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Confidence Intervals and Joint Confidence Regions for the Two-Parameter Exponential Distribution based on Records

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Abstract

Exponential distribution is widely adopted as a lifetime model. Many authors have considered the interval estimation of the parameters of two-parameter exponential distribution based on complete and censored samples. In this paper, we consider the interval estimation of the location and scale parameters and the joint confidence region of the parameters of two-parameter exponential distribution based on upper records. A simulation study is done for the performance of all proposed confidence intervals and regions. We also propose the predictive intervals of the future records. Finally, a numerical example is given to illustrate the proposed methods.

Keywords: Two-parameter exponential distribution, joint confidence region, joint confidence interval, record values.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed(iid) random variables with cumulative distribution function(cdf) $F(x;\theta)$ and probability density function(pdf) $f(x;\theta)$, where θ is an unknown parameter. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record values if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower record values. If $\{U(n),\ n \geq 1\}$ is defined by

$$U(1) = 1, \quad U(n) = \min\{j : j > U(n-1),\ X_j > X_{U(n-1)}\}, \tag{1.1}$$

for $n \geq 2$, then the sequence $\{X_{U(n)},\ n \geq 1\}$ provides a sequence of upper record statistics. The sequence $\{U(n),\ n \geq 1\}$ represents the record times. For more details and applications of record values, see Arnold *et al.* (1998).

The exponential distribution is applied in a wide variety of statistical procedures, especially in lifetime data analysis and reliability. Significant research has been done on statistical inference for the exponential distribution, and a very good summary of this work can be found in Johnson *et al.* (1994). The cumulative distribution function(cdf) of the two-parameter exponential distribution is given by

$$F(x;\mu,\sigma) = 1 - \exp\left(-\frac{x-\mu}{\sigma}\right), \quad x \geq \mu,\ \sigma > 0, \tag{1.2}$$

where μ and σ are the location and scale parameters, respectively.

Recently, some authors have considered the interval estimation of the parameters of a two-parameter exponential distribution based on censored samples. Sun *et al.* (2008) obtained confidence intervals for the scale parameter of exponential distribution based on Type-II doubly censored samples. Wu

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(2007, 2010) proposed the interval estimation of the scale parameter and the joint confidence region of the parameters of two-parameter exponential distribution based on a doubly Type-II censored sample and a progressively Type-II censored sample, respectively. Interval estimation of the parameters of two-parameter exponential distribution has not yet been studied based on record values. In this paper, we address the construction of confidence intervals and joint confidence regions for the parameters of two-parameter exponential distribution based on records.

The paper is organized as follows: In Section 2, confidence intervals for μ and σ are considered based on upper record values. We also present two exact joint confidence regions for the parameters μ and σ. In Section 3, a simulation study is performed to compare the proposed confidence intervals and regions. In addition to the estimation of the two parameters, the predictive intervals of the future record based on observed upper records are given in Section 4. Finally in Section 5, a numerical example is presented to illustrate the proposed methods.

2. Confidence Intervals and Regions

Let $X_{U(1)} < X_{U(2)} < \cdots < X_{U(m)}$ be the first m observed upper record values from the two-parameter exponential distribution. For notation simplicity, we will write X_i for $X_{U(i)}$. Let $Y_i = (X_i - \mu)/\sigma$, $(i = 1, 2, \ldots, m)$, then $Y_1 < Y_2 < \cdots < Y_m$ are the first m upper record values from a standard exponential distribution. Moreover, $Z_1 = Y_1$ and $Z_i = Y_i - Y_{i-1}$, for $i = 2, \ldots, m$, are iid standard exponential random variables. Hence

$$U = 2\,Z_1 = 2\,Y_1 \tag{2.1}$$

has a chi-square distribution with 2 degrees of freedom and

$$V = 2\sum_{i=2}^{m} Z_i = 2(Y_m - Y_1) \tag{2.2}$$

has a chi-square distribution with $2m - 2$ degrees of freedom. We can also find that U and V are independent random variables. Now, let

$$T_1 = \frac{U/2}{V/2(m-1)} = \frac{(m-1)Y_1}{Y_m - Y_1} \tag{2.3}$$

and

$$T_2 = U + V = 2\,Y_m. \tag{2.4}$$

It is easy to show that T_1 has an F distribution with 2 and $2m - 2$ degrees of freedom and T_2 has a chi-square distribution with $2m$ degrees of freedom. Furthermore, T_1 and T_2 are independent (see Johnson *et al.*, 1994, p.350).

