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Integrated Inventory-Distribution Planning in a (1 : N) Supply Chain System with Heterogeneous Vehicles Incorporated*

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ABSTRACT

This paper considers an integrated inventory-distribution system with a fleet of heterogeneous vehicles employed where a single warehouse distributes a single type of products to many spatially distributed retailers to satisfy their dynamic demands. The problem is to determine order planning at the warehouse, and also vehicle schedules and delivery quantities for the retailers with the objective of minimizing the sum of ordering cost at the warehouse, inventory holding cost at both the warehouse and retailers, and transportation cost. For the problem, we give a Mixed Integer Programming formulation and develop a Lagrangean heuristic procedure for computing lower and upper bounds on the optimal solution value. The Lagrangean dual problem of finding the best Lagrangrean lower bound is solved by subgradient optimization. Computational experiments on randomly generated test problems showed that the suggested algorithm gives relatively good solutions in a reasonable amount of computation time.

Keywords: SCM, Inventory, Distribution Planning, Integer Programming, Lagrangean Relaxation, Heuristics

1. Introduction

Nowadays, effective management of the supply chain is recognized as a key factor of competitiveness and success for most manufacturing industries. One part of the

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supply chain that has received increasing attention is coordination of distribution planning and inventory management. The activities of distribution planning and inventory management are often interrelated to each other. For example, shipping smaller-quantity and higher-frequency may lead to reduction in inventory investment but require more distribution cost, while shipping larger-quantity and lower-frequency reduces transportation cost but increases inventory costs. Hence, one may be motivated to develop a systematic and efficient methodology for determining delivery schedule and quantities considering the trade-off between transportation costs and inventory costs.

This paper considers an integrated inventory-distribution system where a fleet of heterogeneous vehicles are employed to distribute a single type of products from a warehouse to spatially distributed retailers to satisfy their demands. In order to satisfy demands from the retailers, the warehouse has to place orders for some amount of the product from a higher echelon, say a supplier (plant or market). The warehouse and the retailers are allowed to hold inventory. The problem is to determine order planning at the warehouse, and also vehicle schedules and delivery quantities for the retailers with the objective of minimizing the sum of ordering cost at the warehouse, inventory holding cost at both the warehouse and retailers, and transportation cost.

There are many researchers who considered trade-off between transportation costs and inventory costs in distribution planning. For an extensive survey about the trade-off between transportation costs and inventory costs, we refer to Bertazzi and Speranza [2]. Bertazzi *et al.* [1] presented a heuristic algorithm for finding an order-up-to level strategy to minimize inventory and transportation costs in a distribution network with one supplier and several retailers. Qu *et al.* [10] developed an algorithm for making a trade-off between transportation costs and inventory costs by iteratively solving a separate inventory problem and a vehicle routing problem. Cetinkaya and Lee [4] presented a model for coordinating inventory management and shipment frequencies in a vendor-managed inventory system. Kim and Kim [8] studied a multi-period inventory and distribution planning problem with one warehouse and multiple retailers. They presented a Lagrangean heuristic for the problem of minimizing the sum of holding costs, distance-dependent and quantity-dependent linear transportation costs. Vroblefski *et al.* [12] considered for serially distributed warehouses a single-product lot-sizing model over an infinite horizon that includes holding costs and unit transportation costs that are stepwise decreasing per interval of the size of the order. Their model assumes constant and continuous demand. They presented algorithms to find efficient power-of-two and integer ratio policies. Bertazzi *et al.* [3] presented a model to make a trade-off between transportation and holding cost by determining a cyclic scheme of cost minimizing transport frequencies for multiple products transported over a single link. Their transportation cost function is linear in the number of trucks needed to transport the products. Van Norden and van de velde [10] analyzed a multi-product lot-sizing problem with a transportation capacity reservation contract. They developed a Lagrangean algorithm for computing lower and upper bounds on the optimal solution value.

In this paper, we present the problem description and formulation in Section 2. Section 3 analyzes some solution properties and proposes an algorithm based on the Lagrangean relaxation. In Section 4, an experimental test is performed. Some concluding remarks and directions for future research are given in Section 5.

2. Problem Description

We consider a $(1:N)$ supply chain system composed of one warehouse and spatially distributed multiple retailers. In order to satisfy demands of retailers, the warehouse has to place orders for some amount of the product from a higher echelon, say a manufacturing plant or a supplier (market). The warehouse employs a fleet of heterogeneous vehicles (different loading capacities) to distribute the product to retailers. Each vehicle is allowed to make round trips between the warehouse and retailers. The time required for a round trip between the warehouse and each retailer is assumed to be less than the length of a period. Each vehicle is allowed to make several round trips to the same retailer or other retailers in a single period if the total travel time does not exceed the length of each period. Per-period demand of each retailer is assumed to be known in advance, and have to be satisfied without backlogging from the warehouse. Both the warehouse and retailers can hold inventory and there is no limit on the storage capacity for inventory.

Transportation cost is composed of distance-based cost and quantity-based cost. The distance-based transportation cost is incurred when a vehicle makes a round trip to a retailer, and is proportional to the associated distance between the warehouse and the retailer. The quantity-based transportation cost can be computed as the delivery quantity multiplied by the unit variable cost. The distance-based cost depends on vehicles used for delivery, but the quantity-based cost is independent of vehicles. The fixed ordering cost at the warehouse is independent of any amount ordered, but it may differ period by period. The inventory holding cost is charged according to inventory quantities at the end of each period. The unit holding cost may also differ at the warehouse and each retailer period by period.

