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A REFINEMENT OF LYAPUNOV-TYPE INEQUALITY FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract. Some new Lyapunov-type inequalities for a class of nonlinear differential systems, which are natural refinements and generalizations of the well-known Lyapunov inequality for linear second order differential equations, are given. The results of this paper cover some previous results on this topic.

1. Introduction

As is well-known, the Lyapunov inequality [1] for the following second-order linear differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1}$$

states that if $q \in C[a, b]$ and $x(t)$ is a solution of (1) such that $x(a) = x(b) = 0$, $x(t) \neq 0$ for $t \in (a, b)$, then the following inequality holds:

$$(b-a)\int_a^b q^+(t)dt > 4, \tag{2}$$

where $q^+(t) = \max\{q(t), 0\}$ and the constant 4 is sharp, which means that it can not be replaced by a larger number. In [5], Hartman obtained the following inequality:

$$\int_a^b q^+(t)(t-a)(b-t)dt > (b-a), \tag{3}$$

which implies (2) since for $t \in [a, b]$, we have

$$(t-a)(b-t) \leq \frac{(b-a)^2}{4}.$$

Over the past few decades, there have been many new proofs and generalizations of the inequality (2). It has been generalized to nonlinear second order equations

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[3, 10, 11], to delay differential equations[2], to higher order differential equations [9, 13, 14], to discrete linear Hamiltonian systems [4], and so on [6, 7, 8, 12].

In this paper, inspired by the work of Hartman [5], we obtain some new Lyapunov-type inequalities which cover (2) and (3) and some previous results on this topic.

2. Main Result and its Proof

Consider the following nonlinear differential system:

$$\begin{aligned} x' &= a_1(t)x + a_2(t)\phi_{p'}(y), \\ y' &= -a_3(t)\phi_p(x) - a_1(t)y, \end{aligned} \tag{4}$$

where $a_k \in C([a,\, b], \mathbb{R})$ for $1 \le k \le 3$, $a_2(t) > 0\,\forall\, t \in [a,\, b]$, $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and $p' = \frac{p}{p-1} > 1$ is the exponent conjugate to p.

The main result of this paper is the following theorem:

Theorem 1. *If $x(t)$ is a nonzero solution of (4) such that $x(a) = x(b) = 0$ and $x(t) \neq 0\,\forall\, t \in (a,\, b)$, then the following inequalities hold:*

$$\begin{aligned} &4^{1-p}\left(\int_a^b a_3^+(t)e^{pA_1(t)}dt\right)^2\left(\int_a^b g(t)dt\right)^{2(p-1)} \\ &> \int_a^b a_3^+(t)e^{pA_1(t)}dt\int_a^b a_3^+(t)e^{pA_1(t)}\left(\int_a^t g(s)ds\int_t^b g(s)ds\right)^{p-1}dt \\ &> 4, \end{aligned} \tag{5}$$

where

$$A_1(t) = \int_a^t a_1(s)ds, \quad g(t) = e^{-p'A_1(t)}a_2(t).$$

If $a_1(t) \equiv 0$ in the system (4), then (4) reduces to the following nonlinear second order equation:

$$\big(r(t)\phi_p(x')\big)' + q(t)\phi_p(x) = 0, \tag{6}$$

where $r(t) = \frac{1}{a_2^{p-1}(t)}$, $q(t) = a_3(t)$. Hence, the inequalities in (5) reduce to the inequalities in the following Corollary 1.

Corollary 1. *If $x(t)$ is a solution of (6) such that $x(a) = x(b) = 0$ and $x(t) \neq 0\,\forall\, t \in (a,\, b)$, then we have*

$$
\begin{aligned}
&4^{1-p}\left(\int_a^b q^+(t)dt\right)^2\left(\int_a^b \frac{dt}{r^{\frac{1}{p-1}}(t)}\right)^{2(p-1)} \\
&> \int_a^b q^+(t)dt\int_a^b q^+(t)\left(\int_a^t \frac{ds}{r^{\frac{1}{p-1}}(s)}\int_t^b \frac{ds}{r^{\frac{1}{p-1}}(s)}\right)^{p-1} dt \\
&> 4.
\end{aligned}
\tag{7}
$$

In particular, if $r(t)\equiv 1$ in the equation (6), then (6) reduces to the following equation

$$
(\phi_p(x'))' + q(t)\phi_p(x) = 0, \tag{8}
$$

and the inequalities in (7) further reduce to the inequalities in the following Corollary 2:

Corollary 2. *If $x(t)$ is a solution of (8) such that $x(a) = x(b) = 0$ and $x(t)\neq 0\,\forall\, t\in(a,\, b)$, then we have*

$$
\begin{aligned}
&4^{1-p}\left(\int_a^b q^+(t)dt\right)^2 (b-a)^{2(p-1)} \\
&> \int_a^b q^+(t)dt\int_a^b q^+(t)\left[(t-a)(b-t)\right]^{p-1} dt \\
&> 4.
\end{aligned}
\tag{9}
$$

Remark 1. Note that if $p = 2$, then (8) reduces to the equation (1). Thus, the inequalities (5), (7) and (9) are natural refinements and generalizations of the inequalities (2) and (3).