The distributions of the pivotal quantities U, V, T_1 and T_2 are independent of parameters. To make use of these pivotal quantities, we can construct the confidence intervals and regions for parameters. Using the pivotal quantity T_1, we can construct a confidence interval for the location parameter μ as follows:

Theorem 1. *Suppose that $X_1 < X_2 < \cdots < X_m$ be the first m upper record values from the two-parameter exponential distribution in (1.2). Then, for any $0 < \alpha < 1$, a $100(1-\alpha)\%$ confidence* *interval for the location parameter μ is given by*

$$\left(X_1 - \frac{(X_m - X_1)F_{\frac{\alpha}{2}}(2, 2m-2)}{m-1} < \mu < X_1 - \frac{(X_m - X_1)F_{1-\frac{\alpha}{2}}(2, 2m-2)}{m-1}\right),$$

where $F_{(\alpha/2)}(2, 2m-2)$ is the right-tailed $(\alpha/2)^{th}$ quantile for F distribution with 2 and $2m-2$ degrees of freedom.

Proof: From (2.3), we know that

$$T_1 = \frac{(m-1)(X_1 - \mu)}{X_m - X_1},$$

has an F distribution with 2 and $2m-2$ degrees of freedom, hence we have

$$\begin{aligned}1 - \alpha &= P\left(F_{1-\frac{\alpha}{2}}(2, 2m-2) < T_1 < F_{\frac{\alpha}{2}(2,2m-2)}\right) \\ &= P\left(X_1 - \frac{(X_m - X_1)F_{\frac{\alpha}{2}}(2, 2m-2)}{m-1} < \mu < X_1 - \frac{(X_m - X_1)F_{1-\frac{\alpha}{2}}(2, 2m-2)}{m-1}\right).\end{aligned}$$

The proof is thus obtained. □

Make use of the pivotal quantity V, we can construct a confidence interval for the scale parameter σ as follows:

Theorem 2. *Based on the first m upper record values $X_1 < X_2 < \cdots < X_m$, a $100(1-\alpha)\%$ confidence interval for σ is given by*

$$\left(\frac{2(X_m - X_1)}{\chi^2_{\frac{\alpha}{2}}(2m-2)} < \sigma < \frac{2(X_m - X_1)}{\chi^2_{1-\frac{\alpha}{2}}(2m-2)}\right),$$

where $\chi^2_{(\alpha/2)}(2m-2)$ is the right-tailed $(\gamma/2)^{th}$ quantile for chi-square distribution with $2m-2$ degrees of freedom.

Proof: From (2.2), we note that

$$V = \frac{2(X_m - X_1)}{\sigma} \tag{2.5}$$

has a chi-square distribution with $2m-2$ degrees of freedom. Hence, we have

$$\begin{aligned}1 - \alpha &= P\left(\chi^2_{1-\frac{\alpha}{2}}(2m-2) < V < \chi^2_{\frac{\alpha}{2}}(2m-2)\right) \\ &= P\left(\frac{2(X_m - X_1)}{\chi^2_{\frac{\alpha}{2}}(2m-2)} < \sigma < \frac{2(X_m - X_1)}{\chi^2_{1-\frac{\alpha}{2}}(2m-2)}\right).\end{aligned}$$

□

Using the set of pivotal quantities U and V, we can construct a confidence region for the parameters μ and σ in the following Theorem. We call it Method 1.

Theorem 3. ***(Method 1)*** *Based on the pivotal quantities U and V, a $100(1-\alpha)\%$ joint confidence region for (μ, σ) is given by*

$$\begin{cases} X_1 - \dfrac{\sigma}{2}\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2) < \mu < X_1 - \dfrac{\sigma}{2}\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2), \\ \dfrac{2(X_m - X_1)}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2)} < \sigma < \dfrac{2(X_m - X_1)}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2)}. \end{cases}$$

Proof: Since U and V are independent and $U = 2(X_1 - \mu)/\sigma$ has a chi-square distribution with 2 degrees of freedom and $V = 2(X_m - X_1)/\sigma$, has a chi-square distribution with $2m-2$ degrees of freedom, then for any $0 < \alpha < 1$, we have

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2) < U < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2)\right) = \sqrt{1-\alpha}$$

and

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2) < V < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2)\right) = \sqrt{1-\alpha}.$$

From these relationships and independence of U and V, we obtain

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2) < \frac{2(X_1-\mu)}{\sigma} < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2),\ \chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2) < \frac{2(X_m - X_1)}{\sigma} < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2)\right)$$
$$= \sqrt{1-\alpha}\sqrt{1-\alpha} = 1-\alpha.$$

Equivalently,

$$P\left(X_1 - \frac{\sigma}{2}\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2) < \mu < X_1 - \frac{\sigma}{2}\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2),\ \frac{2(X_m - X_1)}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2)} < \sigma < \frac{2(X_m - X_1)}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2)}\right) = 1-\alpha.$$

Thus the theorem follows. □

Now, make use of the set of pivotal quantities T_1 and T_2, we can construct another confidence region for the parameters of μ and σ in the following Theorem. We call it Method 2.

