Dominance, Potential Optimality, and Strict Preference Information in Multiple Criteria Decision Making[*]

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ABSTRACT

The ordinary multiple criteria decision making (MCDM) approach requires two types of input, alternative values and criterion weights, and employs two schemes of alternative prioritization, dominance and potential optimality. This paper allows for incomplete information on both types of input and gives rise to the dominance relationships and potential optimality of alternatives. Unlike the earlier studies, we emphasize that incomplete information frequently takes the form of strict inequalities, such as strict orders and strict bounds, rather than weak inequalities. Then the issues of rising importance include: (1) The standard mathematical programming approach to prioritize alternatives cannot be used directly, because the feasible region for the permissible decision parameters becomes an open set. (2) We show that the earlier methods replacing the strict inequalities with weak ones, by employing a small positive number or zeroes, which closes the feasible set, may cause a serious problem and yield unacceptable prioritization results. Therefore, we address these important issues and develop a useful and simple method, without selecting any small value for the strict preference information. Given strict information on both types of decision parameters, we first construct a nonlinear program, transform it into a linear programming equivalent, and finally solve it via a two-stage method. An application is also demonstrated herein.

Keywords: Multi-Criteria Decision Making, Incomplete Information, Strict Preference Information, Dominance, Potential Optimality

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1. Introduction

The fundamental intent of discrete multi-criteria decision making (MCDM) is to prioritize a finite set of decision alternatives whose consequences are characterized by multiple attributes or criteria, in order to select an optimal or acceptable alternative. Multi-attribute value theory (MAVT) forms a significant body of theory, and represents the most widely used framework for MCDM [17, 32]. Typically, this theory requires two types of input, alternative values (or marginal values) and criterion weights, for the evaluation of alternatives. Although a sizeable number of formally elegant procedures have been developed to assess these values precisely, the implementations of those procedures are often difficult and time-consuming in practice, for many reasons like those we describe below.

Frequently, the decision-maker is only able to partially rank order the alternatives on a criterion according to their performance, rather than to assign them precise marginal values. She may only be able to rank the criteria according to their importance, rather than assigning them numerical weights. She might instead categorize the performance as "good," "moderate," or "poor" and the importance as "very important," "important," or "less important." Including such information, incomplete information is also referred to as imprecise or partial information. Its basic assumption is that the decision-maker may be unwilling or unable to provide exact estimations of decision parameters such as marginal values and weights. This assumption is realistic in situations involving time pressure, lack of knowledge, intrinsically intangible or non-monetary nature of criteria, and/or in which the decision-maker has limited attention and information processing capability [33]. In addition, Barron and Barrett [5] previously stated that various methods for eliciting exact weights from the decision-maker may suffer on several counts, because the weights depend heavily on the elicitation method and there is no agreement as to which method yields more accurate results since the "true" weights remain unknown.

After initial investigations by Fishburn [14], many approaches have been developed that obviate the need for precise information in the MAVT framework. For example, Kmietowicz and Pearman [20] demonstrated a linear programming (LP) approach to establishing dominance relations between alternatives in a situation involving imprecise weights and exact marginal values. Sage and White [29] allowed for imprecise knowledge of both marginal values and weights in determining pair wise

dominance. Lee *et al*. [22] proposed another LP approach that makes possible the identification of potential optimality as well as dominance, when both the marginal values and weights are imprecise. Note that dominance refers to a preference relation between the *actual* alternatives under consideration, whereas potential optimality represents a convex dominance, which includes dominance between *virtual* alternatives that are formed by convex combinations of two or more actual alternatives. Park [26] provided a review in which the characteristics of the LP methods used to establish various preference relations, including dominance and potential optimality, were elucidated.

Besides, the LP approach has been applied in several situations involving incomplete information. The following are a few examples of these. Anandalingam and Olsson [2] demonstrated a real-world application in which a fresh water supply project was selected, in which ordinal weight information was used in their LPs to obtain non-dominated projects. Moskowitz *et al*. [25] emphasized the need to use incomplete information on both weights and marginal values when evaluating public policies, and provided the case problem of selecting a nuclear repository. Kirkwood and Corner [18] demonstrated the use of ordinal weights in the prioritization of several oil tanker standards, and Athanassopoulos and Podinovski [4] employed bounded weights in a computer choice problem. More recently, Mateos *et al*. [24] presented an application involving the disposal of surplus weapons-grade plutonium in which bounded information regarding both weights and marginal values were employed, and they [16] also developed a software package for use in MCDM with incomplete information.

However, no comprehensive treatments of strict preference information can be found in the relevant MCDM literature. Why is this treatment important? First, as previously mentioned, ordinal data frequently arise in practical decision making; these data can be viewed as *strict* rather than weak inequality relations. Second, attributing even bounded data to strict relations is more appealing in that the decision parameters need not be anchored at the extreme upper and lower bounds that might generally evidence little desirability. Third, replacing strict relations with weak ones, while tractable especially for the LP approach, may cause a serious problem and may yield unacceptable results for reasons such as those below.

