J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 18, Number 4 (November 2011), Pages 361–368

CLOZ-COVERS OF TYCHONOFF SPACES

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Abstract. In this paper, we construct a cover $(\mathcal{L}(X), c_X)$ of a space X such that for any cloz-cover (Y, f) of X, there is a covering map $g : Y \longrightarrow \mathcal{L}(X)$ with $c_X \circ g = f$. Using this, we show that every Tychonoff space X has a minimal cloz-cover $(E_{cc}(X), z_X)$ and that for a strongly zero-dimensional space X, $\beta E_{cc}(X) = E_{cc}(\beta X)$ if and only if $E_{cc}(X)$ is $z^{\#}$-embedded in $E_{cc}(\beta X)$.

1. Introduction

All spaces in this paper are assumed to be Tychonoff spaces and βX denotes the Stone-Čech compactification of a space X.

Iliadis constructed the absolutes of Hausdorff spaces, which are exactly the minimal extremally disconnected covers of Hausdorff spaces and they turn out to be the perfect onto projective covers([5]).

There have been many ramifications from the minimal extremally disconnected covers of spaces. That is, to generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces has been introduced and their covers have been studied by various aurthors([3], [4], [8]). In these ramifications, minimal covers of compact spaces can be nisely characterized.

In particular, Henriksen, Vermeer and Woods ([4]) introduced the notion of cloz-spaces and they showed that every compact space X has a minimal cloz-cover $(E_{cc}(X), z_X)$. Open questions in the theory of cloz-spaces concerns with the minimal cloz-covers of non-compact spaces and the relation between $E_{cc}(\beta X)$ and $E_{cc}(X)$([4]). For this problem, we have the partial answer in [6]. Indeed, it is shown that for a weakly Lindelöff space X, X has a minimal cloz-cover $(E_{cc}(X), z_X)$ and that $E_{cc}(X)$ is a dense subspace of the minimal cloz-cover cover $E_{cc}(\beta X)$ of βX.

Received by the editors August 31, 2011. Accepted October 23, 2011.
2000 *Mathematics Subject Classification.* 54D80, 54G05, 54C10.
Key words and phrases. Stone-space, cloz-space, covering map.
The present research was conducted by the research fund of Dankook university in 2011.

In this paper, we first construct a cover $(\mathcal{L}(X), c_X)$ of a space X and show that for any cloz-cover (Y, f) of X, there is a covering map $g : Y \longrightarrow \mathcal{L}(X)$ such that $c_X \circ g = f$. Moreover, we show that $\mathcal{L}(X)$ is a cloz-space if and only if $(\mathcal{L}(X), c_X)$ is the minimal cloz-cover of X. Using these and the transfinite induction, we show that every space X has the minimal cloz-cover $(E_{cc}(X), z_X)$. Finally, we show that if X is a strongly zero-dimensional space, then $\beta E_{cc}(X) = E_{cc}(\beta X)$ if and only if $E_{cc}(X)$ is $z^{\#}$-embedded in $E_{cc}(\beta X)$.

For the terminology, we refer to [1], [2] and [7].

2. Cloz-covers of Tychonoff Spaces

Let X be a space. It is well-known that the collection $\mathcal{R}(X)$ of all regular closed sets in a space X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows :

For any $A \in \mathcal{R}(X)$ and any $\mathcal{F} \subseteq \mathcal{R}(X)$,

$\bigvee \mathcal{F} = cl_X(\cup\{F \mid F \in \mathcal{F}\})$,

$\bigwedge \mathcal{F} = cl_X(int_X(\cap\{F \mid F \in \mathcal{F}\}))$, and

$A' = cl_X(X - A)$.

A sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and finite meets([7]). Let $Z(X)^\sharp = \{cl_X(int_X(A)) \mid A$ is a zero-set in $X\}$. Then $Z(X)^\sharp$ is a sublattice of $R(X)$.

Recall that a map $f : Y \longrightarrow X$ is called *a covering map* if it is an onto continuous, perfect, and irreducible map([7]).

Lemma 2.1 ([6]). *Let X be a dense subspace of Y.*

(1) *The map $\phi : R(Y) \longrightarrow R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism.*

(2) *Let $f : Y \longrightarrow X$ be a covering map. Then the map $\psi : R(Y) \longrightarrow R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism.*

In the above lemma, the inverse map $\phi^{-1} : R(X) \longrightarrow R(Y)$ of ϕ is given by $\phi^{-1}(B) = cl_Y(B)$ $(B \in R(X))$ and the inverse map $\psi^{-1} : R(X) \longrightarrow R(Y)$ of ψ is given by $\psi^{-1}(B) = cl_Y(int_Y(f^{-1}(B))) = cl_Y(f^{-1}(int_X(B)))$ $(B \in R(X))$.