Theorem 4. ***(Method 2)*** *Based on the pivotal quantities T_1 and T_2, a $100(1-\alpha)\%$ joint confidence region for (μ, σ) is given by*

$$\begin{cases} X_1 - \dfrac{(X_m - X_1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2, 2m-2)}{m-1} < \mu < X_1 - \dfrac{(X_m - X_1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2, 2m-2)}{m-1}, \\ \dfrac{2(X_m - \mu)}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)} < \sigma < \dfrac{2(X_m - \mu)}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}. \end{cases}$$

Proof: From (2.3) and (2.4), we know that

$$T_1 = \frac{(m-1)(X_1 - \mu)}{X_m - X_1}$$

Table 1: The simulated average confidence length and confidence area.

m	Length		Area	
	μ	σ	Method 1	Method 2
5	4.4208	2.4191	29.7751	26.9818
6	4.1192	2.0251	20.4781	18.7235
7	3.8209	1.7197	14.8141	13.6641
8	3.7085	1.5484	12.3008	11.4338
9	3.5660	1.3948	10.0895	9.4405
10	3.4964	1.2908	8.8476	8.3251
11	3.4299	1.2024	7.8700	7.4409
12	3.3870	1.1330	7.1541	6.7922
13	3.3577	1.0760	6.5934	6.2824
14	3.2830	1.0113	5.9769	5.7130
15	3.3110	0.9832	5.7680	5.5286

has an F distribution with 2 and $2m-2$ degrees of freedom, and

$$T_2 = \frac{2}{\sigma}(X_m - \mu)$$

has a chi-square distribution with $2n$ degrees of freedom, and it is independent of T_1. Next, for $0 < \alpha < 1$, we have

$$P\left(F_{\frac{1+\sqrt{1-\alpha}}{2}}(2, 2m-2) < T_1 < F_{\frac{1-\sqrt{1-\alpha}}{2}}(2, 2m-2)\right) = \sqrt{1-\alpha}$$

and

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m) < T_2 < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)\right) = \sqrt{1-\alpha}.$$

From these relationships, we obtain

$$P\left(F_{\frac{1+\sqrt{1-\alpha}}{2}}(2, 2m-2) < \frac{(m-1)(X_1-\mu)}{X_m - X_1} < F_{\frac{1-\sqrt{1-\alpha}}{2}}(2, 2m-2),\right.$$
$$\left.\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m) < \frac{2}{\sigma}(X_m-\mu) < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)\right) = 1-\alpha.$$

Then, the following inequalities determine the $100(1-\alpha)\%$ joint confidence region of two parameters μ and σ,

$$\begin{cases} X_1 - \dfrac{(X_m - X_1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2, 2m-2)}{m-1} < \mu < X_1 - \dfrac{(X_m - X_1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2, 2m-2)}{m-1}, \\ \dfrac{2(X_m-\mu)}{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)} < \sigma < \dfrac{2(X_m-\mu)}{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}. \end{cases}$$

The proof is thus completed. □

3. Simulation

In this section, we carry out a Monte Carlo simulation to illustrate all the methods discussed in this paper. The computations are performed using Visual S-PLUS(V8). In this simulation, we randomly generated upper record values $X_1 < X_2 < \cdots < X_m$ from the standard exponential distribution with $\mu = 0$ and $\sigma = 1$. We then computed 90% confidence intervals and regions using Theorems 1–4. We replicated the process 5000 times. For $m = 5(1)15$, Table 1 shows that the simulated average confidence length and confidence area with the confidence coefficient $1-\alpha = 0.90$. From Table 1, we observe that when m increases, the average confidence lengths for μ and σ, and the average confidence area in both Methods 1 and 2 are decreased. We also observe that Method 2 has a better performance than Method 1, since it provides the smaller confidence area.

4. Prediction Interval for the Future Record X_{m+1}

In this section, we present a prediction interval for the future record X_{m+1} based on the observed record values $X_1, \ldots, X_m$. Since the statistic

$$S = \frac{(m-1)(X_{m+1} - X_m)}{X_m - X_1}, \tag{4.1}$$

follows an F distribution with 2 and $2m-2$ degrees of freedom, we can construct a prediction interval for X_{m+1} as follows:

Theorem 5. *Based on the observed record values $X_1 < X_2 < \cdots < X_m$, a $(1-\alpha)100\%$ prediction interval for the future record X_{m+1} is given by*

$$\left(X_m + \left(\frac{X_m - X_1}{m-1}\right) F_{1-\frac{\alpha}{2}}(2, 2m-2), X_m + \left(\frac{X_m - X_1}{m-1}\right) F_{\frac{\alpha}{2}}(2, 2m-2)\right). \tag{4.2}$$

