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SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. In this article, we study generalized slant cylindrical surfaces (GSCS's) with pointwise 1-type Gauss map of the first and second kinds. Our main results state that GSCS's with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right cones are the only polynomial kind GSCS's with pointwise 1-type Gauss map of the second kind.

1. Introduction and Preliminaries

The notion of finite type submanifolds in Euclidean or pseudo-Euclidean space, introduced by B.-Y. Chen during the late 1970's, has become a useful tool for investigating and characterizing many important submanifolds (cf. [3, 4]). In [1, 2, 6] the notion of finite type was extended to differential maps, in particular, to Gauss map of submanifolds.

If a submanifold M of Euclidean or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C, where Δ is the Laplace operator corresponding to the induced metric on M (cf [1, 2, 9]). However, the Laplacian of the Gauss map of several important surfaces such as helicoids, catenoids and right cones take a somewhat different form; namely,

$$\Delta G = f(G + C) \tag{1.1}$$

for some non-constant function f and some constant vector C. For this reason, a submanifold is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (1.1) for some smooth function f on M and vector C. A submanifold with pointwise 1-type Gauss map is said to be *of the first kind* if the vector C in (1.1) is the zero

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vector. Otherwise, the pointwise 1-type Gauss map is said to be *of the second kind* ([5]).

Let M be a surface of Euclidean 3-space $\mathbb{E}^3$. The map $G : M \to S^2 \subset \mathbb{E}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of the surface M, where S^2 is the unit sphere in $\mathbb{E}^3$ centered at the origin.

For the matrix $g = (g_{ij})$ consisting of the components of the metric on M, we denote by $g^{-1} = (g^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{\mathcal{G}}\ g^{ij} \frac{\partial}{\partial x^j}\Big). \tag{1.2}$$

Here, we give an example of surfaces of revolution with pointwise 1-type Gauss map of the second kind.

Example 1.1. Consider the right cone C_a which is parameterized by

$$x(u, v) = (v \cos u, v \sin u, av), \quad a \geq 0.$$

Then the Gauss map G and its Laplacian ΔG are respectively given by

$$G = \frac{1}{\sqrt{1+a^2}}(a \cos u, a \sin u, -1)$$

and

$$\Delta G = \frac{1}{v^2}\Big(G + \Big(0, 0, \frac{1}{\sqrt{1+a^2}}\Big)\Big).$$

It implies that the right cone has pointwise 1-type Gauss map of the second kind.

In [5], B.-Y. Chen, M. Choi and Y. H. Kim studied surfaces of revolution with pointwise 1-type Gauss map. In [7], U. Dursun studied flat surfaces in Euclidean 3-space with pointwise 1-type Gauss map.

The author and Y. H. Kim introduced the class of generalized slant cylindrical surfaces (GSCS's) in [8]. This class includes surfaces of revolution and cylindrical surfaces as special cases. Thus, we need to consider the GSCS's in $\mathbb{E}^3$ with pointwise 1-type Gauss map.

In this paper, we study the GSCS's with pointwise 1-type Gauss map. In particular, we prove that GSCS's with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right cones are the only polynomial kind GSCS's with pointwise 1-type Gauss map of the second kind.

Hereafter, all objects are assumed to be connected and smooth unless mentioned otherwise.

2. Generalized Slant Cylindrical Surfaces

For a fixed unit speed plane curve $X(s) = (x(s), y(s), 0)$, let $T(s) = X'(s)$ and $N(s) = (-y'(s), x'(s), 0)$ denote the unit tangent and principal normal vector, respectively. The curvature $\kappa(s)$ of $X(s)$ is defined by $T'(s) = \kappa(s)N(s)$ and we have $T(s) \times N(s) = V$, where V denotes the unit vector $(0, 0, 1)$. For a constant θ, we let $Y_\theta(s) = \cos\theta N(s) + \sin\theta V$. Then the ruled surface M defined by

$$F(s,t) = X(s) + tY_\theta(s) \tag{2.1}$$

is regular at (s,t) where $1 - \cos\theta\kappa(s)t$ does not vanish. This ruled surface M is called a *slant cylindrical surface(SCS)* over $X(s)$. For the unit normal vector $G = -\sin\theta N(s) + \cos\theta V$, M satisfies

$$\langle F_s, F_t\rangle = 0, \langle F_{st}, G\rangle = 0.$$

This shows that the coordinate lines of F are lines of curvature of M with corresponding principal curvatures

$$k_1(s,t) = \frac{-\kappa(s)\sin\theta}{1 - \kappa(s)t\cos\theta}, k_2(s,t) = 0, \tag{2.2}$$

respectively. The SCS with $\sin\theta = 0$ or $\cos\theta = 0$ is nothing but a parametrization of either a plane or a cylindrical surface.