Definition 2.2. Let X be a space.

(1) A cozero-set C in X is said to be *a complemented cozero-set in X* if there

is a cozero-set D in X such that $C \cap D = \emptyset$ and $C \cup D$ is a dense subset of X. In case, $\{C, D\}$ is called *a complemented pair of cozero-sets in* X.

(2) Let $\mathcal{G}(X) = \{cl_X(C) \mid C$ is a complemented cozero-set in $X\}$.

Let X be a space and $\{C, D\}$ a complemented pair of cozero-sets in X. Then $cl_X(C) = cl_X(X - D)$ and since $cl_X(X - D) \in Z(X)^{\#}$, $cl_X(C) \in Z(X)^{\#}$. Hence $\mathcal{G}(X) = \{A \in Z(X)^{\#} \mid A' \in Z(X)^{\#}\}$ and $\mathcal{G}(X)$ is a Boolean subalgebra of $R(X)$.

Definition 2.3 ([4])**.** A space X is called *a cloz-space* if every element of $\mathcal{G}(X)$ is a clopen set in X.

A space X is a cloz-space if and only if βX is a cloz-space([4]).

Definition 2.4. Let X be a space.

(1) A pair (Y, f) is called *a cloz-cover of* X if Y is a cloz-space and $f : Y \longrightarrow X$ is a covering map.

(2) A cloz-cover (Y, f) of X is called *a minimal cloz-cover of* X if for any cloz-cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ with $f \circ h = g$.

Henriksen, Vermeer and Woods showed that every compact space X has the minimal cloz-cover $(E_{cc}(X), z_X)$. Let X be a compact space, $\mathcal{S}(\mathcal{G}(X))$ the Stone-space of $\mathcal{G}(X)$ and $E_{cc}(X)$ the subspace $\{(\alpha, x) \mid x \in \cap\{A \mid A \in \alpha\}\}$ of the product space $\mathcal{S}(\mathcal{G}(X)) \times X$. Then $(E_{cc}(X), z_X)$ is the minimal cloz-cover of X, where $z_X((\alpha, x)) = x$([4]).

A space X is called *a weakly Lindelöff space* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is dense in X.

Let X be a weakly Lindelöff space,

$$E_{cc}(X) = \{(\alpha, x) \in \mathcal{G}(\beta X) \times X \mid x \in \cap\{A \mid A \in \alpha\}\}$$

the subspace of $\mathcal{S}(\mathcal{G}(\beta X)) \times X$ and $z_X(\alpha, x) = x$. Then $(E_{cc}(X), z_X)$ is the minimal cloz-cover of X and $E_{cc}(X)$ is a dense subspace of $E_{cc}(\beta X)$([6]).

Definition 2.5. Let X be a space and $\mathcal{B}$ a sublattice of the power set $P(X)$ of X. A $\mathcal{B}$-filter α is called *fixed* if $\cap\{A \mid A \in \alpha\} \neq \emptyset$.

Let X be a space and $\mathcal{L}(\mathcal{G}(X)) = \{\alpha \mid \alpha$ is a fixed $\mathcal{G}(X)$-ultrafilter$\}$. For any $A \in \mathcal{G}(X)$, let $\lambda_A = \{\alpha \in \mathcal{L}(\mathcal{G}(X)) \mid A \in \alpha\}$. Then $\{\lambda_A \mid A \in \mathcal{G}(X)\}$ is a base for closed sets of some topology on $\mathcal{L}(\mathcal{G}(X))$. Let $\mathcal{L}(X) = \{(\alpha, x) \in \mathcal{L}(\mathcal{G}(X)) \times X \mid x \in \cap\{A \mid A \in \alpha\}\}$ be the subspace of the product space $\mathcal{L}(\mathcal{G}(X)) \times X$ of $\mathcal{L}(\mathcal{G}(X))$(endowed with topology generated by $\{\mathcal{L}(\mathcal{G}(X)) - \lambda_A \mid A \in \mathcal{G}(X)\}$) and X. Define a map $c_X : \mathcal{L}(X) \longrightarrow X$ by $c_X(\alpha, x) = x$.