Suppose that the first criterion was more important than the second one (i.e., $w_1 > w_2 > 0$). Consider two alternatives' value vectors (1, 0.5) and (0.5, 1) on these two crite-

ria in order. Their total scores then each become $w_1+0.5w_2$ and $0.5w_1+w_2$. The difference is $0.5(w_1-w_2) > 0$, and hence the first alternative may be preferable to the second one, which is consistent with the result of Lexicographical ordering. What if we now replace the strict inequality with a weak one on the weights ($w_1 \geq w_2 \geq 0$)? Obviously, equal weights ($w_1 = w_2$) are possible, so it is feasible that no preference relation occurs between the two alternatives. In fact, no dominance relation will result when we use the aforementioned LP approach for this weak inequality, because this approach evaluates every alternative in light of the best weight scenario [26]. In addition to the lower discrimination power, another possible problem is associated with the zero weight. $w_2 = 0$ may possibly hold even though the sum-to-unity condition ($w_1+w_2 = 1$) applies. This implies the serious problem of disregarding the second criterion from the evaluation. The zero weights often occur in view of the trait of the LP approach. We will also show similar problems occurring under the condition of incomplete marginal values.

Why is the treatment of strict information difficult? First, the standard algorithms for solving LP problems cannot be used directly, because the feasible region for the permissible decision parameters becomes an open set. Second, consider a method of employing a small positive number, say epsilon, which represents the gap between both sides of the inequality (e.g., $w_1-w_2 \geq \varepsilon > 0$). This closes the feasible set, but a problem arises as to the proper method of selecting an appropriate number for epsilon. There is no hard and fast rule for choosing the factual small value and, even if the choice is possible, the resultant prioritizations depend heavily on that choice. Third, one might rely on a sensitivity analysis or a simulation study to change the epsilon value in various ways. This seemingly practical approach requires a great deal of computational time and effort.

Nevertheless, the majority of the previous studies have treated incomplete information as weak inequality relations. Some of them have elected to replace strict inequalities with weak ones and then to employ a specific epsilon value like 10^{-4} during prioritization. It should be noted, however, that a few exceptions exist. Kirkwood and Sarin [19] dealt with strict partial and full orders on criterion weights and developed a special algorithm to establish dominance. Their method did not employ an epsilon, but it does not work for other forms of incomplete information such as strict bounds and the strength of preferences as proposed by Malakooti [23]. Hazen [15] employed strict orders and strict bounds on weights, and not only made use of a LP

but also developed an efficient algorithm for identifying potential optimality as well as dominance. Without using an epsilon, his LP approach makes it possible to address any strict inequality types of incomplete weight information. However, his approach can delineate only the dichotomous results of dominance and potential optimality, and cannot show the decision-maker to what extent one alternative is inferior to another. Moreover, the two studies by Kirkwood and Sarin and Hazen dealt only with incomplete weights, while assuming constant marginal values. By way of contrast with these studies, Cook and Kress [9] and their associates [8] handled strict ordinal data regarding both marginal values and weights, and proposed a LP method for prioritizing alternatives. They introduced binary indicators that were to be specified according to the rank positions of alternatives on every criterion. This implies that their approach is unusable in situations in which the rank position is not explicitly known for any one alternative. Many types of incomplete information might be consistent with this case, including partial orders, strength of preferences, and bounded data in which overlapping occurs.

Therefore, we develop a different LP approach to dominance and potential optimality, when both marginal values and weights are imprecise. Our approach can handle arbitrary strict inequality forms of incomplete information. This provides the decision-maker with enhanced freedom of choice and rich collection of preference. We also demonstrate a novel application to a foreign market entry decision, a vital practical subject in international marketing [11, 12], in which a set of countries are evaluated with regard to market potential to make the decision as to which market to enter. We show that our methodology can be utilized effectively in this application.

This paper is organized as follows. First, a brief review of incomplete information is shown. We then present our LP approach for the case in which both marginal values and weights are incomplete. This is followed by an application to a market entry decision problem. Finally, we close this paper with a summary and a brief sketch of some further research opportunities and directions.

2. Incomplete Information

Let there be $i = 1, \cdots, m$ available alternatives, $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$. Every alternative is

to be evaluated on $j = 1, \cdots, n$ criteria. Let $\mathbf{v}_i = (v_{i1}, \cdots, v_{in})$ be the vector of marginal values of alternative $\mathbf{x}_i = (x_{i1}, \cdots, x_{in})$. If the value function of criterion j is $\tau_j : x_{ij} \to [0, 1]$ for all i, then $\tau_j(x_{ij}) = v_{ij}$ in $\mathbf{v}_i$ holds. The vector of criterion weights is denoted by $\mathbf{w} = (w_1, \cdots, w_n)^{\mathsf{T}}$. Based on MAVT, $\mathbf{v}_i\mathbf{w}$ can then represent the overall value of $\mathbf{x}_i$. The sum-to-unity condition $\mathbf{1w} = 1$ and $\mathbf{w} > \mathbf{0} \in \Re^n$ apply, where $\mathbf{1} = (1, \cdots, 1)$ is the sum vector in $\Re^n$. See Keeney and Raiffa [17] and von Winterfeldt and Edwards [32] for the additive value function in detail.

The decision-maker then seeks to select an alternative $\mathbf{x}^* \in X$, the optimal or most preferred one such that $\mathbf{v}^*\mathbf{w} \geq \mathbf{v}_i\mathbf{w}$ for all i. This choice-making task will be straight-forward if the weight and marginal value vectors are all identified precisely. How-ever, as noted in the introduction, many authors have pointed out how difficult it is to elucidate those vectors precisely.