In general, we consider another unit speed plane curve $W(t) = (z(t), w(t))$. If we let $Y_s(t) = z(t)N(s) + w(t)V$, then the parametrized surface defined by

$$H(s,t) = X(s) + Y_s(t) \tag{2.3}$$

is regular at (s,t) where $1 - \kappa(s)z(t)$ does not vanish. This parametrized surface M is called a *generalized slant cylindrical surface(GSCS)* over $X(s)$. For the unit normal vector $G(s,t) = -w'(t)N(s) + z'(t)V$, M satisfies

$$\langle H_s, H_t\rangle = 0, \langle H_{st}, G\rangle = 0.$$

This shows that $H(s,t)$ is a principal curvature coordinate system of M with corresponding principal curvatures

$$k_1(s,t) = \frac{-\kappa(s)w'(t)}{1 - \kappa(s)z(t)}, k_2(s,t) = \kappa(t), \tag{2.4}$$

respectively, where $\kappa(t) = z'(t)w''(t) - z''(t)w'(t)$ denotes the curvature of $W(t)$.

If $W(t)$ is a straight line, then the GSCS $H(s,t)$ is nothing but a SCS. If $X(s)$ is a straight line, then the GSCS $H(s,t)$ is nothing but a cylindrical surface. Furthermore, we have the following ([8]).

Proposition 2.1. *If $X(s)$ is a circle, then GSCS M over $X(s)$ is a surface of revolution.*

Therefore cylindrical surfaces and surfaces of revolution are special cases of GSCS's.

Now we give the following:

Proposition 2.2. *Let M denote a GSCS given by (2.3). Then we have the following.*

(1) *If the mean curvature H is constant, then M is a surface of revolution.*
(2) *If the Gaussian curvature K is constant, then M is either a surface of revolution or an SCS.*

Proof. It follows from (2.4) that

$$2H = \kappa(t) + \frac{-\kappa(s)w'(t)}{1-\kappa(s)z(t)}, \quad K = \frac{-\kappa(s)\kappa(t)w'(t)}{1-\kappa(s)z(t)}. \tag{2.5}$$

Hence we have

$$\kappa(t) - 2H = \kappa(s)\{\kappa(t)z(t) - 2Hz(t) + w'(t)\}, \tag{2.6}$$

and

$$K = \kappa(s)\{Kz(t) - \kappa(t)w'(t)\}. \tag{2.7}$$

Suppose that H is constant. If $\kappa(t) - 2H \neq 0$, then (2.6) shows that $\kappa(s)$ is a nonzero constant, and hence M is a surface of revolution. If $\kappa(t) - 2H = 0$, then (2.5) implies $\kappa(s)w'(t) = 0$. In case $\kappa(s_0) \neq 0$ for some s_0, $w'(t)$ vanishes identically, and hence M is a part of a plane. Otherwise, $\kappa(s)$ vanishes identically. Hence $X(s)$ is a straight line. Thus M is a part of a plane ($H = 0$) or a circular cylinder ($H \neq 0$).

Now suppose that K is constant. If $K \neq 0$, it follows from (2.7) that $\kappa(s)$ is a nonzero constant, and hence M is a surface of revolution. In case $K = 0$ and $\kappa(s_0) \neq 0$, (2.7) shows that $\kappa(t)$ vanishes identically, and hence M is an SCS. In case $K = 0$ and $\kappa(s)$ vanish identically, then M is a cylindrical surface. □

3. GSCS's with Pointwise 1-type Gauss Map of the First Kind

Let $X(s) = (x(s), y(s), 0)$ be a unit speed plane curve with the Frenet frame $\{T(s), N(s)\}$. We consider GSCS's parametrized by

$$H(s,t) = X(s) + Y_s(t), \tag{3.1}$$

where $W(t) = (z(t), w(t))$ is a unit speed plane curve, $Y_s(t) = z(t)N(s) + w(t)V$, and $V = (0,0,1)$. Then $H(s,t)$ is regular at (s,t) where $Q(s,t) = 1 - \kappa(s)z(t)$ does not vanish and we get

$$\begin{aligned} &H_s = Q(s,t)T(s), \quad H_t = z'(t)N(s) + w'(t)V, \\ &G(s,t) = -w'(t)N(s) + z'(t)V. \end{aligned} \tag{3.2}$$

The Laplacian Δ on M is given by

$$\Delta f = -Q^{-3}\{\kappa'(s)z(t)f_s + Qf_{ss} - Q^2\kappa(s)z'(t)f_t + Q^3 f_{tt}\}. \tag{3.3}$$