The problem is to determine ordering planning at the warehouse, vehicle schedules, and delivery quantities for retailers. The objective is to minimize the sum of ordering cost at the warehouse, system-wide inventory holding costs and transportation costs. The associated parameters and variables are listed as follows.

Parameters

i : index for vehicles $(1, \cdots, M)$

j : index for retailers $(1, \cdots, N)$ with 0 representing the warehouse

t : index for time periods $(1, \cdots, T)$

c_{jt}^{Q} : quantity-based transportation cost for delivering a unit to retailer j in period t

c_{ijt}^{D} : distance-based transportation cost of vehicle i for a round trip between the warehouse and retailer j in period t

h_{jt} : per-unit inventory holding cost for inventory at the end of period t at retailer j

d_{jt} : demand quantity at retailer j during period t

o_t : fixed ordering cost at the warehouse for the product to be ordered from a higher echelon in period t

V_i : loading capacity of vehicle i

τ_j : transportation time between the warehouse and retailer j

τ : length of each period

Variables

q_t : amount ordered at the warehouse in period t

I_{jt} : inventory level at retailer j in period t

x_{jt} : amount delivered to retailer j in period t

y_{ijt} : number of round trips made by vehicle i to serve retailer j in period t

z_t : 0-1 variable representing whether the warehouse places an order for the product in period t

The integrated inventory-distribution planning problem can be mathematically expressed as in the following mathematical programming:

Problem P :

$$Z_{opt} = Min \ \sum_t o_t z_t + \sum_j \sum_t h_{jt} I_{jt} + \sum_j \sum_t c_{jt}^Q x_{jt} + \sum_i \sum_j \sum_t c_{ijt}^D y_{ijt} \tag{1}$$

subject to

$$I_{0t} = I_{0t-1} + q_t - \sum_j x_{jt} \qquad \forall t \tag{2}$$

$$I_{jt} = I_{jt-1} + x_{jt} - d_{jt} \qquad \forall j,\ t \tag{3}$$

$$q_t \le M_t \cdot z_t \qquad \forall t \tag{4}$$

$$x_{jt} \le \sum_i V_i y_{ijt} \qquad \forall j,\ t \tag{5}$$

$$\sum_j \tau_j \, y_{ijt} \le \tau \qquad \forall i,\ t \tag{6}$$

$$y_{ijt} \in \{0,\ 1,\ 2,\ \cdots\} \qquad \forall i,\ j,\ t \tag{7}$$

$$z_t \in \{0,\ 1\} \qquad \forall t \tag{8}$$

$$x_{jt},\ I_{jt},\ q_t \ge 0 \qquad \forall j,\ t \tag{9}$$

Expression (1) specifies the objective of minimizing the sum of ordering cost at the warehouse, system-wide inventory holding costs and transportation costs. Constraints (2) and (3) represent the inventory balance constraints at the warehouse and retailers, respectively. Constraint (4) enforces binary variable z_t to be positive if the warehouse places an order in period t. Note that M_t is a very large positive number. Constraint (5) ensures that the quantity delivered to a retailer in a period does not exceed the sum of the loading capacities of the vehicles that visit the retailer. Constraint (6) ensures that the total travel time of each vehicle during each period does not exceed the length of each period.

3. Solution Approach

Since Problem P is a mixed integer programming problem, it may require excessive amount of time to find the optimal solution directly using a commercial software for integer programs. In this paper, we use a Lagrangean relaxation approach to develop a heuristic solution procedure for the proposed problem. For the details of the Lagrangean relaxation theory, we refer to Geoffrion [6] and Fisher [5].

3.1 Lagrangean Relaxation

The original problem P is relaxed by dualizing constraint 950 with Lagrangean multipliers $\lambda_{jt} \geq 0$. The resulting relaxed problem is $L(\lambda)$.

Problem $L(\lambda)$:

$$Z_L(\lambda) = Min \quad \sum_t o_t z_t + \sum_j \sum_t h_{jt} I_{jt} + \sum_j \sum_t \left(c_{jt}^Q + \lambda_{jt}\right) x_{jt} + \sum_i \sum_j \sum_t \left(c_{ijt}^D - V_i \lambda_{jt}\right) y_{ijt}$$

subject to

$$\text{(2), (3), (4), (6), (7), (8), (9) and} \quad \lambda_{jt} \geq 0 \quad \forall j,\, t \quad (10)$$

The relaxed problem $L(\lambda)$ can be decomposed into two independent subproblems, $L_1(\lambda)$ and $L_2(\lambda)$.

Subproblem $L_1(\lambda)$:

$$Z_{L_1}(\lambda) = Min \quad \sum_t o_t z_t + \sum_j \sum_t h_{jt} I_{jt} + \sum_j \sum_t \left(c_{jt}^Q + \lambda_{jt}\right) x_{jt}$$

subject to

(2), (3), (4), (8), (9) and (10)

Subproblem $L_2(\lambda)$:

$$Z_{L_2}(\lambda) = Min \quad \sum_i \sum_j \sum_t \left(c_{ijt}^D - V_i \lambda_{jt} \right) y_{ijt}$$

subject to

(6), (7) and (10)

The first subproblem, $L_1(\lambda)$, is the problem of determining ordering quantity at the warehouse, delivery quantities to the retailers, and inventory levels at the warehouse and retailers for each period. The second subproblem, $L_2(\lambda)$, is that of making schedules of vehicles. These subproblems can be solved independently and their solutions can be used to obtain a lower bound on the optimal solution value of the original problem. For given $\lambda_{jt} \geq 0$, a lower bound for the original problem can be obtained as $Z_L(\lambda) = Z_{L_1}(\lambda) + Z_{L_2}(\lambda)$ where $Z_{L_1}(\lambda)$ and $Z_{L_2}(\lambda)$ denote the optimal values of Subproblem $L_1(\lambda)$ and Subproblem $L_2(\lambda)$, respectively, where λ denotes the vector of $\left\{ \lambda_{jt} \geq 0 \right\}$.