We first show concrete examples of strict *ordinal* data that are of use in both the-ory and practice, and how these data can be made applicable to the MAVT frame-work. The foundation of measurement can be traced back to Krantz *et al*. [21], who asserted that questions based on binary and quaternary relationships could be used to establish a value function such as τ_j. Recall criterion j, on which several alternatives are evaluated. The binary relationship implies that x_{1j} is preferable to x_{2j} if and only if $\tau_j(x_{1j}) > \tau_j(x_{2j})$. For the quaternary relationship, the difference in preference between x_{1j} and x_{2j} exceeds that between x_{3j} and x_{4j} if and only if $\tau_j(x_{1j}) - \tau_j(x_{2j}) > \tau_j(x_{3j}) - \tau_j(x_{4j})$. We do not know the functional form of τ_j, and yet we have incomplete information on mar-ginal values, such that $v_{1j} > v_{2j}$ and $v_{1j} - v_{2j} > v_{3j} - v_{4j}$. We can represent these relations al-ternatively by:

$$v_{1j} - v_{2j} \geq \varepsilon \tag{1}$$

$$v_{1j} - v_{2j} - v_{3j} + v_{4j} \geq \varepsilon \tag{2}$$

where in we employ the non-Archimedean element $\varepsilon > 0$, which is an *unknown* con-stant. Type (1) is referred to as *strict* order and, if $\varepsilon = 0$, becomes a *weak* order. Type (2) is referred to as the strength of preference or strict difference order. We refer to both as strict ordinal data.

Basic to the techniques of eliciting both types are the concepts of order and rela-tion. Common to those techniques is the concept of *paired comparisons* within a set of alternatives. These concepts and techniques function as a foundation of measurement

and are utilized rather widely in the MCDM literature, for example, to describe a measurable value function [13] and to elicit a precise value function [17] or imprecise data [27, 29]. Besides the theoretical touchstone, both types have an appealing practical usefulness. We can regard type (1) as a one-level strength of preference and type (2) as a two-level strength of preference. Namely, Type (1) results from the fact that $\mathbf{x}_1$ is *just* preferred to $\mathbf{x}_2$ on criterion j. Considering that, for the same criterion, $\mathbf{x}_1$ is *strongly* preferred to $\mathbf{x}_2$ and $\mathbf{x}_3$ *weakly* to $\mathbf{x}_4$, it yields type (2). The quaternary relationship reduces to binary and, hence, the decision-maker does not need to respond to the direct preference difference questions complicated in the quaternary relationship.

Malakooti [23] investigated this, demonstrating that the two levels of strength of preference could be extended to more levels, such as a five-level: For criterion j,

A(i, k): $\mathbf{x}_i$ is *very strongly* preferred to $\mathbf{x}_k$.

B(i, k): $\mathbf{x}_i$ is *strongly* preferred to $\mathbf{x}_k$.

C(i, k): $\mathbf{x}_i$ is *moderately* preferred to $\mathbf{x}_k$.

D(i, k): $\mathbf{x}_i$ is *weakly* preferred to $\mathbf{x}_k$.

E(i, k): $\mathbf{x}_i$ is *very weakly* preferred to $\mathbf{x}_k$.

Using this five-level system, we can construct many relations of type (2) as well as (1). For example, given A(1, 2) and B(3, 4), relation (2) holds in addition to (1) and $v_{3j} > v_{4j}$. Additionally, a different number of levels, such as three or four, may be selected as deemed appropriate. This measurement system can be readily understood by management and can be widely utilized to elicit imprecise data. It is also worth noting that the use of type (2) considerably reduces the feasible region of unknown variables [27].

The information in (1) and (2) form a system of linear inequalities such that

$$\mathbf{B}_j\mathbf{v}^j \geq \varepsilon\mathbf{1} \tag{3}$$

where $\mathbf{v}^j = (v_{1j}, \cdots, v_{mj})^\mathsf{T}$, the vector of marginal value variables of all alternatives on criterion j, and $\mathbf{1} = (1, \cdots, 1)$ in appropriate sizes, depending on the number of prescribed inequalities or constraints for $j = 1, \cdots, n$. The incomplete marginal value information in (3) will be used later for prioritizing alternatives.

Under other circumstances, bounded data may occur because the outcome of an alternative x_{ij} or a marginal value function τ_j can be imprecise. For example, as noted

previously by Salo and Hamalainen [30], the decision-maker may only specify a range of $l < x_{ij} < u$ for the possible outcomes of $\mathbf{x}_i$ on criterion j. The bounded marginal value $\tau_j(l) < v_{ij} < \tau_j(u)$ can then be derived from the known increasing value function τ_j. Conversely, if x_{ij} is estimated as an exact number, but the marginal value function is imprecisely known such that there is a range of possible values for the estimate, the lower and upper bounds on the marginal value will be the result. Additionally, other bounded data remain possible. The decision-maker may say that, on criterion j, $\mathbf{x}_1$ is more than 80% and less than 90% relative to the referent $\mathbf{x}_2$. This yields a ratio bound $0.8 < v_{1j}/v_{2j} < 0.9$. These data also constitute a linear system as is seen in (3).

Along with incomplete marginal values, we can have incomplete weights in various ways, including swing methods [32] and holistic judgments. For instance, we can derive the relationship $w_1 > w_2$ from the assessment that swing x_1 to its best level is preferred to swing x_2 to its best level, $(x_1^*, x_2^o) > (x_1^o, x_2^*)$, where $x_j^* = 1$ and $x_j^o = 0$, the best and worst levels on criterion j. If a holistic judgment is possible for a pair of alternatives, then we may have $(\mathbf{v}_k - \mathbf{v}_i)\mathbf{w} > 0$, where $\mathbf{v}_k$ and $\mathbf{v}_i$ are the exact value vectors of hypothetical alternatives. Without any loss of generality, these incomplete weights can thus be represented by the system of linear inequalities,

$$\mathbf{A}\mathbf{w} \geq \varepsilon \mathbf{1} \tag{4}$$

This also includes the positivity conditions $\mathbf{w} > \mathbf{0}$, which prevents any one of the criteria from being ignored in the prioritization of alternatives.

