unique solution of the Equation (1.3). $\sigma^2 = (\sigma_t^2)_{t\geq 0}$ of (2.1) is a time homogeneous Markov process whose sample path is càdlàg. Let $p_t(x, dy)$ be the probability transition function of $(\sigma_t^2)_{t\geq 0}$. Note that σ^2 is a nonexplosive Borel right process since p_t maps Borel functions to Borel functions for each $t \geq 0$.

Theorem 1. *(1) $\int_{|z|>1} \log|z|\nu(dz) < \infty$ holds if and only if σ_t^2 converges in distribution to a finite random variable σ_∞^2 as $t \to \infty$. In this case $\sigma_\infty^2 \overset{D}{=} \int_0^\infty e^{-s} dZ_s$. (2) Let π be the distribution of σ_∞^2. Then $p_t(x, A) \to \pi(A)$ as $t \to \infty$ for every $x \in R$ and $A \in \mathcal{B}(R)$. Here π is the unique invariant distribution of σ^2. (3) If the distribution of σ_0^2 is π, then σ^2 is strictly stationary.*

Proof: See Sato (1999), Chapter 17 and Klüppelberg *et al.* (2006), Theorem 2. □

$\mathcal{A}$ is called an extended generator of a Markov process $\Phi = (\Phi_t)_{t\geq 0}$ associated with a function $f : R \to R$ if for each $x \in R$, $t \geq 0$,

$$\int_0^t E_x\left[|\mathcal{A}f(\Phi_s)|\right] ds < \infty \tag{2.2}$$

and

$$E_x[f(\Phi_t)] = f(x) + E_x\left[\int_0^t \mathcal{A}f(\Phi_s)ds\right]. \tag{2.3}$$

The following Theorem plays a crucial role to prove our main results.

Theorem 2. *(Theorem 5.2 in Down et al. (1995)) Let Φ be a ϕ-irreducible, aperiodic Markov process. Suppose that for constants $c > 0$, $d > 0$, a petite set C in $\mathcal{B}(R)$ and a measurable function $V \geq 1$, the following inequality holds with some extended generator $\mathcal{A}$:*

$$\mathcal{A}V \leq -cV + dI_C. \tag{2.4}$$

Then Φ is uniformly ergodic.

Now define the following integro-differential equation:

$$\mathcal{A}V(x) = (\lambda b - \lambda x)V'(x) + \lambda \int_R \left(V(x+z) - V(x) - zV'(x)I_{\{|z|\leq 1\}}\right) \nu(dz), \tag{2.5}$$

where $\mathcal{A}$ acts on real valued C^2-function which denotes the class of functions with continuous first and second partial derivatives. $\mathcal{A}$ in (2.5) is an infinitesimal generator of σ^2 given by the equation (1.3) (see Sato and Yamazato, 1984)

Theorem 3. *Let $\int_{|z|>1} |z|^p \nu(dz) < \infty$ and $\int_{|z|>1} |z|^p \eta(dz) < \infty$ for some $p > 0$ with η as its initial distribution. Then σ^2 is uniformly ergodic.*

Proof: For each positive integer m, let $T^m = \inf\{t \geq 0 : |\sigma_t^2| \geq m\}$.

For some p $(0 < p < 1)$, we define C^2-function $V : R \to R^+$ by $V(x) = |x|^p + 1$, $|x| > 1$ and V, V', and V'' are continuous and bounded on $|x| \leq 1$.

Recall that $|x+z|^p \leq |x|^p + |z|^p$ $(0 < p \leq 1)$ and $V(x+z) - V(x) - zV'(x) = 1/2z^2V''(x_1)$, $x_1 = x + \alpha z$, $0 \leq \alpha \leq 1$, by Lagrange remainder theorem. In this proof, we use $K < \infty$ as the universal constant and K may vary from line to line.