3.1.1 Solving Subproblem $L_1(\lambda)$ with λ Given

With λ given, Subproblem $L_1(\lambda)$ is similar to the single facility, uncapacitated, multi-period warehouse ordering problem. Due to binary variable z_t, it may require excessive amount of time to solve Subproblem $L_1(\lambda)$ optimally. Therefore, we relax an integrality constraint (8) for making the problem a simple linear programming problem to solve optimally via the simplex method.

As mentioned earlier, the sum of solution values of the two subproblems can give a lower bound for the original problem. Therefore, to obtain a good lower bound, we derive some valid inequalities for Subproblem $L_1(\lambda)$. Since backlogging is not allowed in the proposed problem, the maximum ordering quantity at the warehouse in period *t* can be up to the sum of the overall net demands of all the retailers over the periods from *t* through *T*. As a result, we have that $M_t = \sum_{k=t}^{T} \sum_j d_{jk}$.

Lemma 1: *The following inequalities are valid for Subproblem $L_1(\lambda)$.*

$$\sum_j I_{jk-1} \geq \sum_{t=k}^{l} \left(\sum_j d_{jt} \right) \left(1 - z_k - \cdots - z_t \right) \quad for\ 1 \leq k \leq l \leq T$$

Proof: Suppose that no ordering is made at the warehouse in period 2. Since backlogging is not allowed, the demands of all the retailers for period 2 must be satisfied by the total inventory in period 1. This implies that $\sum_j I_{j1} \geq \sum_j d_{j2}$ if $z_2 = 0$, which can be converted into a valid inequality as $\sum_j I_{j1} \geq \sum_j d_{j2}(1-z_2)$. Observe that in the case of $z_2 = 1$, the inequality reduces to $\sum_j I_{j1} \geq 0$ which is valid for the initial problem.

Similarly, we expand the above valid inequality by considering the case that no ordering is made at the warehouse in periods 2 and 3. Then, it follows that

$$\sum_j I_{j1} \geq \left(\sum_j d_{j2} + \sum_j d_{j3}\right)(1-z_2-z_3) = \sum_j d_{j2}(1-z_2-z_3) + \sum_j d_{j3}(1-z_2-z_3).$$

Comparing the two valid inequalities, $\sum_j I_{j1} \geq \sum_j d_{j2}(1-z_2)$ and $\sum_j I_{j1} \geq \sum_j d_{j2}(1-z_2-z_3) + \sum_j d_{j3}(1-z_2-z_3)$, it can be observed that $\sum_j d_{j2}(1-z_2)$ in the first inequality is stronger than $\sum_j d_{j2}(1-z_2-z_3)$ in the second inequality, which leads to a new valid inequality

$$\sum_j I_{j1} \geq \sum_j d_{j2}(1-z_2) + \sum_j d_{j3}(1-z_2-z_3).$$

Therefore, the general form of the valid inequality is

$$\sum_j I_{jk-1} \geq \sum_{t=k}^{l} \left(\sum_j d_{jt}\right)(1-z_k-\cdots-z_t) \quad \textit{for } 1 \leq k \leq l \leq T.$$

This completes the proof. □

With the valid inequalities of Lemma 1 and the upper bound M_t, we include $I_{j0} = 0$ for all j and $z_1 = 1$ as constraints because the initial inventory at the warehouse and retailers is assumed to be zero. Then, an LP relaxed problem of Subproblem $L_1(\lambda)$, $LPRL_1(\lambda)$, can be expressed as follows.

Problem $LPRL_1(\lambda)$ (LP relaxed problem of Subproblem $L_1(\lambda)$):

$$Z_{LPRL_1}(\lambda) = \ Min \quad \sum_t o_t z_t + \sum_j \sum_t h_{jt} I_{jt} + \sum_j \sum_t \left(c^Q_{jt} + \lambda_{jt} \right) x_{jt}$$

subject to

$$I_{0t} = I_{0t-1} + q_t - \sum_j x_{jt} \qquad \forall t \qquad (11)$$

$$I_{jt} = I_{jt-1} + x_{jt} - d_{jt} \qquad \forall j,\ t \qquad (12)$$

$$q_t \le \left(\sum_{k=t}^{T} \sum_j d_{jk} \right) \cdot z_t \qquad \forall t \qquad (13)$$

$$\sum_j I_{jk-1} \ge \sum_{t=k}^{l} \left(\sum_j d_{jt} \right) \left(1 - z_k - \cdots - z_t \right) \qquad \textit{for } 1 \le k \le l \le T \qquad (14)$$

$$I_{j0} = 0 \qquad \forall j \qquad (15)$$

$$z_1 = 1 \qquad (16)$$

$$0 \le z_t \le 1 \qquad \forall t$$

$$x_{jt},\ I_{jt},\ q_t \ge 0 \qquad \forall j,\ t$$

Note that Problem $LPRL_1(\lambda)$ is a simple linear programming problem with λ given, so that it can be solved optimally by the simplex method.