3. Methodology

3.1 Potential Optimality

Given the incomplete marginal values in (3) and the incomplete weights in (4), we can then consider the following non-LP model for the potential optimality of the $\mathbf{x}_k$ alternative:

$$\min z \text{ subject to}$$
$$(\mathbf{v}_k - \mathbf{v}_i)\mathbf{w} + z \geq 0, \qquad i = 1, \cdots, m$$

$$\mathbf{B}_j \mathbf{v}^j \geq \varepsilon \mathbf{1}, \, j = 1, \cdots, n$$
$$-\mathbf{I}\mathbf{v}^j \geq -\mathbf{1}, \, j = 1, \cdots, n \tag{5}$$
$$\mathbf{A}\mathbf{w} \geq \varepsilon \mathbf{1}; \, \mathbf{1}\mathbf{w} = 1$$
$$\mathbf{v}^j \geq \mathbf{0}, \, j = 1, \cdots, n$$

While using (3) and (4), the constraints $\mathbf{0} \leq \mathbf{v}^j \leq \mathbf{1}$ for all j are added such that all of the marginal values fall within a range from zero to one. From the first m constraints, we know that $z \geq 0$ when $i = k$. Thus the $\mathbf{x}_k$ alternative becomes potentially optimal if and only if $z^* = 0$, meaning that this alternative is superior to all the other alternatives simultaneously, while satisfying all the given marginal value and weight constraints.

Using the transformation technique proposed by Park [26],

$$\mathbf{y}_j = (y_{1j}, \cdots, y_{mj})^\mathrm{T} = w_j \mathbf{v}^j \text{ for every } j \tag{6}$$

we can reduce the non-LP model in (5) to the following LP equivalent:

$$
\begin{aligned}
&\min z \text{ subject to}\\
&\sum_{j=1}^{n}\left(y_{kj} - y_{ij}\right) \quad +z \quad\quad \geq \quad 0, \quad \forall i\\
&\mathbf{B}_j \mathbf{y}_j \quad\quad\quad\quad -w_j \varepsilon \mathbf{1} \geq \quad \mathbf{0}, \quad \forall j\\
&-\mathbf{I}\mathbf{y}_j \quad\quad\quad\quad +w_j \mathbf{1} \quad \geq \quad \mathbf{0}, \quad \forall j \\
&\quad\quad\quad\quad\quad\quad \mathbf{A}\mathbf{w} \quad \geq \quad \varepsilon \mathbf{1}\\
&\quad\quad\quad\quad\quad\quad \mathbf{1}\mathbf{w} \quad = \quad 1\\
&\quad\quad\quad\quad\quad\quad \mathbf{y}_j \quad \geq \quad \mathbf{0}, \quad \forall j
\end{aligned}
\tag{7}
$$

The problem is how to handle the unknown epsilon in model (7). The use of zeros or any other small positive number in place of ε may be unacceptable. To address this problem, we now modify model (7) to

$$
\begin{aligned}
&\min z \text{ subject to}\\
&\sum_{j=1}^{n}\left(\mathbf{E}_k - \mathbf{I}\right)\mathbf{y}_j \quad +z\mathbf{1} \quad\quad \geq \quad 0\\
&\mathbf{B}_j \mathbf{y}_j \quad\quad\quad\quad\quad\quad \geq \quad \varepsilon \mathbf{1}, \quad \forall j\\
&-\mathbf{I}\mathbf{y}_j \quad\quad\quad +w_j \mathbf{1} \quad \geq \quad \mathbf{0}, \quad \forall j\\
&\quad\quad\quad\quad \sum_{j=1}^{n}\mathbf{a}_j w_j \quad \geq \quad \varepsilon \mathbf{1}\\
&\quad\quad\quad\quad \sum_{j=1}^{n} w_j \quad\quad = \quad 1\\
&\quad\quad\quad\quad\quad\quad \mathbf{y}_j \quad\quad \geq \quad \mathbf{0}, \quad \forall j
\end{aligned}
\tag{8}
$$

Here, $\mathbf{E}_k$ is the $m{\times}m$ matrix having all ones only in the kth column and all zeros in the other columns, and $\mathbf{I}$ is the $m{\times}m$ identity matrix. The term $w_j\varepsilon > 0$ in (7) becomes another non-Archimedean element like ε_j, the unknown positive value that is smaller than any positive real number, and thus we can simply replace this by the same ε for every j. The $\mathbf{a}_j$ is the jth column vector of $\mathbf{A}$. However, the problem of epsilon still remains before the LP in (8) can be solved.

Dual to LP (8) becomes

$$\max h + \varepsilon\left(\mathbf{1g} + \sum\nolimits_{j=1}^{n}\mathbf{1p}_j\right) \text{ subject to}$$
$$\mathbf{It} - \mathbf{B}_j^{\mathrm{T}}\mathbf{p}_j + \mathbf{Iq}_j \geq \mathbf{e}_k, \quad \forall j$$
$$\mathbf{a}_j^{\mathrm{T}}\mathbf{g} + \mathbf{1q}_j + h = 0, \quad \forall j \qquad\qquad (9)$$
$$\mathbf{1t} = 1$$

where all variable values are non-negative except for h. The variable vectors are such that $\mathbf{q}_j = (q_{1j}, \cdots, q_{mj})$, $\mathbf{t} = (t_1, \cdots, t_m)$, and the sizes of $\mathbf{p}_j$ and $\mathbf{g}$, respectively, are determined according to those of $\mathbf{B}_j$ and $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_n)$. The $\mathbf{e}_k \in \Re^m$ is the unit vector with one in the kth element. The specification of this model is thus relatively straightforward once the data matrices $\mathbf{B}_j$ and $\mathbf{A}$ have been identified.