Proposition 2.6. *Let X be a space. Then there is a homeomorphism $h : \mathcal{L}(X) \longrightarrow z_{\beta X}^{-1}(X)$ such that $z_{\beta X}^0 \circ h = c_X$, where $z_{\beta X}^0 : z_{\beta X}^{-1}(X) \longrightarrow X$ is the restriction and corestriction of $z_{\beta X} : E_{cc}(\beta X) \longrightarrow \beta X$ with respect to $z_{\beta X}^{-1}(X)$ and X, respectively.*

Proof. Since the map $\phi : R(\beta X) \longrightarrow R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism, the restriction and corestriction $\phi_{\mathcal{G}(\beta X)} : \mathcal{G}(\beta X) \longrightarrow \mathcal{G}(X)$ of ϕ with respect to $\mathcal{G}(\beta X)$ and $\mathcal{G}(X)$, respectively is a Boolean isomorphism. Hence for any $\alpha \in \mathcal{L}(\mathcal{G}(X))$, there is a unique $\alpha_\beta \in \mathcal{L}(\mathcal{G}(\beta X))$ such that $\{B \cap X \mid B \in \alpha_\beta\} = \alpha$. Let $(\alpha, x) \in \mathcal{L}(X)$. Then $x \in \cap\{A \mid A \in \alpha\}$ and $x \in \cap\{A \mid A \in \alpha_\beta\}$. Hence $(\alpha_\beta, x) \in z_{\beta X}^{-1}(X)$.

Define a map $h : \mathcal{L}(X) \longrightarrow z_{\beta X}^{-1}(X)$ by $h(\alpha, x) = (\alpha_\beta, x)$. Then clearly, h is an one-to-one, onto map. Let $A \in \mathcal{G}(X)$ and U be an open set in βX. Then $cl_{\beta X}(A) \in \mathcal{G}(\beta X)$. Let

$$B = [(\mathcal{L}(\mathcal{G}(X)) - \lambda_A) \times (U \cap X)] \cap \mathcal{L}(X)$$

and

$$C = [(\mathcal{L}(\mathcal{G}(\beta X)) - \lambda_{cl_{\beta X}(A)}) \times U] \cap z_{\beta X}^{-1}(X).$$

Let $(\alpha, x) \in B$. Since $\alpha \notin \lambda_A$, $\alpha_\beta \notin \lambda_{cl_{\beta X}(A)}$ and hence $h(\alpha, x) = (\alpha_\beta, x) \in C$. So $h(B) \subseteq C$. Similarly $C \subseteq h(B)$. Since $h(B) = C$, h is a homeomorphism. Clearly, $z_{\beta X}^0 \circ h = c_X$ □

For any covering map $f : Y \longrightarrow X$ and a subspace S of X, the restriction and corestriction $f^0 : f^{-1}(S) \longrightarrow S$ of f with respect to $f^{-1}(S)$ and S, respectively is a covering map([7]). Hence we have the following :

Corollary 2.7. *For any space X, $c_X : \mathcal{L}(X) \longrightarrow X$ is a covering map.*

Proposition 2.8. *A space X is a cloz-space if and only if $c_X : \mathcal{L}(X) \longrightarrow X$ is a homeomorphism.*

Proof. ($\Rightarrow$) We will show that c_X is an one-to-one map. Let $(\alpha, x) \neq (\gamma, y)$ in $\mathcal{L}(X)$. Suppose that $\alpha \neq \gamma$. Then there are $A, B \in \mathcal{G}(X)$ such that $A \in \alpha$, $B \in \gamma$ and $A \wedge B = \emptyset$. Since X is a cloz-space, $A \wedge B = A \cap B = \emptyset$. Since $x \in A$ and $y \in B$, $x \neq y$ and $c_X(\alpha, x) \neq c_X(\gamma, y)$. Hence c_X is an one-to-one map and since c_X is a covering map, c_X is a homeomorphism.

($\Leftarrow$) Suppose that X is not a cloz-space. Then there are $A, B \in \mathcal{G}(X)$ such that $A \wedge B = \emptyset$ and $A \cap B \neq \emptyset$. Pick $x \in A \cap B$. Then $\alpha = \{G \in \mathcal{G}(X) \mid x \in int_X(G)\} \cup \{A\}$ is a $\mathcal{G}(X)$-filter base and by Zorn's lemma, there is a $\mathcal{G}(X)$-ultrafilter δ such that $\alpha \subseteq \delta$. Suppose that $x \notin \cap\{F \mid F \in \delta\}$. Then there is an $F \in \delta$ such that $x \notin F$.

