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$h-$STABILITY OF PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems.

1. Introduction

The notion of h-stability (hS) was introduced by Pinto [12, 14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. Also, he obtained some properties about asymptotic behavior of solutions of perturbed h-systems, some general results about asymptotic integration and gave some important examples in [13]. Choi and Ryu [3] investigated the important properties about hS for the various differential systems. Recently, Choi et al. [4] and Goo [7] obtained results for hS of nonlinear differential systems via t_∞-similarity. Goo et al. [7, 8] investigated hS for the nonlinear Volterra integro-differential system and for the linear perturbed Volterra integro-differential systems.

In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems .

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. Let $x(t, t_0, x_0)$ denote the

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unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $J = [t_0, \infty)$. Then we consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \ v(t_0) = v_0 \tag{2.2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0. \tag{2.3}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.2).

We recall some notions of h-stability [12] and the notion of t_∞-similarity [9].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called
(hS) $h-$*stable* if there exist $c \geq 1$, $\delta > 0$, and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$,
(hSV) $h-$*stable in variation* if (2.3) (or $z = 0$ of (2.3)) is h-stable.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+ = [0, \infty)$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

Definition 2.2. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 9].

We give some related properties that we need in the sequal.

Lemma 2.3 ([14]). *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{2.5}$$

where $A(t)$ is an $n \times n$ continuous matrix, is hS if and only if there exist $c \geq 1$ and a positive bounded continuous function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0, x_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.6}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0, x_0)$ is a fundamental matrix of (2.5).

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y, Ty), \ y(t_0) = y_0, \tag{2.7}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.4. *If $y_0 \in \mathbb{R}^n$, for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 2.5 ([2, 14]). *If the zero solution of (2.1) is hS, then the zero solution of (2.2) is hS.*

Theorem 2.6 ([4]). *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.2) is hS, then the solution $z = 0$ of (2.3) is hS.*

The following comparison results are well-known.

Lemma 2.7 ([11]). *Let $u(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_0^s g(\tau)u(\tau)d\tau)ds, \ t \in \mathbb{R}^+,$$

holds, where u_0 is a nonnegative constant. Then,

$$u(t) \leq u_0(1 + \int_0^t f(s)\exp(\int_0^s (f(\tau) + g(\tau))d\tau))ds, \ t \in \mathbb{R}^+.$$

We introduce a few of the basic notions involved. Let $C(\mathbb{R}^+)$ denote the space of continuous functions $u \in C[\mathbb{R}^+, \mathbb{R}^+]$ and T be a continuous operater such that T maps $C(\mathbb{R}^+)$ into $C(\mathbb{R}^+)$, in our subsequent discussion it is assumed that, for any two continuous function $u, v \in C[\mathbb{R}^+, \mathbb{R}^+]$ the operator T satisfies the following property:

$$u(t) \leq v(t),\ 0 \leq t \leq t_1,\ t_1 \in \mathbb{R}^+$$

implies

$$Tu(t) \leq Tv(t),\ t = t_1.$$

and

$$|Tu| \leq T|u|.$$

Lemma 2.8 ([3]). *Suppose that $r(t,u,v) \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$ is monotone nondecreasing in u and v for fixed $t \in \mathbb{R}^+$ satisfying*

$$m(t) - \int_{t_0}^{t} r(s, m(s), Tm(s))ds \leq k(t) - \int_{t_0}^{t} r(s, k(s), Tk(s))ds,$$

for $t \geq t_0 \geq 0$ and $m, k \in C[\mathbb{R}^+, \mathbb{R}^+]$. If $m(t_0) \leq k(t_0)$, then $m(t) < k(t)$, for all $t \geq t_0 \geq 0$.

3. Main Results

In this section, we investigate hS for the nonlinear perturbed differential systems.

Theorem 3.1. *Suppose that the solution $x = 0$ of (2.1) is hS with the nondecreasing function h and the perturbed term g in (2.7) satisfies*

$$|\Phi(t,s,z)g(s,y,z)| \leq \gamma(s)(|y| + |z|),\ t \geq t_0 \geq 0,$$

where $\gamma \in C[\mathbb{R}^+, \mathbb{R}^+]$ and $\int_{t_0}^{\infty} \gamma(s)ds < \infty$. Further, suppose that the operator T satisfies the inequality

$$|Ty(t)| \leq \int_{t_0}^{t} q(s)|y(s)|ds,$$

where $q \in C[\mathbb{R}^+, \mathbb{R}^+]$ and $\int_{t_0}^{\infty} q(s)ds < \infty$. Then $y = 0$ of (2.7) is hS.