Since $V(x)$ is bounded on $|x| \leq 1$, for any x,

$$\left|\int_{|z|>1} (V(x+z) - V(x))\nu(dz)\right| \leq K\nu(|z| > 1) + \int_{|z|>1} |z|^p \nu(dz) < \infty. \tag{2.6}$$

For $|x| \leq 2$, we have that

$$\begin{aligned}\left|\int_{|z|\leq 1} (V(x+z) - V(z) - zV'(x))\nu(dz)\right| &= \left|\int_{|z|\leq 1} \frac{1}{2}z^2V''(x_1)\nu(dz)\right| \\ &\leq \frac{1}{2}\left|\sup_{\{0\leq\alpha\leq 1, |z|\leq 1\}} V''(x_1)\right| \int_{|z|\leq 1} z^2\nu(dz) \\ &< \infty. \end{aligned} \tag{2.7}$$

Here the last inequality in (2.7) follows from continuity of V''. If $|x| > 2$, then $\sup_{\{0\le\alpha\le1,|z|\le1\}} |x+\alpha z| > 1$, that is, $|x_1| > 1$ and hence

$$|V''(x_1)| = |p(p-1)||x_1|^{p-2} \le |p(p-1)|. \tag{2.8}$$

Combining (2.5)–(2.8) yields that for each x,

$$\begin{aligned}\mathcal{A}V(x) &\le (\lambda b - \lambda x)V'(x) + K\\ &= (\lambda b - \lambda x)p(\operatorname{sgn} x)|x|^{p-1}I_{|x|>1} + K\\ &= -\lambda p|x|^p I_{|x|>1} + K,\end{aligned}$$

which implies that for any $x \in R$,

$$\mathcal{A}V \le -cV + dI_C, \tag{2.9}$$

with proper positive constants c and d and a compact set C. Hence the drift condition (2.4) holds.

Now we need to prove that $\mathcal{A}$ is an extended generator of σ^2. From Dynkin's formula (Meyn and Tweedie, 1993), we get that for each $x \in R$,

$$E_x\left[V\left(\sigma^2_{t\wedge T^m}\right)\right] = V(x) + E_x\left[\int_0^{t\wedge T^m} \mathcal{A}V\left(\sigma^2_s\right)ds\right]. \tag{2.10}$$

Use (2.9) and (2.10) to show that

$$\begin{aligned}E_x\left[V\left(\sigma^2_{t\wedge T^m}\right)\right] &\le V(x) + E_x\left[\int_0^{t\wedge T^m}\left(cV\left(\sigma^2_s\right)+d\right)\right]ds\\ &\le V(x) + E_x\left[\int_0^t\left(V\left(\sigma^2_{s\wedge T^m}\right)+\frac{d}{c}\right)d(cs)\right]\\ &\le V(x) + \int_0^t E_x\left[V\left(\sigma^2_{s\wedge T^m}\right)+\frac{d}{c}\right]d(cs).\end{aligned} \tag{2.11}$$

Applying Gronwall's inequality (see, *e.g.* Protter, 2005) to (2.11) yields

$$E_x\left[V\left(\sigma^2_{t\wedge T^m}\right)\right] \le \left(V(x) + \frac{d}{c}\right)e^{c\cdot t}. \tag{2.12}$$

On the other hand, choose $\epsilon > 0$ arbitrary small so that $p(1+\epsilon) = p' < 1$ and $\int_{|z|>1}|z|^{p'}\eta(dz) < \infty$ and then by adopting the same processes used to obtain (2.12), we have that

$$\begin{aligned}E_x\left[\left(V\left(\sigma^2_{t\wedge T^m}\right)\right)^{1+\epsilon}\right] &\le 2^{1+\epsilon}E_x\left(\left|\sigma^2_{t\wedge T^m}\right|^{p'}+1\right)\\ &= 2^{1+\epsilon}\left(|x|^{p'}+1+\frac{d'}{c'}\right)e^{c'\cdot t},\end{aligned} \tag{2.13}$$

with some positive constants d' and c'. It follows from given assumptions and (2.13) that $V(\sigma^2_{t\wedge T^m})$ is uniformly integrable and hence $\int_0^t E_x[\mathcal{A}V(\sigma^2_s)]ds$ is bounded and

$$\lim_{m\to\infty} E_x\left[V\left(\sigma^2_{t\wedge T^m}\right)\right] = E_x\left[V\left(\sigma^2_t\right)\right].$$

Taking $m \to \infty$ on both sides of (2.10), Lebesgue dominated convergence theorem yields that

$$E_x\left[V\left(\sigma_t^2\right)\right] = V(x) + E_x\left[\int_0^t \mathcal{A}V\left(\sigma_s^2\right)ds\right].$$