3.1.2 Solving Subproblem $L_2(\lambda)$

For any given λ, Subproblem $L_2(\lambda)$ can be further decomposed into several single-period, single-vehicle scheduling problems $L_2^{it}(\lambda)$ for $\forall i = 1, \cdots, M,\ \forall t = 1,\ T,$ as follows.

Subproblem $L_2^{it}(\lambda)$:

$$Z_{L_2^{it}}(\lambda) = \ Min \quad \sum_j \left(c^D_{ijt} - V_i \lambda_{jt} \right) y_{ijt}$$

subject to

$$\sum_j \tau_j y_{ijt} \le \tau \qquad (17)$$

$$y_{ijt} \in \{0,\ 1,\ 2,\ \cdots\} \qquad \forall j \qquad (18)$$

In each Subproblem $L_2^{it}(\lambda)$, a set of retailers to be served by vehicle *i* in period *t* is to be determined. With the coefficients of the objective function multiplied by -1, each Subproblem $L_2^{it}(\lambda)$ is considered as an integer knapsack problem by treating each retailer as a product item, transportation time to a retailer as the size of the associated item, and the length of a period as the capacity of a knapsack. In this paper, the integer knapsack problem is solved by a dynamic programming algorithm, referring to Nemhauser and Wolsey [9]. The algorithm solves the problem optimally in the complexity order $O(NR)$, where *N* is the number of retailers and *R* is the length of a period.

3.2 Subgradient Optimization Scheme for the Lagrangean Multipliers

In order to obtain a good lower bound, it is important to find good Lagragean multiplier values. In the proposed problem, it is desired to determine the greatest lower bound $Z_L(\lambda^*)$ by $Z_L(\lambda^*) = \max_{\lambda\geq 0} Z_L(\lambda) = \max_{\lambda\geq 0}\left\{Z_{L_1}(\lambda)+Z_{L_2}(\lambda)\right\}$.

Referring to Fisher [5], we use the subgradient optimization algorithm which is known as common method to determine the Lagrangean multiplier values. At iteration *k* in the subgradient optimization, subgradient vector θ^k is determined by $\theta_{jt}^k = x_{jt}^* - \sum_i V_i \cdot y_{ijt}^*$, where x_{jt}^* and y_{ijt}^* are the optimal solutions of $LPRL_1(\lambda)$ and $L_2(\lambda)$ obtained at iteration *k*, respectively. Given the multipliers λ^k at iteration *k*, the multipliers for the next iteration λ^{k+1} are generated by $\lambda^{k+1} = \max\left\{\lambda^k + s_k\theta^k, 0\right\}$, where s_k is a positive scalar, called the step size. We uses a commonly used step size (at iteration *k*) $s_k = \mu_k\left(UB - Z_L\left(\lambda^k\right)\right)/\parallel\theta^k\parallel^2$, where $0 \leq \mu_k \leq 2$ is a positive scalar and *UB* is the best upper bound found until now by the converting procedure in Section 3.3, and $\parallel\cdot\parallel$ denotes the norm of vector.

Initially, μ_k is set to the value 2 $(\mu_0 = 2)$, and then μ_k will be halved if the best lower bound is not improved in a predetermined number of consecutive iterations (20 consecutive iterations in this paper).

3.3 Converting of Infeasible Solutions

Since the solution of the Lagrangean relaxed problem $L(\lambda)$ may be infeasible to the original problem *P*, some procedures of converting such infeasible solutions to feasible ones may be needed, for which we develop three procedures, called Procedure-FS 1, Procedure-FS 2 and Procedure-FS 3. Procedure-FS 1 and Procedure-FS 2 are executed to find a feasible solution which is an upper bound for the original problem *P*. The Procedure-FS 3 checks if any further improvement can be made in vehicle schedules so as to lead to reduction in the distribution and inventory cost while maintaining the feasibility of the solution. In Procedure-FS 1, if there are any retailers whose demands are not satisfied by the solution of Subproblem $LPRL_1(\lambda)$, then the number of trips of vehicles to the associated retailers will increase until the demands of all the retailers are satisfied. In Procedure-FS 2, if the value of z_t in the solution of Subproblem $LPRL_1(\lambda)$ does not satisfy its binary constraint, it will be adjusted to have a binary value by considering trade-off between inventory holding cost at the warehouse and ordering cost. In Procedure-FS 3, the number of trips of vehicles will decrease so as to reduce transportation and inventory cost, while maintaining the feasibility of the solution. We now present a formal description of the heuristic.

Procedure-FS 1

1. Let $\bar{x}_{jt}$ be the solution values of Subproblem $LPRL_1(\lambda)$, and $y_{ijt} = 0$ for all i, j and t.
2. Sort all the retailers in nonincreasing order of their round trip times (τ_j).
3. For j = 1 to N, do;

   3.1 Repeat if $\bar{x}_{jt} > \sum_i V_i y_{ijt}$: Find vehicle i^* such that

   $i^* = \arg\min_i \left\{ c_{ijt}^D - V_i \lambda_{jt} \mid \sum_j \tau_j y_{ijt} + \tau_j \le \tau \quad for \;\; i^* \right\}$, and let $y_{i^*jt} \leftarrow y_{i^*jt} + 1$.
4. Repeat Step 3 for all $t \in T$.