Hence it follows from (3.2) and (3.3) that

$$\begin{aligned} -Q^3\Delta G = {} & \kappa'(s)w'(t)T(s) + Q\{\kappa(s)^2 w'(t) + Q\kappa(s)z'(t)w''(t) \\ & - Q^2 w'''(t)\}N(s) + Q^2\{-\kappa(s)z'(t)z''(t) + Qz'''(t)\}V. \end{aligned} \tag{3.4}$$

Now suppose that M has the pointwise 1-type Gauss map G which satisfies (1.1). Then, letting $C = C_1(s)T(s) + C_2(s)N(s) + C_3V$, we have the following.

$$\kappa'(s)w'(t) = -Q^3 C_1(s)f(s,t), \tag{3.5}$$

$$\kappa(s)^2 w'(t) + Q\kappa(s)z'(t)w''(t) - Q^2 w'''(t) = Q^2 f(s,t)\{w'(t) - C_2(s)\}, \tag{3.6}$$

and

$$\kappa(s)z'(t)z''(t) - Qz'''(t) = Qf(s,t)\{z'(t) + C_3\}. \tag{3.7}$$

Using above, we get the following:

Theorem 3.1. *Let M be a GSCS given by (3.1). Suppose that M has pointwise 1-type Gauss map G of the first kind. Then M is a surface of revolution.*

Proof. Since $C = C_1(s)T(s) + C_2(s)N(s) + C_3V = 0$, it follows from (3.5) that $\kappa'(s)w'(t) = 0$. In case $\kappa'(s_0) \neq 0$ for some s_0, $w(t)$ is constant, and hence M is a part of a plane. Otherwise, κ is constant. If κ is nonzero, then M is a surface of revolution. If $\kappa = 0$, then it follows from (3.6) and (3.7) that

$$z'''(t) + f(s,t)z'(t) = 0, \quad w'''(t) + f(s,t)w'(t) = 0. \tag{3.8}$$

This shows that $\kappa'(t) = 0$. Thus M is a plane or a circular cylinder. $\square$

Combining Theorem 3.1 in [5] and Proposition 2.2, Theorem 3.1 shows directly the following.

Corollary 3.2. *Let M be a GSCS given by (3.1). Then the following are equivalent.*

(1) *M has pointwise 1-type Gauss map G of the first kind.*
(2) *M has constant mean curvature.*
(3) *M is a surface of revolution with constant mean curvature.*

Remark 3.3. Surfaces of revolution with constant mean curvature are also known as *surfaces of Delaunay* (cf. [10, p.115]).

4. GSCS's with Pointwise 1-type Gauss Map of the Second Kind

Consider a GSCS M parametrized by (3.1). If M is not cylindrical, then $W(t)$ can be parametrized by $W(t) = (t, g(t))$ for some function $g = g(t)$. Hence M is given by

$$H(s,t) = X(s) + tN(s) + g(t)V. \tag{4.1}$$

If $g(t)$ is a polynomial in t, Then M is said to be of polynomial kind ([5]). $H(s,t)$ is regular at (s,t) where $Q(s,t) = 1 - t\kappa(s) \neq 0$ and we get

$$\begin{aligned} &H_s = Q(s,t)T(s), H_t = N(s) + g'(t)V, \\ &G(s,t) = \frac{1}{P(t)}\{-g'(t)N(s) + V\}, P(t) = \sqrt{1 + g'(t)^2}. \end{aligned} \tag{4.2}$$

The Laplacian Δ on M is given by

$$\begin{aligned} \Delta f = &- P^{-4}Q^{-3}\{\kappa'(s)tP^4 f_s + P^4 Q f_{ss} \\ &- (P^2Q^2\kappa(s) + Q^3 g'g'')f_t + P^2Q^3 f_{tt}\}. \end{aligned} \tag{4.3}$$

Hence it follows from (4.2) and (4.3) that

$$\begin{aligned} \Delta G = &- \kappa'(s)g'P^{-1}Q^{-3}T(s) \\ &- P^{-7}Q^{-2}\{\kappa(s)^2 g'P^6 + \kappa(s)g''P^2Q \\ &+ g'(g'')^2Q^2 - g'''P^2Q^2 + 3g'(g'')^2Q^2\}N(s) \\ &- P^{-7}Q^{-1}\{(3(g')^2(g'')^2 - (g'')^2 - g'g''' - (g')^3g''')Q + \kappa(s)g'g''P^2\}V. \end{aligned} \tag{4.4}$$

Suppose that the Gauss map G satisfies (1.1) with nonzero constant vector C. Then, letting $C = C_1(s)T(s) + C_2(s)N(s) + C_3V$, we have the following.