Proof. Using the nonlinear variation of constants formula of Alekseev[1], the solutions of (2.1) and (2.7) with the same initial values are related by

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \Phi(t,s,y(s))\, g(s,y(s),T(s))\, ds.$$

By the hypotheses and the nondecreasing property of the function h

$$\begin{aligned}|y(t) &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))g(s,y(s),T(s))|ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} \gamma(s)(h(t)h(s)^{-1}|y(s)| \\ &+ \int_{t_0}^{s} q(\tau)h(t)h(\tau)^{-1}|y(\tau)|d\tau)ds.\end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, it follows from Lemma 2.7 that

$$\begin{aligned}|y(t)| &\leq c_1|y_0|h(t)h(t_0)^{-1}(1 + \int_{t_0}^{t} \gamma(s)\exp(\int_{t_0}^{s}(\gamma(\tau)+q(\tau))d\tau)ds) \\ &\leq c|y_0|h(t)h(t_0)^{-1},\ t \geq t_0,\end{aligned}$$

where $c = c_1(1+\int_{t_0}^{\infty}\gamma(s)\exp(\int_{t_0}^{\infty}(\gamma(\tau)+q(\tau))d\tau)ds)$. Hence, $y = 0$ of (2.7) is hS. □

Corollary 3.2. *Suppose that the solution $x = 0$ of (2.1) is hSV with a nondecreasing function h, and for all $t \geq t_0 \geq 0$,*

$$|\Phi(t,s,z)\, g(s,y,Ty)| \leq \gamma(s)(|y| + |Ty|),$$

and

$$|Ty| \leq \int_{t_0}^{t} q(s)|y(s)|ds,$$

where $\gamma, q \in C[\mathbb{R}^+, \mathbb{R}^+]$, $\int_{t_0}^{\infty}\gamma(s)ds < \infty$, and $\int_{t_0}^{\infty} q(s)ds < \infty$. Then, $y = 0$ of (2.7) is hS.

Proof. It follows from hypothesis that the solution $z = 0$ of (2.3) is hS. Thus, the solution $x = 0$ of (2.1) is hS. Hence, by Theorem 3.1, the solution $y = 0$ of (2.7) is hS. This completes the proof. □

Remark 3.3. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is hS, then the perturbed system

$$y' = A(t)y + g(t,y,Ty),\ y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.1.

We also examine the properties of hS for the perturbed system

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s),Ty(s))ds,\ y(t_0) = y_0, \tag{3.1}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $g(t,0,0) = 0$.

Theorem 3.4. *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.1) is hS with the increasing function h and g in (3.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| \leq \gamma(s)(|y| + |Ty|),\ t \geq t_0 \geq 0,$$

and

$$|Ty| \leq \int_{t_0}^{t} q(s)|y(s)|ds$$

where $\gamma, q \in C[\mathbb{R}^+, \mathbb{R}^+]$, $\int_{t_0}^{\infty} \gamma(s)ds < \infty$, and $\int_{t_0}^{\infty} q(s)ds < \infty$. Then, the solution $y = 0$ of (3.1) is hS.

Proof. Let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,x_0)$. By Theorem 2.5, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.2) is hS. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.3) is hS. By Lemma 2.4 and the increasing property of h, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)d\tau \right| ds \\ &\leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\,h(s)^{-1}\gamma(s)(|y(s)| \\ &\quad + \int_{t_0}^{s} q(\tau)|y(\tau)|d\tau)ds \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Gronwall's inequality, we obtain

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)\,h(t_0)^{-1} \exp c_2 \int_{t_0}^{t} \gamma(s)(1 + \int_{t_0}^{s} q(\tau)d\tau))ds \\ &\leq c|y_0|h(t)\,h(t_0)^{-1}, \quad c = c_1 \exp c_2 \int_{t_0}^{\infty} \gamma(s)(1 + \int_{t_0}^{\infty} q(\tau)d\tau)ds. \end{aligned}$$

It follows that $y = 0$ of (3.1) is hS. Hence, the proof is complete. □

Remark 3.5. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s, y(s), Ty(s))ds,\ y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.4 except the condition of t_∞-similarity.