Since $x \in (X - F) = X - cl_X(X - F') = int_X(F')$, $F' \in \alpha$ and hence $F' \in \delta$. Note that F, $F' \in \delta$ and $F \wedge F' = \emptyset$. Since δ is a $\mathcal{G}(X)$-ultrafilter, this is a contradiction. Hence $x \in \cap\{F \mid F \in \delta\}$.

Similarly, there is a $\mathcal{G}(X)$-ultrafilter γ such that $B \in \gamma$ and $x \in \cap\{D \mid D \in \gamma\}$. Since $(\alpha, x) \neq (\delta, x)$ and $c_X(\delta, x) = c_X(\gamma, x)$, c_X is not one-to-one. Hence c_X is not a homeomorphism. □

Theorem 2.9. *Let X be a space and (Y, f) a cloz-cover of X. Then there is a covering map $g : Y \longrightarrow \mathcal{L}(X)$ such that $c_X \circ g = f$.*

Proof. Let $j_0 : z_{\beta X}^{-1}(X) \longrightarrow E_{cc}(\beta X)$ be the inclusion map. Then $j = j_0 \circ h : \mathcal{L}(X) \longrightarrow E_{cc}(\beta X)$ is a dense embedding and there is a covering map $f^\beta : \beta Y \longrightarrow \beta X$ such that $f^\beta \circ \beta_Y = \beta_X \circ f$. Since Y is a cloz-space, βY is a cloz-space and there is a covering map $k : \beta Y \longrightarrow E_{cc}(\beta X)$ such that $f^\beta = z_{\beta X} \circ k$. Since $\beta_X \circ f = z_{\beta X} \circ k \circ \beta_Y$, there is a continuous map $l : Y \longrightarrow z_{\beta X}^{-1}(X)$ such that $z_{\beta X}^0 \circ l = f$ and $k \circ \beta_Y = j_0 \circ l$. Since f and f^β are covering maps, $f^\beta(\beta Y - Y) \subseteq \beta X - X$ and l is an onto map, because f^β is on onto map([7]). Since $z_{\beta X}^0 \circ l = f$ is a covering map and l is an onto map, l is a covering map. Let $g = h^{-1} \circ l : Y \longrightarrow \mathcal{L}(X)$. Then g is a covering map and $c_X \circ g = f$. □

Corollary 2.10. *Let X be a space such that $\mathcal{L}(X)$ is a cloz-space. Then $(\mathcal{L}(X), c_X)$ is the minimal cloz-cover of X.*

A space X is called *an extremally disconnected space* if every regular closed set in X is open in X.

Let X be a space. Then there is an extremally disconnected space EX and a covering map $k_X : EX \longrightarrow X$ such that for any extremally disconnected space Y and any covering map $g : Y \longrightarrow X$, there is a covering map $h : Y \longrightarrow EX$ such that $k_X \circ h = g$([5]).

For any space X, (EX, k_X) is called *the absolute of X* or *the minimal extremally disconnected cover of X*.

Theorem 2.11. *Every space X has the minimal cloz-cover $(E_{cc}(X), z_X)$.*

Proof. Let $\mathcal{L}_0(X) = X$ and $z_0^0 = 1_X$ be the idetity map on X. Let α be an ordinal. For an ordinal β with $\beta < \alpha$, suppose that

(A) for any ordinal γ with $\gamma \leq \beta$, there is a cover $(\mathcal{L}_\gamma(X), z_0^\gamma)$ of X and that

(B) for any ordinals γ, δ with $\gamma < \delta \leq \beta$, there is covering map $z_\gamma^\delta : \mathcal{L}_\delta(X) \longrightarrow \mathcal{L}_\gamma(X)$ such that $z_0^\delta = z_0^\gamma \circ z_\gamma^\delta$.

Let α be a non-limit ordinal. Then there is an ordinal β with $\alpha = \beta + 1$. Let $\mathcal{L}_\alpha(X) = \mathcal{L}(\mathcal{L}_\beta(X))$ and $z^\alpha_\beta = c_{\mathcal{L}_\beta(X)} : \mathcal{L}(\mathcal{L}_\beta(X)) \longrightarrow \mathcal{L}_\beta(X)$. Then (A) and (B) hold for α.