Procedure-FS 2

1. Let $(\bar{z}_t, \bar{q}_t)$ be the optimal solution value of Subproblem $LPRL_1(\lambda)$, and $\bar{y}_{ijt}$ be the current solution after Procedure-FS 1 is executed.
2. Let t = 1.
3. Find the smallest $t^* > t$ such that $\sum_{k=t}^{t^*} \left(1 - \bar{z}_k\right) \cdot o_k \le \sum_{k=t+1}^{t^*} \left(\sum_{l=t}^{k-1} h_{0l} \right) \cdot \bar{q}_k$.

   Let $z_{t^*} = 1$ and $z_k = 0 \quad for \;\; t < k < t^*$.

Solve the following linear program :

$$Min \quad \sum_t o_t \bar{z}_t + \sum_j \sum_t h_{jt} I_{jt} + \sum_j \sum_t c_{jt}^Q x_{jt} + \sum_i \sum_j \sum_t c_{ijt}^D \bar{y}_{ijt}$$

Subject to

$$q_t \le \left(\sum_{k=t}^{T} \sum_j d_{jk} \right) \cdot \bar{z}_t \qquad \forall t \qquad (19)$$

$$x_{jt} \le \sum_i V_i \bar{y}_{ijt} \qquad \forall j,\, t \qquad (20)$$

(11), (12) and (16)

4. Let $t \leftarrow t^*$, and if $t < T$, then go to Step 3. Otherwise, stop.

Procedure-FS 3

1. Let $\Omega = \left\{ (i,\, j,\, t) \mid \bar{y}_{ijt} > 0,\ \ i \in I,\, j \in J,\, t \in T \right\}$ where $\bar{y}_{ijt}$ is the current solution obtained from Procedure-FS 1.
2. Find $(i,\, j,\, t)^*$ such that $(i,\, j,\, t)^* = \arg\max_{(i,j,t) \in \Omega} \left(c_{ijt}^D - V_i \lambda_{jt} \right)$.

   Let $\Omega = \Omega \setminus (i,\, j,\, t)^*$.

   Solve the following linear program where $\bar{z}_t$ is the current solution obtained from Procedure-FS 2 :

$$Min \quad \sum_t o_t \bar{z}_t + \sum_j \sum_t h_{jt} I_{jt} + \sum_j \sum_t c_{jt}^Q x_{jt} + \sum_i \sum_j \sum_t c_{ijt}^D \bar{y}_{ijt}$$

   subject to

$$x_{jt} \le \sum_i V_i \bar{y}_{ijt} - V_{i^*} \qquad \forall j,\, t \qquad (21)$$

   (11), (12), (16) and (19)

   If the objective value of Problem (LP2) is smaller than the objective value of the current solution, then let $\bar{y}_{(i,j,t)^*} \leftarrow \bar{y}_{(i,j,t)^*} - 1$.

4. Repeat Step 2 until $\Omega = \phi$.

In the above procedures, two of three procedures used to obtain a feasible solution are similar to the heuristics proposed by Kim and Kim [8]. However, in our heuristic, the second step (Procedure-FS 2) is developed to find a binary value of variable z_t.

3.4 Lagrangean Heuristic Procedure

In order to find a lower bound and also an upper bound for the original problem P, we consider a Lagrangean heuristic procedure. In the heuristic procedure, two subproblems, $L_1(\lambda)$ and $L_2(\lambda)$, are solved for finding a lower bound, and Procedure-FS 1 and Procedure-FS 2 are executed for finding an upper bound, for given λ. Since Procedure-FS 3 does not require little time, it is allowed to be executed once in some number of iterations of Procedure-FS 1 and Procedure-FS 2, say K_H iterations (set to the value 30 in this paper). The heuristic procedure is designed to terminate after its number of iterations reaches a predetermined limit, K (set to the value 500 in this paper) or when the lower bound is not improved for a predetermined number of iterations, L (set to the value 200 in this paper). It is also to be terminated when the difference between the best upper bound and the best lower bound found so far is less than a predetermined limit, *called error limit* (set to 1% in this paper). Obviously it can be terminated when an optimal solution is found.

We now present a formal description of Lagrangean heuristic procedure, called Procedure-LH.

Procedure-LH

Step 1: Initialize the Lagrangean multipliers and parameters as follows;

1.1: Set the improvement counter at $k = 0$, the iteration counter at $l = 0$.

1.2: Set the Lagrangean multiplier $\lambda^k = 1$.

1.3: Set the current best lower bound, LB, to negative infinity and the current best upper bound, UB, to positive infinity.

Step 2: Generate a lower bound and use it to update the current best lower bound.

2.1: Given λ^k, solve the Lagrangean Problem $L(\lambda^k)$ using the simplex method and the DP algorithm (referring to Sections 3.1.1 and 3.1.2), and obtain the value for $Z_L(\lambda^k)$ which is a lower bound for the original problem P. If the solutions are feasible to the original problem, terminate and give $Z_L(\lambda^k)$ as the final solution.

2.2: If $Z_L(\lambda^k) > LB$, then let $LB = Z_L(\lambda^k)$ and $l = 0$. Otherwise, let $l = l + 1$.

Step 3: Generate a Lagrangean heuristic solution and use it to update the current best upper bound.

3.1: Generate a Lagrangean heuristic solution by using Procedure-FS 1 and Procedure-FS 2, and compute the feasible solution value of $\overline{Z}_{opt}$ which is an upper bound for the original problem *P*.

3.2: If *k* is a multiple of K_H, execute Procedure-FS 3 and update the feasible solution value of $\overline{Z}_{opt}$. Otherwise, go to Step 3.3.

3.3: If $\overline{Z}_{opt} < UB$, then let $UB = \overline{Z}_{opt}$.