Therefore (2.2) and (2.3) hold and $\mathcal{A}$ is an extended generator of σ^2 associated with the function V. Since $\int_{|z|>1} |z|^p \nu(dz) < \infty$ for some $p > 0$ implies that $\int_{|z|>1} \log|z|\nu(dz) < \infty$, we can apply Theorem 1 to conclude that σ^2 is π-irreducible. Weak Feller property of $p_t(x, dy)$ follows from the Lebesgue dominated convergence theorem. Hence by Theorem 2 (see also Theorem 6.1 in Meyn and Tweedie (1993)), σ_t^2 is uniformly ergodic and σ_t^2 with π as its initial distribution is exponentially β-mixing. Moreover, $\int_R |z|^p \pi(dz) < \infty$. □

Remark 1. Let Q be a $d \times d$ matrix whose eigenvalues have positive real parts, and let Z be a nontrivial d-dimensional Lévy process. Let X be a d-dimensional Ornstein-Uhlenbeck type process given by

$$dX_t = -QX_t dt + dZ_{\lambda t}.$$

Exponential β-mixing property for X_t can be obtained by considering a discrete time skeleton $\{X_{\Delta n}\}$, $\Delta > 0$, $n = 1, 2, \ldots$ (see Masuda, 2004).

3. Exponential α-Mixing for Continuous Time SV Model

In the previous section, we obtain the uniform ergodicity of the volatility process σ^2. Uniform ergodicity implies the geometric ergodicity and β-mixing property of the process. There are many authors who considered the mixing properties for various continuous time stochastic models (see, *e.g.*, Masuda, 2004; Haug and Czado, 2007; Haug *et al.*, 2007; Fasen, 2009). In this section, we examine the mixing properties of the continuous time stochastic volatility models.

We consider the (logarithmic) price process that is given by the following SDE;

$$dG_t = \left\{\mu + \beta\sigma_t^2\right\}dt + \sigma_t dW_t, \quad t \geq 0,\ G_0 = 0, \tag{3.1}$$

where μ and β are constants, σ_t^2 is given in the Equation (2.1) and $(W_t)_{t\geq 0}$ is a standard Brownian motion independent of σ_0^2 and Z. The Itô solution of the Equation (3.1) is given as

$$G_t = \mu t + \beta\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s, \quad t \geq 0. \tag{3.2}$$

It is known that $(\sigma_t^2, G_t)_{t\geq 0}$ is a time homogeneous Markov process and the various probabilistic properties of G_t with the subordinator Z_t are investigated in Barndorff-Nielsen and Shephard (2001) and Klüppelberg *et al.* (2006).

The logarithmic asset returns over time period of length $r > 0$ are then given by $G_t^{(r)} = G_{t+r} - G_t,\ t \geq 0$. If σ_t^2 is stationary, then $(G_t^{(r)})_{t\geq 0}$ are also stationary for each fixed $r > 0$.

Barndorff-Nielsen and Shephard (2001) suggest the following extended version of (3.1) to allow the leverage effect;

$$d\tilde{G}_t = \left\{\mu + \beta\sigma_t^2\right\}dt + \sigma_t dW_t + \rho d\bar{Z}_{\lambda t}, \quad t \geq 0,\ \tilde{G}_0 = 0, \tag{3.3}$$

where $\bar{Z}_t = Z_t - E(Z_t)$ whose solution is given by

$$\tilde{G}_t = \mu t + \beta \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s + \rho \int_0^t d\bar{Z}_{\lambda t}, \quad t \geq 0. \tag{3.4}$$

Let $\tilde{G}_t^{(r)} = \tilde{G}_{t+r} - \tilde{G}_t$, $t \geq 0$.

We now examine the mixing properties of $G_t^{(r)}$ and $\tilde{G}_t^{(r)}$. Recall the definitions of α-mixing, $\tilde{\alpha}$-mixing and β-mixing for a process $Y = (Y_t)_{t\geq 0}$. For $\mathcal{F}_I^Y = \sigma(Y_s : s \in I)$,

$$\alpha_Y(t) = \sup_{u\geq 0} \sup \left\{ |P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_{[0,u]}^Y, B \in \mathcal{F}_{[u+t,\infty)}^Y \right\},$$

$$\tilde{\alpha}_Y(t) = \sup_{u\geq 0} \sup \left\{ \left\| E\left[f \left| \mathcal{F}_{[0,u]}^Y \right. \right] - E(f) \right\|_{L^1(P)} : f \in b\mathcal{F}_{[u+t,\infty)}^Y, \|f\|_\infty \leq 1 \right\}$$