Theorem 3.6. *For the system (3.1), suppose that*

$$\left| \int_{t_0}^{t} g(\tau, y(\tau), Ty(\tau)) d\tau \right| \leq r(t, |y|, |Ty|),$$

where $r \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$ *is strictly increasing in u,v for each fixed* $t \geq t_0 \geq 0$ *with* $r(t, 0, 0) = 0$. *Assume also that* $x = 0$ *of (2.1) is* hSV *with the nonincreasing function h. Consider the scalar differential equation*

$$u' = cr(t, u, Tu), \ u(t_0) = u_0 = c|y_0|. \tag{3.2}$$

If $u = 0$ *of (3.2) is hS, then* $y = 0$ *of (3.1) is also hS whenever* $u_0 = c|y_0|$.

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Lemma 2.4, we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)) d\tau \right| ds,$$

where $\Phi(t, s, y(s))$ is the fundamental matrix of (2.3). Then, by assumptions, we obtain

$$\begin{aligned} |y(t)| &\leq c|y_0| h(t)\, h(t_0)^{-1} + c \int_{t_0}^{t} h(t)\, h(s)^{-1} \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)) d\tau \right| ds \\ &\leq c|y_0| + c \int_{t_0}^{t} r(s, |y(s)|, |Ty(s)|) ds \end{aligned}$$

since $h(t)$ is nonincreasing. Thus we have

$$|y(t)| - c \int_{t_0}^{t} r(s, |y(s)|, |Ty(s)|) ds \leq c|y_0| = u_0 = u(t) - c \int_{t_0}^{t} r(s, u(s), Tu(s)) ds.$$

By Lemma 2.8, we get $|y(t)| < u(t)$ for all $t \geq t_0 \geq 0$. In view of assumption, since $u = 0$ of (3.2) is hS,

$$\begin{aligned} |y(t)| &< u(t) \leq c_1 |u_0| h(t)\, h(t_0)^{-1} \\ &= c_1 c |y_0| h(t)\, h(t_0)^{-1} = M |y_0| h(t) h(t_0)^{-1}, \ M = c_1 c > 1. \end{aligned}$$

This completes the proof. □

Remark 3.7. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is hS, then the perturbed system

$$y' = A(t) y + \int_{t_0}^{t} g(s, y(s), Ty(s)) ds, \ y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.6.

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REFERENCES

1. F. Brauer: Perturbations of nonlinear systems of differential equations II. *J. Math. Anal. Appl.* **17** (1967), 579–591.
2. S.K. Choi & H.S. Ryu: $h-$stability in differential systems. *Bull. Inst. Math. Acad. Sinica* **21** (1993), 245–262.
3. S.K. Choi, N.J. Koo & S.M. Song: Stabilities for nonlinear functional differential equations. *J. Chungcheong Math. Soc.* **9** (1996), 165-174.
4. S.K. Choi, N.J. Koo & H.S. Ryu: h-stability of differential systems via t_∞-similarity. *Bull. Korean. Math. Soc.* **34** (1997), 371–383.
5. R. Conti: Sulla t_∞-similitudine tra matricie l'equivalenza asintotica dei sistemi differenziali lineari. *Rivista di Mat. Univ. Parma* **8** (1957), 43–47.
6. Y.H. Goo: h-stability of the nonlinear differential systems via t_∞-similarity. *J. Chungcheong Math. Soc.* **23** (2010), 383-389.
7. Y.H. Goo & D.H. Ry: h-stability for perturbed integro-differential systems. *J. Chungcheong Math. Soc.* **21** (2008), 511–517.
8. Y.H. Goo, M.H. Ji & D.H. Ry: h-stability in certain integro-differential equations. *J. Chungcheong Math. Soc.* **22** (2009), 81–88.
9. G.A. Hewer: Stability properties of the equation by t_∞-similarity. *J. Math. Anal. Appl.* **41** (1973), 336–344.
10. V. Lakshmikantham & S. Leela: *Differential and Integral Inequalities: Theory and Applications Vol. I.* Academic Press, New York and London, 1969.
11. B.G. Pachpatte: A note on Gronwall-Bellman inequality. *J. Math. Anal. Appl.* **44** (1973), 758-762.
12. M. Pinto: Perturbations of asymptotically stable differential systems. *Analysis* **4** (1984), 161–175.
13. ______: Asymptotic integration of a system resulting from the perturbation of an h-system. *J. Math. Anal. Appl.* **131** (1988), 194-216.
14. ______: Stability of nonlinear differential systems. *Applicable Analysis* **43** (1992), 1–20.

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