Proof of Theorem 1. Multiplying the first equation of (4) by $y(t)$ and the second one by $x(t)$, and adding the results, we get

$$
(x(t)y(t))' = a_2(t)|y(t)|^{p'} - a_3(t)|x(t)|^p.
$$

Integrating the above equation from a to b and using $x(a) = x(b) = 0$, we obtain

$$
\int_a^b a_2(t)|y(t)|^{p'}dt = \int_a^b a_3(t)|x(t)|^p dt. \tag{10}
$$

For any $t\in(a,\, b)$, by using the assumption $x(a) = x(b) = 0$, we obtain from (4) that

$$
\begin{aligned}
x(t) &= e^{\int_a^t a_1(s)ds}\int_a^t e^{-\int_a^s a_1(\tau)d\tau} a_2(s)\phi_{p'}(y(s))ds \\
&= -e^{\int_a^t a_1(s)ds}\int_t^b e^{-\int_a^s a_1(\tau)d\tau} a_2(s)\phi_{p'}(y(s))ds.
\end{aligned}
\tag{11}
$$

Now applying the Hölder inequality to the first equation of (11), we get

$$|x(t)| \le e^{\int_a^t a_1(s)ds} \left(\int_a^t e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{\frac{1}{p'}} \left(\int_a^t a_2(s)|y(s)|^{p'} ds \right)^{\frac{1}{p}}. \tag{12}$$

Similarly, from the second equation of (11) we have

$$|x(t)| \le e^{\int_a^t a_1(s)ds} \left(\int_t^b e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{\frac{1}{p'}} \left(\int_t^b a_2(s)|y(s)|^{p'} ds \right)^{\frac{1}{p}}. \tag{13}$$

Let $d \in (a, b)$ be any fixed number. Then by using (12), we obtain

$$\begin{aligned} &\int_a^d a_3^+(t)|x(t)|^p dt \\ &\le \int_a^d a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} \int_a^t a_2(s)|y(s)|^{p'} dsdt \\ &\le \int_a^d a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} dt \int_a^d a_2(t)|y(t)|^{p'} dt. \end{aligned} \tag{14}$$

Similarly by using (13), we obtain

$$\begin{aligned} &\int_d^b a_3^+(t)|x(t)|^p dt \\ &\le \int_d^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} \int_t^b a_2(s)|y(s)|^{p'} dsdt \\ &\le \int_d^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} dt \int_d^b a_2(t)|y(t)|^{p'} dt. \end{aligned} \tag{15}$$

It is easy to see that the function

$$h_1(x) = \int_a^x a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} dt$$

is nondecreasing for $x \in (a, b)$ and the function

$$h_2(x) = \int_x^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} dt$$

is nonincreasing for $x \in (a, b)$. Now since $h_1(a) = h_2(b) = 0$, $h_1(b) > 0$ and $h_2(a) > 0$, it follows that there exists at least one $c \in (a, b)$ such that $h_1(c) = h_2(c) > 0$, that is,

$$\begin{aligned} &\int_a^c a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} dt \\ &= \int_c^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p' \int_a^s a_1(\tau)d\tau} a_2(s)ds \right)^{p-1} dt. \end{aligned} \tag{16}$$

Let $d = c \in (a, b)$ in (14) and (15). Then (16) holds and by adding (14) and (15), and using (10), we obtain the following:

$$\begin{aligned}
&\int_a^b a_3^+(t)|x(t)|^p dt \\
&< \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt \int_a^c a_2(t)|y(t)|^{p'}dt \\
&\quad + \int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt \int_c^b a_2(t)|y(t)|^{p'}dt \\
&= \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt \int_a^b a_2(t)|y(t)|^{p'}dt \\
&= \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt \int_a^b a_3(t)|x(t)|^{p}dt \\
&\le \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt \int_a^b a_3^+(t)|x(t)|^{p}dt.
\end{aligned} \tag{17}$$

The first strict inequality in the above inequalities holds since $x(t)$ is not a constant solution (zero solution) of (4) and hence at least one inequalities in (14) or (15) is strict. From the equation (10), we have

$$\int_a^b a_3^+(t)|x(t)|^p dt > 0,$$

which, together with (17), yields

$$\begin{aligned}
1 &< \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt \\
&= \int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1} dt.
\end{aligned} \tag{18}$$

From (18), we obtain

$$\begin{aligned}
1 &< \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}dt\left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1} \\
1 &< \int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt\left(\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1},
\end{aligned} \tag{19}$$

which implies that

$$\frac{1}{\displaystyle\int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}dt} < \left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}$$

(20)

$$\frac{1}{\displaystyle\int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt} < \left(\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}.$$

It follows from the first inequality of (18) that

(21)

$$\int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt$$

$$\geq \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}dt$$

$$> \left(\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}.$$

Similarly, from the second inequality of (18) we can show that

(22)

$$\int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt$$

$$\geq \int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt$$

$$> \left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}.$$

Adding (21) and (22), and applying (20), we obtain

(23)

$$\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt$$

$$> \left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1} + \left(\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}$$

$$> \frac{1}{\displaystyle\int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}dt} + \frac{1}{\displaystyle\int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt}$$

$$\geq \frac{4}{\displaystyle\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt}.$$

Since $AB \leq \frac{(A+B)^2}{4}$ for any real numbers A and B, we have

$$\begin{aligned}
&\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds \\
&\le \frac{1}{4}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds+\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^2 \qquad (24)\\
&=\frac{1}{4}\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^2.
\end{aligned}$$

Substituting (24) into (23), we get finally the result of Theorem 1

$$\begin{aligned}
&4^{1-p}\left(\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt\right)^2\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{2(p-1)}\\
&>\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t g(s)ds\int_t^b g(s)ds\right)^{p-1}dt\\
&>4,
\end{aligned}$$

where

$$g(s)=e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s).$$

$\square$

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