and

$$\beta_Y(t) = E_{ess.sup} \left\{ \left| P\left(B \middle| \mathcal{F}_{[0,u]}^Y \right) - P(B) \right| : B \in \mathcal{F}_{[u+t,\infty)}^Y \right\},$$

where $b\mathcal{F}$ denotes the set of all bounded $\mathcal{F}$ measurable random variables. If $\alpha_Y(t) \leq Ke^{-at}$, for some constants $K, a > 0$ for all $t \geq 0$, then Y is called α-mixing with exponential rate. It is well known that (see Doukhan, 1994)

$$2\alpha_Y(t) \leq \beta_Y(t), \qquad \alpha_Y(t) \leq \tilde{\alpha}_Y(t) \leq 6\alpha_Y(t). \tag{3.5}$$

Theorem 4. *Suppose that assumptions in Theorem 2.3 holds. For any $r > 0$, $(G_{nr}^{(r)})_{n\in N}$ and $(\tilde{G}_{nr}^{(r)})_{n\in N}$ obtained from equations (3.1)–(3.4) are exponentially α-mixing.*

Proof: Note that

$$\tilde{G}_t^{(r)} = \tilde{G}_{t+r} - \tilde{G}_t = \mu r + \beta \int_t^{t+r} \sigma_s^2 ds + \int_t^{t+r} \sigma_s dW_s + \int_t^{t+r} \rho dZ_{\lambda s}$$

and

$$\mathcal{F}_{\{1,2,\ldots,l-1\}}^{\tilde{G}^{(r)}} \subset \mathcal{F}_{[0,lr]}^{\sigma^2,dZ,dW}, \qquad \mathcal{F}_{\{k+l,k+l+1,\ldots\}}^{\tilde{G}^{(r)}} \subset \mathcal{F}_{[(k+l)r,\infty)}^{\sigma^2,dZ,dW}. \tag{3.6}$$

Here

$$\mathcal{F}_{\{1,2,\ldots,l-1\}}^{\tilde{G}^{(r)}} = \sigma\left(\tilde{G}_{nr}^{(r)} : n \in \{1, 2, \ldots, l-1\} \right), \qquad \mathcal{F}_I^{dZ} = \sigma\left(Z_{\lambda t} - Z_{\lambda s} : s < t, s, t \in I \right)$$

and

$$\mathcal{F}^{\sigma^2,dZ} := \mathcal{F}^{\sigma^2} \vee \mathcal{F}^{dZ}, \qquad \mathcal{F}^{\sigma^2,dZ,dW} := \mathcal{F}^{\sigma^2} \vee \mathcal{F}^{dZ} \vee \mathcal{F}^{dW}.$$

$$\begin{aligned}
\tilde{\alpha}_{\tilde{G}^{(r)}}(k) &= \sup \left\{ \left\| E\left[f \left| \mathcal{F}_{\{1,2,\ldots,l-1\}}^{\tilde{G}^{(r)}} \right. \right] - E(f) \right\|_{L_1(P)} : f \in b\mathcal{F}_{\{k+l,k+l+1,\ldots\}}^{\tilde{G}^{(r)}}, \|f\|_\infty \leq 1 \right\} \\
&\leq \sup \left\{ \left\| E\left[f \left| \mathcal{F}_{[0,lr]}^{\sigma^2,dZ,dW} \right. \right] - E(f) \right\|_{L_1(P)} : f \in b\mathcal{F}_{[(k+l)r,\infty)}^{\sigma^2,dZ,dW}, \|f\|_\infty \leq 1 \right\} \\
&= \sup \left\{ \left\| E\left[f \left| \mathcal{F}_{[0,lr]}^{\sigma^2,dZ} \right. \right] - E(f) \right\|_{L_1(P)} : f \in b\mathcal{F}_{[(k+l)r,\infty)}^{\sigma^2,dZ,dW}, \|f\|_\infty \leq 1 \right\} \\
&= \sup \left\{ \left\| E\left[f \left| \mathcal{F}_{[0,lr]}^{\sigma^2,dZ} \right. \right] - E(f) \right\|_{L_1(P)} : f \in b\mathcal{F}_{[(k+l)r,\infty)}^{\sigma^2,dZ}, \|f\|_\infty \leq 1 \right\} \\
&= \sup \left\{ \left\| E\left[f \left| \mathcal{F}_{[0,lr]}^{\sigma^2} \right. \right] - E(f) \right\|_{L_1(P)} : f \in b\mathcal{F}_{[(k+l)r,\infty)}^{\sigma^2}, \|f\|_\infty \leq 1 \right\}.
\end{aligned} \tag{3.7}$$