Because of the dual theorem of LP, we can instead use model (9) to check the potential optimality of $\mathbf{x}_k$. This alternative is potentially optimal if and only if $h^* = 0$, $\mathbf{g}^* = \mathbf{0}$, and $\mathbf{p}_j^* = \mathbf{0}$ for all j. We can now utilize the two-stage method of Arnold $et\ al$. [3] to solve problem (9), involving non-Archimedean epsilon only in the objective function: First, maximize h subject to the same constraints in (9), thus generating h^*. Second, maximize $(\mathbf{1g}+\Sigma_j\mathbf{1p}_j)$ subject to the same constraints but using h^* in place of h, there by deriving $\mathbf{g}^*$ and $\mathbf{p}_j^*$ for all j. Therefore, without specifying the epsilon value, we can establish the potential optimality of all alternatives. The key to this two-stage method is that the preemptive priority is given to maximizing the Archimedean h value. This is followed by maximizing the non-Archimedean $\varepsilon(\mathbf{1g}+\Sigma_j\mathbf{1p}_j)$ item, where this epsilon is regarded as a positive value smaller than any positive real number.

Indeed, the two-stage method has been customarily employed in data envelopment analysis (DEA). As has been well-known, DEA is a LP approach to evaluations of organizational efficiency (see Cooper $et\ al$. [10]). Many authors have explored the similarity between DEA and MCDM [4, 6, 7, 31]. However, no one directly employs the two-stage method for the MCDM with strict preference information. As a matter

of fact, the earlier LP approach as in (7) does not pursue further its dual model as in (9) and the two-stage method; thus, it suffers from epsilon. Furthermore, we will show later that the same two-stage method can also be employed to identify dominance even when both the marginal values and weights are imprecise.

En route to solving problem (9) using the two-stage method, the objective function and all the constraints remain the same, but only change takes place in $\mathbf{e}_k$ according to the $\mathbf{x}_k$ alternative to be evaluated: For instance, $\mathbf{e}_1 = (1, 0, \cdots, 0)$ when alternative 1 is evaluated and $\mathbf{e}_2 = (0, 1, 0, \cdots, 0)$ when alternative 2 is evaluated. Besides the classification of potential optimality, model (9) provides the referent alternatives (Theorem 1), but does not supply a clear path for improvement because the marginal values are all assumed to be strict ordinal data.

Theorem 1: Assume that $\mathbf{x}_k$ is not potentially optimal from model (9). This alternative is then dominated by the alternative $\sum \mathbf{v}_i^* t_i^*$, representing the convex combination of the $\mathbf{v}_i^*$, where the referent alternatives, i, are such that $t_i^* > 0$.

Proof: We multiply both sides of the first n constraints in (9) by a feasible vector $\mathbf{y}_j \in \{\boldsymbol{\pi}_j \in \mathfrak{R}^m: \mathbf{B}_j \boldsymbol{\pi}_j \geq \varepsilon\mathbf{1}; \ \mathbf{0} \leq \boldsymbol{\pi}_j \leq w_j\mathbf{1}\}$ to yield $(\mathbf{I}\mathbf{y}_j)^\mathsf{T}\mathbf{t} - (\mathbf{B}_j\mathbf{y}_j)^\mathsf{T}\mathbf{p}_j + (\mathbf{I}\mathbf{y}_j)^\mathsf{T}\mathbf{q}_j = \mathbf{e}_k^\mathsf{T}\mathbf{y}_j$. Applying the complimentary slackness theorem, we have $(\mathbf{I}\mathbf{y}_j)^\mathsf{T}\mathbf{t}^* - \varepsilon\mathbf{1}\mathbf{p}_j^* + w_j^*\mathbf{1}\mathbf{q}_j^* = \mathbf{e}_k^\mathsf{T}\mathbf{y}_j^*$ for every $j = 1, \cdots, n$. The total sum of these n equations yields

$$\sum\nolimits_{j=1}^{n}\left(\sum\nolimits_{i=1}^{m} y_{ij}^* t_i^* - y_{kj}^*\right) = \varepsilon\sum\nolimits_{j=1}^{n}\mathbf{1}\mathbf{p}_j^* - \sum\nolimits_{j=1}^{n} w_j^*\mathbf{1}\mathbf{q}_j^* \tag{10}$$

Similarly, multiplying both sides of the second n constraints in (9) by a feasible weight w_j, we obtain $(\mathbf{a}_j w_j)^\mathsf{T}\mathbf{g} + w_j\mathbf{1}\mathbf{q}_j + w_j h = 0$ for all j. The sum of these n equations results in $(\mathbf{A}\mathbf{w})^\mathsf{T}\mathbf{g} + \sum_j w_j\mathbf{1}\mathbf{q}_j + h = 0$. We then apply the complementary slackness theorem to yield $\varepsilon\mathbf{1}\mathbf{g}^* + h^* = -\sum_j w_j^*\mathbf{1}\mathbf{q}_j^*$. Substituting this into the right of (10) yields

$$\sum\nolimits_{j=1}^{n}\left(\sum\nolimits_{i=1}^{m} y_{ij}^* t_i^* - y_{kj}^*\right) = h^* + \varepsilon\left(\mathbf{1}\mathbf{g}^* + \varepsilon\sum\nolimits_{j=1}^{n}\mathbf{1}\mathbf{p}_j^*\right) \tag{11}$$