3.4: If $\left(\frac{UB - LB}{LB}\right) \times 100 \le 1$, terminate and give *UB* as the final solution. Otherwise, go to Step 4.

Step 4: If $k > K$ or $l > L$, terminate and give *UB* as the final solution. Otherwise, go to Step 5.

Step 5: Update the Lagrangean multipliers using the subgradient optimization method (referring to Section 3.2). Set $k = k + 1$ and go to Step 2.

4. Computational Results

In this section, we experimentally test the quality of solutions found for the proposed problem by the Lagrangean heuristic. The heuristic is coded in C++ language and run on a personal computer with a Core2-Duo 2.40GHz processor. The linear programming problem in the heuristic is solved by CPLEX 12.1, a commercial solver. We use a data generation scheme of Kim and Kim [7, 8], which reflects the regional distribution planning problem of Korean oil companies. For example, the length of planning horizon is set to seven periods and the length of each period (τ) is set at 32 time units (one day) where one unit time corresponds to 15 minutes. Three levels of the loading capacity of vehicles (small, medium and large) are considered. The capacities of vehicles are selected randomly at an equal probability. Transportation time between the warehouse and each retailer (τ_j) is generated from DU [$0.2\mu_1$, μ_1], where DU[l, u] is the discrete uniform distribution over the interval [l, u] and $\mu_1 = \tau \cdot M / N$.

We evaluate the performance of the proposed heuristic by considering the impact of: number of retailers (J), number of vehicles (I), demand of retailers (d_{jt}), loading capacities of vehicles (V_i), distance-based costs (c_{ijt}^D), quantity-based costs (c_{jt}^Q), ordering cost of the warehouse (o_t), inventory holding cost of the warehouse (h_{0t}), and inventory holding costs of retailers (h_{jt}). To test the effects of varying number of retailers (J), number of vehicles (I) and demand of retailers (d_{jt}), we let $J \in \{40, 60, 80, 100, 120, 160\}$, $I \in \{20, 40, 60\}$ and d_{jt}~DU[100, 500], d_{jt}~DU[100, 700], d_{jt}~DU[100, 1000]. Also, costs associated with the vehicles, loading capacities of vehicles (V_i), distance-based costs (c_{ijt}^D) and quantity-based costs (c_{jt}^Q), may affect the performance of the heuristic. We let the loading capacities of small, medium and large size vehicles to be (μ_2, 1.25μ_2, 1.5μ_2), (1.5μ_2, 1.75μ_2, 2μ_2) and (2μ_2, 2.25μ_2, 2.5μ_2) where $\mu_2 = \left\lfloor \left(\sum_j \sum_t d_{jt}\tau_j\right) / \left(\tau \cdot N\right) \right\rfloor$, and c_{ijt}^D~U[μ_3, 10μ_3], c_{ijt}^D~U[μ_3, 30μ_3] and c_{ijt}^D~U[μ_3, 50μ_3], and c_{jt}^Q~U[10, 20], c_{jt}^Q~U [10, 40], c_{jt}^Q~U[10, 60] where U[l, u] is the uniform distribution over the interval [l, u]. Finally, the performance of the heuristic may be affected by ordering cost of the warehouse (o_t), inventory holding cost of the warehouse (h_{0t}) and inventory holding costs of retailers (h_{jt}). We let o_t~U[μ_4, 5μ_4], o_t~U[μ_4, 10μ_4] and o_t~U[μ_4, 15μ_4] where $\mu_4 = 3 \cdot \left(\sum_j \sum_t d_{jt}\right)/T$, and h_{0t}~U[1, 5], h_{0t}~U[1, 10] and h_{0t}~U[1, 15], and h_{jt}~U[5, 10], h_{jt}~U[10, 15], h_{jt}~U[15, 20], h_{jt}~U[10, 20], h_{jt}~U[20, 30], h_{jt}~U[30, 40], h_{jt}~U[15, 30], h_{jt}~U[30, 45] and h_{jt}~U[45, 60].

As the performance measure, we use mean relative error which is defined as $100 \times \left(UB - LB\right)/LB$ where UB and LB are the solution from the Lagrangean heuristic and a lower bound, respectively. The lower bound can be obtained by solving the Lagrangean relaxed problem $L(\lambda)$. Each table entry represents the average of its associated 20 instances. Times are given in seconds and only include computation time.

Three sets of experiments were run. In the first set, d_{jt}, I and J were allowed to vary, while V_i = (1.5μ_2, 1.75μ_2, 2μ_2), c_{ijt}^D~U[μ_3, 10μ_3], c_{jt}^Q~U[10, 20], o_t~U[μ_4, 5μ_4], h_{0t}~U [1, 5] and h_{jt}~U[5, 10]. These results are presented in Table 1. In the second set, V_i, c_{ijt}^D and c_{jt}^Q were allowed to vary, while I = 20, J = 60, d_{jt}~DU[100, 700], o_t~U[μ_4, 5μ_4], h_{0t}~ U[1, 5] and h_{jt}~U[5, 10]. These results are presented in Table 2. In the third set, o_t, h_{0t} and h_{jt} were allowed to vary, while I = 20, J = 60, d_{jt}~DU[100, 700], V_i = (1.5μ_2, 1.75μ_2, 2μ_2), c_{ijt}^D~U[μ_3, 10μ_3] and c_{jt}^Q~U[10, 20]. These results are presented in Table 3.