The inequality in (3.7) follows from (3.6) and the fact that if $\mathcal{G}_1' \subset \mathcal{G}_1,\ \mathcal{G}_2' \subset \mathcal{G}_2$, then $\tilde{\alpha}(\mathcal{G}_1', \mathcal{G}_2') \leq \tilde{\alpha}(\mathcal{G}_1, \mathcal{G}_2)$. The second equality in the third line holds since (σ^2, dZ) and dW are independent. For any $f \in b\mathcal{F}^{\sigma^2,dZ,dW}_{[(k+l)r,\infty]}$, define $g = E[f|\mathcal{F}^{\sigma^2,dZ}_{[(k+l)r,\infty)}]$. Then g is bounded and $\mathcal{F}^{\sigma^2,dZ}_{[(k+l)r,\infty)}$- measurable function. From the properties of conditional expectation, we have that $E(f) = E(g)$ and

$$\begin{aligned}
E\left[f\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,lr]}\right] &= E\left[E\left[f\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,lr]}\right]\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,\infty)} \vee \mathcal{F}^{dW}_{[0,\infty)}\right] \\
&= E\left[E\left[f\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,lr]}\right]\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,\infty)}\right] \\
&= E\left[E\left[f\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,\infty)}\right]\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,lr]}\right] \\
&= E\left[E\left[f\middle|\mathcal{F}^{\sigma^2,dZ}_{[(k+l)r,\infty)}\right]\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,lr]}\right] \\
&= E\left[g\middle|\mathcal{F}^{\sigma^2,dZ}_{[0,lr]}\right].
\end{aligned}$$

Hence, we have the third equality. The last inequality can be obtained using the same method adopted in the proof of Theorem 3.5, Haug *et al.* (2007) and we have that

$$\tilde{\alpha}_{\tilde{G}^{(r)}}(k) \leq \tilde{\alpha}_{\sigma^2}(kr). \tag{3.8}$$

Therefore α-mixing of $\tilde{G}^{(r)}$ follows from the β-mixing property of σ^2 and inequalities in (3.5) and (3.8). In this case the mixing coefficient for returns are less than or equal to the mixing coefficient of σ^2.

On the other hand, we have that

$$G_t^{(r)} = G_{t+r} - G_t = \mu r + \beta \int_t^{t+r} \sigma_s^2 ds + \int_t^{t+r} \sigma_s dW_s$$

and

$$\mathcal{F}^{G^{(r)}}_{\{1,2\ldots,l-1\}} \subset \mathcal{F}^{\sigma^2,dW}_{[0,lr]}, \qquad \mathcal{F}^{G^{(r)}}_{\{k+l,k+l+1,\ldots\}} \subset \mathcal{F}^{\sigma^2,dW}_{[(k+l)r,\infty)}.$$

Note that σ^2 and W are independent. Though the same methods as those used to prove the previous case, we have that

$$\tilde{\alpha}_{G^{(r)}}(k) \leq \tilde{\alpha}_{\sigma^2}(kr). \tag{3.9}$$

(3.9) together with the inequalities in (3.5) and β-mixing property of σ^2 implies that $G^{(r)}$ is also exponential α-mixing. □

4. Conclusion

Recently, a stochastic volatility model has been treated mostly in continuous time. Barndorff-Nielsen and Shephard (2001) introduce a continuous time stochastic volatility model for financial assets, where the volatility process is modelled as an Ornstein-Uhlenbeck(OU) type process driven by a subordinator and the price process is obtained using a standard Brownian motion independent of the Lévy process. It is known that a Levy driven OU type process $(\sigma_t^2)_{t\geq 0}$ can capture heavy tails, volatility jumps and volatility clusters on high levels, provided the driving process has regularly varying tails.

In this paper, two important continuous time log asset price $(G_t)_{t\geq 0}$ and $(\tilde{G}_t)_{t\geq 0}$ together with the volatility process $(\sigma_t^2)_{t\geq 0}$ driven by a Levy process are considered. We first obtain the uniform ergodicity and β-mixing property of volatility process $(\sigma_t^2)_{t\geq 0}$ via a drift condition and extended generator. Then the exponential α-mixing properties of the log price processes are obtained.

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