$$PQ^3C_1(s)f(s,t) + \kappa'(s)g'(t) = 0, \tag{4.5}$$

$$P^6Q^2f(s,t)\{-g'(t)+PC_2(s)\}+\kappa(s)^2g'P^6 \\ +\kappa(s)g''P^2Q+g'(g'')^2Q^2-g'''P^2Q^2+3g'(g'')^2Q^2=0, \tag{4.6}$$

and

$$P^6Qf(s,t)\{1+C_3P\}+\{3(g')^2(g'')^2 \\ -(g'')^2-g'g'''-(g')^3g'''\}Q+\kappa(s)g'g''P^2=0. \tag{4.7}$$

It follows from (4.5) and (4.7) that

$$\begin{aligned} &C_3\kappa'(s)g'P^6+\kappa'(s)g'P^5 \\ &=C_1(s)Q^3\{3(g')^2(g'')^2-(g')^3g'''\}+C_1(s)\kappa(s)g'g''P^2Q^2 \\ &\quad -C_1(s)Q^3\{(g'')^2+g'g'''\} \end{aligned} \tag{4.8}$$

Suppose that M is a GSCS of polynomial kind, that is, $g(t)$ is a polynomial in t. Denote by $\deg g(t)$ the degree of $g(t)$.

If $\deg g(t)=n\geq 2$, then P^2 is a polynomial of degree $2n-2$. By comparing the degree of both sides of (4.8), we see that $C_3\kappa'(s)=0$, and hence we get

$$\kappa'(s)g'P^5=C_1(s)Q^3\{3(g')^2(g'')^2-(g')^3g'''\}+C_1(s)\kappa(s)g'g''P^2Q^2 \\ -C_1(s)Q^3\{(g'')^2+g'g'''\}. \tag{4.9}$$

By comparing the degree of both sides of (4.9), we see that $\kappa'(s)=0$. Thus, if $\kappa\neq 0$, M is a surface of revolution. If $\kappa=0$, then T,N are constant vectors and M is a cylindrical surface over a plane curve $W(t)$. Since $Q=1$, we have from (4.4)

$$\begin{aligned} \Delta G=&-P^{-7}\{g'(g'')^2-g'''P^2+3g'(g'')^2\}N \\ &-P^{-7}\{3(g')^2(g'')^2-(g'')^2-g'g'''-(g')^3g'''+\}V. \end{aligned} \tag{4.10}$$

Using (1.1), we get $C_1=0, C_2'=C_3'=0$, and

$$\{1+(g')^2\}\{C_2A-C_2B-C_3D\}^2=\{g'A-g'B+D\}^2, \tag{4.11}$$

where

$$A=3(g')^2(g'')^2-(g')^3g''',\quad B=(g'')^2+g'g''', \\ D=4g'(g'')^2-g'''-(g')^2g'''. \tag{4.12}$$

By comparing the coefficient of highest degree of both sides of (4.11), we get $C_2^2=1$, and hence again we get $C_3=0$. This shows that the coefficient of highest degree of $g'AD$ becomes zero, which is a contradiction.

If $\deg g(t)=1$, then M is a slant cylindrical (non-cylindrical) surface. Note that $P=\sqrt{1+a^2}$, where $g'(t)=a\neq 0$. By applying (4.5) and (4.6), we get

$$PQ^3C_1(s)f(s,t)+a\kappa'(s)=0, \tag{4.13}$$

and

$$Q^2 f(s,t)\{PC_2(s) - a\} + a\kappa(s)^2 = 0. \tag{4.14}$$

Suppose that $\kappa'(s_0) \neq 0$ for some s_0. Then on an interval I, we have $\kappa'(s) \neq 0$. On I, $f(s,t)$ is given by

$$f(s,t) = \frac{-a\kappa'(s)}{PQ^3 C_1(s)}. \tag{4.15}$$

Hence, by applying $Q = 1 - \kappa(s)t$, it follows from (4.13) and (4.14) that

$$aP\kappa(s)^2 C_1(s) - aP\kappa'(s)C_2(s) + a^2\kappa'(s) - aP\kappa(s)^3 C_1(s)t = 0. \tag{4.16}$$

The coefficient of t in (4.16) must vanish, and hence $C_1(s) = 0$ on I, which contradicts to (4.13). This contradiction shows that $\kappa(s)$ is a constant. Therefore M is a plane or a right circular cone.

Summarizing above, we obtain

Theorem 4.1. *Suppose that a GSCS M of polynomial kind has pointwise* 1*-type Gauss map G of the second kind. Then M is a surface of revolution.*

Hence, combining Theorem 4.1 in [5], we get

Corollary 4.2. *A GSCS M of polynomial kind has the pointwise* 1*-type Gauss map G of the second kind if and only if it is a plane or a right circular cone.*

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