Table 1. Performance of the Heuristic for Demands of Retailers, and Number of Retailers and Vehicles

d_{jt}	I	J	Mean relative error (%)			CPU time (s)		
			Min	Mean	Max	Min	Mean	Max
DU[100, 300]	20	40	4.15	7.32	12.32	9.65	27.12	39.75
		60	3.54	4.58	5.88	26.17	68.56	100.87
		80	2.86	3.72	4.38	55.57	83.76	113.01
	40	80	6.69	8.58	10.59	28.81	54.84	90.00
		100	4.06	4.93	5.63	91.28	133.57	168.71
		120	4.51	4.98	6.13	114.46	238.28	318.14
	60	120	4.39	5.50	7.06	83.92	162.03	212.07
		140	4.83	5.55	6.78	209.21	251.67	329.26
		160	4.57	5.06	5.53	294.14	352.09	396.26
DU[100, 700]	20	40	3.42	8.22	12.80	7.57	22.33	32.89
		60	3.24	4.04	5.04	39.14	74.75	105.40
		80	3.35	3.79	4.44	64.87	92.76	113.00
	40	80	5.58	7.80	11.17	30.03	50.30	101.40
		100	3.73	4.82	6.08	55.64	111.27	172.50
		120	3.73	4.40	4.81	167.15	234.83	303.45
	60	120	3.64	4.87	5.78	88.62	164.26	218.09
		140	4.20	5.33	6.66	209.93	268.18	351.48
		160	4.19	4.72	5.58	247.17	379.02	471.48
DU[100, 1000]	20	40	3.96	7.66	11.93	10.60	26.30	36.87
		60	2.86	4.10	5.83	24.18	59.96	87.65
		80	3.02	3.39	3.61	64.12	91.77	117.29
	40	80	8.67	10.14	12.61	32.46	45.70	59.54
		100	4.21	4.95	5.54	62.14	110.52	175.34
		120	3.79	4.53	5.37	183.87	233.48	290.36
	60	120	3.75	4.74	6.46	74.37	122.05	229.60
		140	4.17	4.79	6.35	146.17	258.06	341.54
		160	3.82	4.46	5.45	243.39	350.15	410.60

Table 1 shows that the proposed heuristic provides a relative good solution in reasonable time. It can be observed that the mean relative error decreases as J/I increases, and that the distribution of retailer's demand does not have impact on the performance of the heuristic.

Table 2 shows that the mean relative error decreases as the loading capacities of vehicles increases. Also, the heuristic performs better when both distance-based costs and quantity-based costs decrease.

Table 2. Performance of the Heuristic for Loading Capacities of Vehicles, Distance−Based Costs and Quantity−Based Costs

V_i	c^D_{ijt}	c^Q_{jt}	Mean relative error (%)		
			Min	Mean	Max
Small = μ_2 Medium = $1.25\mu_2$ Large = $1.5\mu_2$	U[μ_3, $10\mu_3$]	U[10, 20]	0.60	0.84	1.15
		U[10, 40]	1.49	2.68	4.18
		U[10, 60]	2.25	2.97	4.06
	U[μ_3, $30\mu_3$]	U[10, 20]	0.77	1.17	1.78
		U[10, 40]	2.03	2.71	3.26
		U[10, 60]	1.92	3.56	4.10
	U[μ_3, $50\mu_3$]	U[10, 20]	1.24	1.79	3.05
		U[10, 40]	2.42	3.75	5.03
		U[10, 60]	2.52	3.30	4.30
Small = $1.5\mu_2$ Medium = $1.75\mu_2$ Large = $2\mu_2$	U[μ_3, $10\mu_3$]	U[10, 20]	1.30	1.69	3.08
		U[10, 40]	1.95	2.85	4.36
		U[10, 60]	2.14	2.75	3.39
	U[μ_3, $30\mu_3$]	U[10, 20]	1.49	2.77	5.07
		U[10, 40]	1.95	3.23	5.31
		U[10, 60]	3.04	3.66	5.19
	U[μ_3, $50\mu_3$]	U[10, 20]	2.88	3.43	4.24
		U[10, 40]	2.14	3.64	5.48
		U[10, 60]	2.81	3.83	5.15
Large = $2.5\mu_2$ Medium = $2.25\mu_2$ Small = $2\ \mu_2$	U[μ_3, $10\mu_3$]	U[10, 20]	1.77	2.32	3.98
		U[10, 40]	2.13	2.96	4.02
		U[10, 60]	2.39	3.59	4.89
	U[μ_3, $30\mu_3$]	U[10, 20]	2.87	3.63	5.04
		U[10, 40]	2.49	3.67	4.77
		U[10, 60]	1.97	3.94	5.22
	U[μ_3, $50\mu_3$]	U[10, 20]	3.96	5.23	7.19
		U[10, 40]	3.00	4.68	5.71
		U[10, 60]	2.69	4.17	6.23

Table shows that the mean relative error decreases as ordering costs of warehouse decrease. Also, it can be observed that the heuristic performs better as the ratio of inventory holding cost of retailers over inventory holding cost of warehouse increases, and as inventory holding cost of warehouse increases.

To evaluate the quality of the heuristic solutions in terms of the deviation from an optimal solution, we generate twenty small sized test problems with 10 vehicles, 25 retailers and 7 periods. The optimal solutions are obtained using CPLEX (a commercial software package for mixed integer programming problems) for small sized problems.