Let α be a limit ordinal. Let $I = \{\beta \mid \beta \text{ is an ordinal with } \beta < \alpha\}$. Define an inverse limit system $D : I \longrightarrow \underline{TOP}$ as follow : for any ordinal β, γ in I with $\gamma < \beta$, let $D(\beta) = \mathcal{L}_\beta(X)$ and $D(\gamma < \beta) = z^\beta_\gamma$, where $\underline{TOP}$ is the category of topological spaces and continuous maps. Let $(\mathcal{L}_\alpha(X), z^\alpha_\beta)_{\beta<\alpha}$ be the inverse limit of D. Then (A) and (B) hold for α.

By transfinite induction, (A) and (B) hold for all ordinals.

By Theorem 2.9, for any ordinal α, there is a covering map $g_\alpha : EX \longrightarrow \mathcal{L}_\alpha(X)$ such that $z^\alpha_0 \circ g_\alpha = k_X$. Hence for any ordinal α, $\mathcal{L}_\alpha(X)$ lie between X and EX and there is a smallest ordinal δ such that $z^{\delta+1}_\delta : \mathcal{L}(\mathcal{L}_\delta(X)) \longrightarrow \mathcal{L}_\delta(X)$ is a homeomorphism. Since $z^{\delta+1}_\delta = c_{\mathcal{L}_\delta(X)}$, by Proposition 2.8, $\mathcal{L}_\delta(X)$ is a cloz-space. Let $E_{cc}(X) = \mathcal{L}_\delta(X)$ and $z_X = z^\delta_0$. Then $(E_{cc}(X), z_X)$ is the minimal cloz-cover of X. □

Let $f : Y \longrightarrow X$ be a covering map. Then f is called *a $z^\#$-irreducible map* if $\{f(A) \mid A \in Z(Y)^\#\} = Z(X)^\#$. Note that for any $B \in Z(X)^\#$,

$$cl_Y(f^{-1}(int_X(B))) \in Z(Y)^\# \text{ and } f(cl_Y(f^{-1}(int_X(B)))) = B.$$

Hence f is $z^\#$-irreducible if and only if for any $A \in Z(Y)^\#$, $f(A) \in Z(X)^\#$.

A space X is called *a quasi-F space* if for any A, $B \in Z(X)^\#$, $A \wedge B = A \cap B$. For any compact space X, there is a quasi-F space $QF(X)$ and a $z^\#$-irreducible map $\Phi_X : QF(X) \longrightarrow X$([3]).

Let X be a space. Since every quasi-F space is a cloz-space, $(QF(\beta X), \Phi_{\beta X})$ is a cloz-cover of βX and so there is a covering map $m_{\beta X} : QF(\beta X) \longrightarrow E_{cc}(\beta X)$ such that $z_{\beta X} \circ m_{\beta X} = \Phi_{\beta X}$. Clearly, $z_{\beta X}$ and $m_{\beta X}$ are $z^\#$-irreducible, because $\Phi_{\beta X}$ is $z^\#$-irreducible. Moreover, $\{z_{\beta X}(A) \mid A \in Z(\mathcal{G}(\beta X))^\#\} = \mathcal{G}(\beta X)^\#$.

Recall that a subspace S of a space X is called *C^*-embedded ($z^\#$-embedded, resp.) in* X if for any real-valued continuous map f on S ($A \in Z(S)^\#$, resp.), there is a real-valued continuous map g on X ($B \in Z(X)^\#$, resp.) such that $g\,|_X = f(A = B \cap S$, resp.). Every dense C^*-embedded subspace of a space X is $z^\#$-embedded in X.

Consider the following conditions for a space X

(C_1) $\beta E_{cc}(X) = E_{cc}(\beta X)$, that is, $E_{cc}(X)$ is C^*-embedded in $E_{cc}(\beta X)$

(C_2) $\mathcal{L}(X)$ is $z^\#$-embedded in $E_{cc}(\beta X)$.

(C_3) $\{z_X(A) \mid A \in (\mathcal{G}(\mathcal{L}(X)))\} = \mathcal{G}(X)$.

Proposition 2.12. *Let X be a space. Then (C_1) implies (C_2) and (C_2) implies (C_3). Moreover, if (C_3) holds, then $(\mathcal{L}(X), c_X)$ is the minimal cloz-cover of X.*

Proof. (C_1) $(\Rightarrow)$ (C_2) By Theorem 2.9, there is a covering map $g : E_{cc}(X) \longrightarrow \mathcal{L}(X)$ such that $c_X \circ g = z_X$. Note that $z_{\beta X}$ is $z^\#$-irreducible and $E_{cc}(X)$ is $z^\#$-embedded in $E_{cc}(\beta X)$. Hence z_X is $z^\#$-irreducible and c_X is $z^\#$-irreducible. Since c_X is $z^\#$-irreducible, $\mathcal{L}(X)$ is $z^\#$-embedded in $E_{cc}(\beta X)$.