Applying the definition in (6) to the left of (11), we obtain

$$\left(\sum\nolimits_{i=1}^{m} \mathbf{v}_i^* t_i^* - \mathbf{v}_k^*\right)\mathbf{w} = h^* + \varepsilon\left(\mathbf{1}\mathbf{g}^* + \varepsilon\sum\nolimits_{j=1}^{n}\mathbf{1}\mathbf{p}_j^*\right) \tag{12}$$

By the assumption that x_k is not potentially optimal, we know that $\sum v_i^* t_i^*$ dominates v_k^*. Since $\mathbf{1t} = 1$ and $t_i \geq 0$ for all i, the virtual value vector $\sum v_i^* t_i^*$ is the convex combination of the vectors v_i^*, wherein i is such that $t_i^* > 0$ are effective in this combination as the referent alternatives for the evaluation of x_k. □

3.2 Dominance Test

Given (3) and (4), the following model can be employed to check dominance between two alternatives k and i:

$$\min z_{ki} = (\mathbf{v}_k - \mathbf{v}_i)\mathbf{w} \text{ subject to}$$
$$\mathbf{B}_j\mathbf{v}^j \geq \varepsilon\mathbf{1};\ \mathbf{0} \leq \mathbf{v}^j \leq \mathbf{1},\ j = 1,\cdots, n \qquad (13)$$
$$\mathbf{Aw} \geq \varepsilon\mathbf{1};\ \mathbf{1w} = 1$$

If $z_{ki}^* \geq 0$, then x_k dominates x_i. Note that this objective value can be determined when $\mathbf{v}_k$ is under the worst scenario for the admissible weights and marginal values, but $\mathbf{v}_i$ under the best scenario.

The transformation technique in (6) can also be used to convert the non-LP model in (13) into a LP equivalent. However, the problem of epsilon still remains and many LP problems (with the epsilon problem) should be solved, up to $m(m\text{-}1)$, to identify adequately the dominance structure of all alternatives.

Thus, we propose herein an alternative method which is both computationally efficient and circumvents the epsilon problem:

$$\max h + \varepsilon\left(\mathbf{1g} + \sum\nolimits_{j=1}^{n} \mathbf{1p}_j\right) \text{ subject to}$$
$$\mathbf{It} - \mathbf{B}_j^{\mathrm{T}}\mathbf{p}_j + \mathbf{Iq}_j \geq \mathbf{e}_k,\ \ \forall j$$
$$\mathbf{a}_j^{\mathrm{T}}\mathbf{g} + \mathbf{1q}_j + h = 0,\ \ \forall j \qquad (14)$$
$$\mathbf{1t} = 1$$
$$t_i = 0 \text{ or } 1,\ \forall i$$

where all variable values except for h are non-negative. Only the bivalent condition for t_i is added to model (9) designed to check potential optimality. This leads to the dominance test between *actual* alternatives (see Theorem 2 and Lemma 1, below). We can still use the two-stage method for problem (14) with no specification of the epsi-

lon value. In order to assess the dominance of alternative $\mathbf{x}_k$, we solve only two zero-one integer problems. The program settings for both dominance and potential optimality are convenient because only the bivalent condition is removed or added. We can also readily capture the set-inclusive relationship between the non-dominated alternative set and the potentially optimal alternative set (Lemma 2).

Theorem 2: Models (13) and (14) yield the same dominance results.
Proof: We know from (12) in Theorem 1 that

$$\left(\sum_{i=1}^{m}\mathbf{v}_i^* t_i^* - \mathbf{v}_k^*\right)\mathbf{w} = h^* + \varepsilon\left(\mathbf{1g}^* + \varepsilon\sum_{j=1}^{n}\mathbf{1p}_j^*\right)$$

in model (9), which can also be applied to model (14). While preserving the original optimal solution, we can thus modify model (14) to

$$\max\left(\sum_{i=1}^{m}\mathbf{v}_i t_i - \mathbf{v}_k\right)\mathbf{w} \quad \text{subject to}$$

$$\mathbf{B}_j\mathbf{v}^j \geq \varepsilon\mathbf{1}; \, 0 \leq \mathbf{v}^j \leq \mathbf{1}, \, j = 1, \cdots, n$$

$$\mathbf{Aw} \geq \varepsilon\mathbf{1}; \, \mathbf{1w} = 1$$

$$\mathbf{1t} = 1; \, t_i = 0 \text{ or } 1, \, \forall i$$

The exhaustive enumeration of possible values for t_i generates the (m-1) programs equivalent to those in (13). □

Lemma 1: In model (14), alternative $\mathbf{x}_k$ is non-dominated if and only if $h^* = 0$, $\mathbf{g}^* = \mathbf{0}$, and $\mathbf{p}_j^* = \mathbf{0}$ for all j. (This is apparent from Theorems 1 and 2). □
Lemma 2: The set of non-dominated alternatives implies the set of potentially optimal alternatives (This is apparent from comparing the sets of constraints for models (9) and (14)). □

Therefore, the advantages of model (14) we proposed to check dominance can be summarized as follows. (a) The epsilon problem is avoided. (b) The use of model (14) is computationally efficient. To evaluate the dominance of alternative $\mathbf{x}_k$ using the two-stage method, we solve only two zero-one integer problems. Using the LP equi-

valent to model (13), we solve (m-1) LP problems. The number of problems we solve is reduced greatly, independently of the number of alternatives. Moreover, efficient algorithms for zero-one integer programs exist without the need for tedious exhaustive enumerations of zero-one variables. (c) The program settings for both dominance and potential optimality are convenient in that only the bivalent condition is removed or added. (d) The set-inclusive relationship between the non-dominated alternative set and the potentially optimal alternative set can readily be captured.