Table 3. Performance of the Heuristic for Ordering Cost at Warehouse, Inventory Holding Costs at Warehouse and Retailers

o_t	h_{0t}	h_{jt}	Mean relative error (%)		
			Min	Mean	Max
U[μ_4, 5μ_4]	U[1, 5]	U[5, 10]	2.53	3.71	5.29
		U[10, 15]	1.43	2.23	2.89
		U[15, 20]	0.96	1.39	1.84
	U[1, 10]	U[10, 20]	1.45	2.24	3.76
		U[20, 30]	0.92	1.18	1.67
		U[30, 40]	0.82	0.98	1.20
	U[1, 15]	U[15, 30]	1.17	1.46	1.70
		U[30, 45]	0.84	1.17	2.50
		U[45, 60]	0.82	1.15	2.44
U[μ_4, 10μ_4]	U[1, 5]	U[5, 10]	2.95	4.40	6.43
		U[10, 15]	0.95	2.88	7.26
		U[15, 20]	1.34	2.72	4.86
	U[1, 10]	U[10, 20]	1.35	3.05	5.64
		U[20, 30]	1.12	1.99	4.15
		U[30, 40]	0.81	1.23	3.41
	U[1, 15]	U[15, 30]	1.16	2.06	3.75
		U[30, 45]	0.64	0.97	1.27
		U[45, 60]	0.78	1.27	3.88
U[μ_4, 15μ_4]	U[1, 5]	U[5, 10]	2.78	6.45	13.31
		U[10, 15]	1.32	3.23	6.93
		U[15, 20]	0.99	4.00	7.98
	U[1, 10]	U[10, 20]	1.55	3.24	5.21
		U[20, 30]	0.90	2.50	6.36
		U[30, 40]	0.69	2.74	8.08
	U[1, 15]	U[15, 30]	1.32	3.60	8.93
		U[30, 45]	0.76	1.32	3.05
		U[45, 60]	0.70	1.57	5.72

Table 4 gives the resulting objective values, percentage errors and computation times, associated with the proposed Lagrangean heuristic algorithm and the optimal algorithm. The percentage error is defined as $100\times(UB-OP)/OP$ where UB and OP

are the objective values obtained from the proposed Lagrangean heuristic algorithm and the optimal algorithm, respectively. The average percentage error is 1.06% with the worst case error of 1.93%. In the test, optimal solutions are not found for three problems (shown in Table 2) out of 20 problems within the time limit of two hours. In these cases, the best feasible solutions obtained within the time limit are used for the associated performance comparison. The average computation time taken by the proposed heuristic algorithm is 34.88 seconds, while that of CPLEX is 1711.45 seconds.

Table 4. Performance Comparison Between the Proposed Heuristic Algorithm and the Optimal algorithm

	Objective Values		Mean relative error (%)	CPU time (s)	
	Heuristic	CPLEX		Heuristic	CPLEX
1	1579797.083	1559744.648	1.29	35.13	48.13
2	1613814.788	1597675.502	1.01	33.92	7200.00[a]
3	1583998.405	1569937.321	0.90	32.30	1343.11
4	1594620.381	1577083.968	1.11	36.56	64.48
5	1576569.190	1557837.240	1.20	38.03	82.69
6	1583704.682	1571188.293	0.80	35.98	7200.00[a]
7	1668590.851	1655777.770	0.77	36.34	7200.00[a]
8	1649410.512	1632901.343	1.01	33.08	3234.97
9	1602486.425	1584523.675	1.13	33.69	24.48
10	1610225.602	1597212.568	0.81	35.19	2700.16
11	1641816.165	1626178.985	0.96	35.00	283.53
12	1630590.729	1621207.220	0.58	35.86	159.58
13	1551889.626	1546025.258	0.38	32.25	51.36
14	1596771.719	1584296.015	0.79	30.58	26.34
15	1607720.048	1577320.506	1.93	34.60	262.61
16	1561524.459	1548712.396	0.83	30.98	245.59
17	1583087.551	1556041.678	1.74	36.42	75.77
18	1568538.977	1541751.160	1.74	41.81	3504.16
19	1547027.192	1531117.543	1.04	38.17	200.05
20	1640912.190	1622238.892	1.15	31.75	321.89
Avg.			1.06	34.88	1711.445

Note) [a] The best feasible solutions found by CPLEX with the time limit (2 hours).

5. Conclusion

This paper studies an integrated inventory-distribution problem with a fleet of heterogeneous vehicles employed to distribute a single type of product from a single warehouse to spatially distributed retailers to satisfy their dynamic deterministic demands. The objective of this study is to determine order planning at the warehouse, and also vehicle schedules and delivery quantities for the retailers with the goal of minimizing the sum of ordering cost at the warehouse, inventory holding cost at both the warehouse and retailers, and transportation cost.

As a solution approach, we give a Mixed Integer Programming formulation for the problem, and develop a Lagrangean heuristic procedure for computing lower and upper bounds on the optimal solution value. In the approach, the relaxed problem is divided into two independent subproblems. To solve the subproblems more efficiently, some valid inequalities are proposed. The Lagrangean dual problem of finding the best Lagrangrean lower bound is solved by subgradient optimization. In order to evaluate the effectiveness and efficiency of the proposed algorithm, computational experiments are performed with some numerical instances that are randomly generated. The experiment results show that the proposed algorithm gives good solution within a reasonable time.

As a further study, an extended problem with demand being non-deterministic may be interesting, in the situation where demand is somewhat irregular and hard to forecast accurately in practice. Consideration of storage capacities at the warehouse and retailers may also be interesting in view of practicality. Moreover, it may be needed to consider multiple warehouses and multiple types of products.

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