(C_2) $(\Rightarrow)$ (C_3) Let $F \in \mathcal{G}(\mathcal{L}(X))$. Then F, $F' \in Z(\mathcal{L}(X))^\#$ and since $\mathcal{L}(X)$ is $z^\#$-embedded in $E_{cc}(\beta X)$, there is an $H \in \mathcal{G}(E_{cc}(\beta X))$ such that $F = H \cap \mathcal{L}(X)$ and $F' = H' \cap \mathcal{L}(X)$. Since $z_{\beta X}$ is $z^\#$-irreducible, $z_{\beta X}(H) \in \mathcal{G}(\beta X)$ and since

$$z_X(F) = z_X(H \cap E_{cc}(X)) = z_{\beta X}(H) \cap X,$$

$z_X(F) \in \mathcal{G}(X)$. Hence $\{z_X(A) \mid A \in \mathcal{G}(\mathcal{L}(X))\} \subseteq \mathcal{G}(X)$ and clearly, $\mathcal{G}(X) \subseteq \{z_X(A) \mid A \in \mathcal{G}(\mathcal{L}(X))\}$.

Suppose that (C_3) holds. Let $S \in \mathcal{G}(\mathcal{L}(X))$. Then there is an $A \in \mathcal{G}(\beta X)^\#$ sush that $z_X(S) = A \cap X$. Since $z_{\beta X}$ is $z^\#$-ireducible, there is a $B \in \mathcal{G}(E_{cc}(\beta X))$ such that $z_{\beta X}(B) = A$. By Lemma 2.1, $S = B \cap \mathcal{L}(X)$. Since $E_{cc}(\beta X)$ is a cloz-space, B is a clopen set in $E_{cc}(\beta X)$ and $S = B \cap \mathcal{L}(X)$ is a clpoen set in $\mathcal{L}(X)$. Hence $\mathcal{L}(X)$ is a cloz-space. □

A space X is called *a stonrgly zero-dimensional space* if βX is a zero-dimensional space.

Theorem 2.13. *Let X be a strongly zero-dimensional space. Then $\beta E_{cc}(X) = E_{cc}(\beta X)$ if and only if $E_{cc}(X)$ is $z^\#$-embedded in $E_{cc}(\beta X)$.*

Proof. $(\Leftarrow)$ Note that there is a continuous map $f : \beta E_{cc}(X) \longrightarrow \beta X$ such that $\beta_X \circ z_X = f \circ \beta_{\beta E_{cc}(X)}$. Since $E_{cc}(X)$ is dense in $\beta E_{cc}(X)$ and $\beta E_{cc}(X)$ is a compact space, f is a covering map and since $\beta E_{cc}(X)$ is a cloz-space, there is a covering map $g : \beta E_{cc}(X) \longrightarrow E_{cc}(\beta X)$ such that $z_{\beta X} \circ g = f$. Since $E_{cc}(X)$ is $z^\#$-embedded in $E_{cc}(\beta X)$, z_X is $z^\#$-irreducible. Clearly, f is $z^\#$-irreducible, because $\beta_X \circ z_X = f \circ \beta_{\beta E_{cc}(X)}$. Since $z_{\beta X} \circ g = f$ is $z^\#$-irreducible, g is $z^\#$-irreducible.

Take any $p \neq q$ in $\beta E_{cc}(X)$. Since βX is a zero-dimensional space and f is a covering map, $\beta E_{cc}(X)$ is a zero-dimensinal space([7]). Hence there is a clopen set B in $\beta E_{cc}(X)$ such that $p \in B$ and $q \notin B$. Hence $B \in \mathcal{G}(\beta E_{cc}(X))$ and $g(B) \in \mathcal{G}(E_{cc}(\beta X))$. Since $E_{cc}(\beta X)$ is a cloz-space, $g(B) \cap g(B') = \emptyset$, $g(p) \in g(B)$ and $q \in g(B')$. Since $g(p) \neq g(q)$, g is a homeomorphism.

$(\Rightarrow)$ It is trivial. □

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