3.3 Illustrative Example

Table 1 provides an example involving three alternatives evaluated with regard to three criteria. The marginal values in the first column are symbolic and are employed only to represent the strict ordinal relations, such that $\{v_{11} > v_{21} > v_{31}\}$. Therefore, we obtain

$$\mathbf{B}_1^{\mathrm{T}} \mathbf{p}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}$$

The imprecise values in the final two columns are assessed using the previously described five-level system. The relations for the second criterion are $\{v_{22} > v_{12} > v_{32}; v_{22} - v_{12} > v_{12} - v_{32}\}$. Those for the final criterion are $\{v_{13} > v_{23} > v_{33}; v_{13}\text{-}v_{33} > v_{13}\text{-}v_{23} > v_{23}\text{-}v_{33}\}$. We thus obtain:

$$\mathbf{B}_2^{\mathrm{T}} \mathbf{p}_2 = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} p_{12} \\ p_{22} \\ p_{32} \end{pmatrix}$$

$$\mathbf{B}_3^{\mathrm{T}} \mathbf{p}_3 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} p_{13} \\ p_{23} \\ p_{33} \end{pmatrix}$$

Assume the strict ordinal weights, $w_1 > w_2 > w_3 > 0$.

We can then readily specify model (9) for potential optimality and model (14) for dominance. Using the two-stage method, we obtain Table 2. Alternatives 1 and 2 be-

come potentially optimal. Alternative 3 is compared to alternative 1 and its inferiority measures are determined. We obtain the same result for dominance.

Finally, it should be noted that if we instead use weak ordinal relations for the incomplete marginal values in Table 1, alternative 3 turns out to be potentially optimal. These weak ordinal data wipe out all of the p_{ij} variables in the objective and, hence, play no role as inferiority measures in this prioritization. We then find that h^* and $\mathbf{g}^*$ are all zeros for alternative 3. However, looking at the information in Table 1, we can see that alternative 3 is dominated by both alternatives 1 and 2. Consequently, treating ordinal data as only weak orderings generates an unacceptable result. In addition, the use of a small positive number in place of epsilon will always be controversial and the prioritization results will depend heavily on that choice.

Table 1. Example for the Incomplete Marginal Value Case

Alternatives	Criterion 1	Criterion 2	Criterion 3
1	1	A(2, 1)	A(1, 3)
2	2		B(1, 2)
3	3	B(1, 3)	C(2, 3)

Table 2. Potential Optimality Results for the Data in Table 1

Alternative	Inferiority measures*								Reference
i	p_{11}^*	p_{21}^*	p_{12}^*	p_{22}^*	p_{32}^*	p_{13}^*	p_{23}^*	p_{33}^*	$t_i^* > 0$
1	0	0	0	0	0	0	0	0	$t_1^* = 1$
2	0	0	0	0	0	0	0	0	$t_2^* = 1$
3	1	1	0	1	1	0	2	1	$t_1^* = 1$

Note: * All the h^* and $\mathbf{g}^*$ values were zeros for all three alternatives.

4. Application

Samchuly Bicycle is the largest manufacturer of bicycles in Korea. While the company has aggressively exported bicycles overseas over the past several decades, its exports were targeted at developed countries in North America and Western Europe. Recently, owing to an economic slowdown in its primary markets, Samchuly elected to expand its sales by marketing its products in a new market. The company's top management is considering setting up a production facility in Asia, owing to the

region's untapped market potential for bicycle products. With special consideration given to geographic proximity and economic growth, the company has developed a long-term plan to penetrate the Southeast Asian market.

The Asian region encompasses a number of countries which tend to differ widely along demographic, economic, cultural and political dimensions. Considering the diversity of national markets in Southeast Asia, therefore, it was deemed important to narrow the list of countries down into a manageable set for further consideration. In order to determine which markets to enter, the company's international business executives adopted a two-stage process for evaluating and comparing market opportunities in Asian countries.

The preliminary assessment proceeded with special consideration provided to national economic environments among the Asian countries. In this initial stage, the company employed a set of indicators for initial screening, including population, per capita income, and economic growth rate. First, population represents the overall market size (tapped and untapped demand) for bicycles. Second, the level of personal income is another important issue for bicycle products, which may reflect the purchasing power of local consumers in each country. Consumers with very low levels of personal income may not have sufficient purchasing power to buy Samchully bicycles. Finally, the rate of economic growth is reflective of the long-term attractiveness of each country. For instance, a country's rapidly growing economy can be considered a general signal for market attractiveness.

After initial scanning for alternative locations, the international business executive has shortlisted five Asian countries. In the second stage, therefore, the company evaluates these countries in greater depth in terms of market attractiveness. In fact, as we describe below, our methodology can best be employed in this stage.

As far as international markets are concerned, a company can seldom acquire all the information its managers would like. The lack, obsolescence, and inaccuracy of data regarding many countries render research in this area difficult [12]. Given such issues with inaccurate and unavailable data regarding foreign countries, and developing markets in particular, a company will frequently have to rank order alternatives, draw rough estimates, or even make projections concerning some of the key criteria for the decisions to be made for international markets [11]. In international market research, therefore, it is common practice to employ qualitative factors, given the difficulty of their data collection and the unavailability of accurate information

with regard to foreign countries. Nevertheless, almost all approaches to market entry decision-making encountered in the area of international business have assumed exact data regarding every criterion [11, 12], which reveals a sizeable gap between research and practice.