Proof: Since $S \sim F(2, 2m-2)$, then we have

$$\begin{aligned} 1-\alpha &= P\left(F_{1-\frac{\alpha}{2}}(2, 2m-2) < \frac{(m-1)(X_{m+1} - X_m)}{X_m - X_1} < F_{\frac{\alpha}{2}}(2, 2m-2)\right) \\ &= P\left(X_m + \left(\frac{X_m - X_1}{m-1}\right) F_{1-\frac{\alpha}{2}}(2, 2m-2) < X_{m+1} < X_m + \left(\frac{X_m - X_1}{m-1}\right) F_{\frac{\alpha}{2}}(2, 2m-2)\right). \end{aligned}$$

The proof is thus completed. □

5. Numerical Example

In this example, we consider a simulated sample of size $n = 15$ from the Exponential distribution in (1.2) with parameters $\mu = 2.5$ and $\sigma = 1$. The simulated upper records are as follows:

```
2.7127  3.1841  3.4990  4.0002  4.4863  4.6226  5.3125  5.9537
5.9724  6.0238  6.0357  6.8307  6.8957  7.4296  8.0229.
```

To find 95% confidence intervals for μ and σ, and joint confidence regions for (μ, σ), we need the percentiles:

$$\begin{aligned} F_{0.025(2,28)} &= 4.22052, & F_{0.975(2,28)} &= 0.02534, \\ F_{0.0127(2,28)} &= 5.12792, & F_{0.9873(2,28)} &= 0.01275, \\ \chi^2_{0.0127(2)} &= 8.73857, & \chi^2_{0.9873(2)} &= 0.02548, \\ \chi^2_{0.025(28)} &= 44.46079, & \chi^2_{0.975(28)} &= 15.30786. \\ \chi^2_{0.0127(28)} &= 47.32321, & \chi^{28}_{0.9873(28)} &= 13.9786 \end{aligned}$$

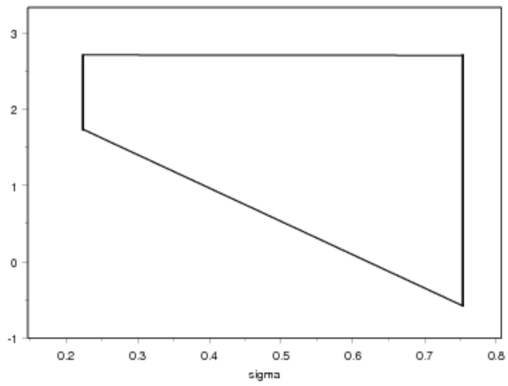

Figure 1: *Joint confidence region for* (μ, σ) *by Method 1.*

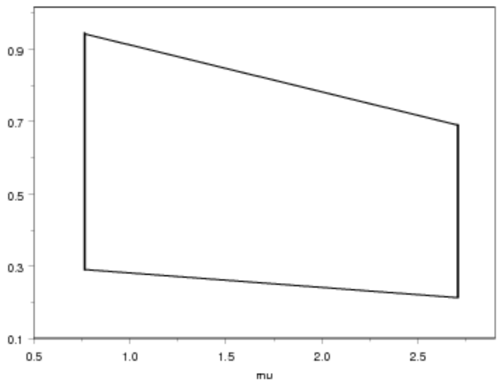

Figure 2: *Joint confidence region for* (μ, σ) *by Method 2.*

and

$$\chi^2_{0.0127(30)} = 49.9138, \qquad \chi^2_{0.9873(30)} = 15.39033.$$

By Theorems 1 and 2, the 95% confidence intervals for μ and σ are obtained as (1.11185, 2.70309) and (0.23887, 0.69379) with confidence lengths 1.59124 and 0.45492, respectively.

By Theorem 3 (Method 1), the 95% joint confidence region for (μ, σ) is given by

$$\begin{cases} 2.7127 - \frac{\sigma}{2}8.73857 < \mu < 2.7127 - \frac{\sigma}{2}0.02548, \\ 0.22442 < \sigma < 0.75976, \end{cases}$$

with area 1.14763. Applying Theorem 4 (Method 2), the confidence region is given by

$$\begin{cases} 0.76768 < \mu < 2.70786, \\ \frac{2(8.0229 - \mu)}{49.9138} < \sigma < \frac{2(8.0229 - \mu)}{15.3903}, \end{cases}$$

with area 1.09604. Figures 1 and 2, show the 95% joint confidence regions for parameters μ and σ. By Theorem 5, the 95% prediction interval for X_{16} is obtained as (8.03251, 9.62374).

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