In the second stage, the five countries selected in the first stage were compared using four criteria representing market attractiveness-*competitive risk*, *political stability*, *cultural similarity*, and the *need for product adaptation*. We do not reveal the countries' names primarily because they might be sensitive to the assessments of some of these criteria. We just denote the five countries as i = A, B, C, D, E.

As shown in Table 3, we evaluated the five countries in terms of the four criteria. It will be virtually impossible to assign exact numerical judgments to these criteria, although they are to be considered very important in such a market entry decision [12]. (a) *Competitive risk* represents the nature of competition among bicycle companies in a given country. For instance, a market can be highly risky when the level of competition is significant and also when strong players already dominate the local market. The executive has evaluated competitive risk for the five countries and then ordered them by rank according to the level of risk (see the first row of Table 3, and its footnote describing the ordinal data mathematically). (b) The assessment of *political stability* involves a myriad of country-level issues such as the nature of the political system, government type, racial composition, military situation, and religious diversity. The executive finds it more reasonable to classify the political stability into three categories (see the second row of Table 3). (c) *Cultural similarity* reflects the extent to which the national culture of a foreign country (i.e., host country) resembles that of the home country (i.e., Korea). The concept of national culture is sufficiently broad that even anthropologists may find it difficult to assign numerical values to the degrees of overall similarity. The executive thus categorizes the five countries into three groups (see the third row). Finally, (d) the *need for product adaptation* is attributed to national market conditions such as product usage conditions (e.g., leisure or transportation), consumer preferences (e.g., product design), and physical environments (e.g., road conditions). If the extent of adaptation is significant, then the company will need to incur additional costs for modifications of product design and manufacturing processes. The executive classifies the overall need for product adaptation into three categories (see the fourth row). In addition to the marginal value assessments made for the four criteria, the executive also provides weight information, but could only

rank-order the relative importance of these criteria, as shown in the final footnote of Table 3.

Table 3. Information and Evaluation Results for the Market Entry Decision

Countries / Criteria	A	B	C	D	E
1. Competitive risk	2	1	3	5	4
2. Political stability	High	High	Moderate	Low	Low
3. Cultural similarity	Moderate	High	Low	High	Moderate
4. Product adaptation	High	High	Moderate	Moderate	Low
Inferiority h^*	0	0	0	0	0
Inferiority $\mathbf{g}^*$, $\mathbf{p}_j^*$	0	0	0	0	Positive
Reference	A	B	C	D	D

Notes: the mathematical expression for.

    1. Competitive risk: $1 = v_{D1} > v_{E1} > v_{C1} > v_{A1} > v_{B1} = 0$.

    2. Political stability: $1 = v_{A2} = v_{B2} > v_{C2} > v_{D2} = v_{E2} = 0$.

    3. Cultural similarity: $1 = v_{B3} = v_{D3} > v_{A3} = v_{E3} > v_{C3} = 0$.

    4. Product adaptation: $1 = v_{A4} = v_{B4} > v_{C4} = v_{D4} > v_{E4} = 0$.

      Weight information: $w_1 > w_2 > w_3 > w_4$.

As summarized in Table 3, we have incomplete information on both marginal values and weights, and hence employ models (9) and (14) to evaluate the potential optimality and dominance of the five countries, respectively. The final three rows of Table 3 summarize the results of potential optimality. The four countries A-D appear to be potentially optimal. Only one country (E) is not potentially optimal, because it evidences positive values for several variables in $\mathbf{g}^*$ and $\mathbf{p}_j^*$. We have the same results for the dominance tests. The same four countries are non-dominated, and E is dominated by D.

Although the selection of only one best country is not shown in this application, the techniques described herein can be well utilized for this issue of market entry decision-making. It is worth mentioning that, in general, one might fail to have only one best alternative when both marginal values and weights are provided only in the form of orders. Perhaps this is because the feasible region becomes too large for one best alternative to be identified. One might consider an interactive procedure, in a way that requests more specific information that can reduce the number of decision candidates until a final decision can be made.

5. Conclusions

We have developed an effective and general approach to establishing the dominance and potential optimality of alternatives involving strict preference information about marginal values and weights. Special emphasis is placed on the fact that treating ordinal data about marginal values and weights only as weak orderings may yield unacceptable results. In fact, our approach is sufficiently general to deal with arbitrary linear forms of preference information. This is a significant improvement on earlier methods, and circumvents many of their associated problems. We have also demonstrated its practical applicability via application to a foreign market entry decision-making, a vital practical subject in international marketing. The possible areas of application are numerous, and include product design evaluation, project selection, location and policy analysis, and managerial performance evaluation, among others.

Revisiting the study of Ahn [1] and Park [26], we can identify a path for future research. As shown in the numerical examples (see Table 1 and Table 2) and the application (Table 3), we might fail to have only one best alternative when both marginal values and weights are provided only in the form of orders. Thus, there arises the need for a stronger evaluation scheme, like the concept of strength of preference measure proposed by Ahn [1] and the notion of strong potential optimality described by Park [26]. This might give rise to a further prioritization between competitive alternatives. Alternatively, one might consider an interactive procedure, in a way that requests more specific information that can reduce the number of decision candidates until a final decision can be made. Extensions of our developments to such a strong evaluation and interactive approach might prove interesting topics for future research.

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