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SOME RESULTS ON MONOGENIC (R, S)-GROUPS

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Abstract. In this paper, we denote that R is a near-ring and G an R-group. We initiate the study of the substructures of R and G. Next, we investigate some properties of R-groups, d.g. near-rings and monogenic (R, S)-groups.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in R. We note that obviously, $a0 = 0$ and $a(-b) = -ab$ for all a, b in R, but in general, $0a \neq 0$ and $(-a)b \neq -ab$.

If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R.

For example, if R is a near-ring with unity 1, then 0 and 1 are clearly distributive elements.

We consider the following substructures of near-rings: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of R,

$$R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a,\ for\ all\ r \in R\} = \{0a \in R \mid a \in R\}$$

which is called the *constant part* of R, and $R_d = \{a \in R \mid a\ is\ distributive\}$ which is called the *distributive part* of R.

A non-empty subset S of a near-ring R is said to be a *subnear-ring* of R, if S is a near-ring under the operations of R, equivalently, for all a, b in S, $a - b \in S$ and $ab \in S$. This is called a *criterion of subnear-rings*. Sometimes, we denote it by $S < R$.

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We note that R_0 and R_c are subnear-rings of R, on the other hand, R_d is a multiplicative semigroup of R (see following Lemma 2.2), but not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

Let $(G, +)$ be a group (not necessarily abelian). We may obtain some useful examples of near-rings as following, there are three kinds of trivial near-rings:

Example 1.1. Let G be an additive group.

(1) If we define a multiplication on G by $xy = 0$ for all x, y in G, then $(G, +, \cdot)$ becomes a near-ring, which is called the trivial near-ring on G.
(2) If we define a multiplication on G by $xy = y$ for all x, y in G, then $(G, +, \cdot)$ becomes a near-ring, because $(xy)z = z = x(yz)$ and $x(y + z) = y + z = xy + xz$, for all x, y, z in G, but in general, $0x = 0$ and $(x + y)z = xz + yz$ are not true. This kind of near-ring is constant near-ring, which is called the *trivial constant near-ring on G.*
(3) If we define a multiplication on G by $xy = 0$ if $x = 0$, $= y$ otherwise in G, then $(G, +, \cdot)$ becomes a zero symmetric near-ring, which is called the *trivial zero symmetric near-ring on G.*

Next is in [6], in the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* on G.

Also, we can define the substructures of $(M(G), +, \cdot)$ as following: $M_0(G) = \{f \in M(G) \mid 0f = 0\}$ and $M_c(G) = \{f \in M(G) \mid f \textit{ is constant}\}$. Then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring. Moreover, $M_0(G) = M(G)_0$ and $M_c(G) = M(G)_c$.

The ideas in Example 1.1 inspire the following notation which we will use.

Example 1.2. Let G be an additive group.

(1) If $x \in G$, then we define $\alpha_x \in M(G)$ by $g\alpha_x = x$ for all $g \in G$.
(2) If $x \in G$, then we define $\beta_x \in M_0(G)$ by $o\beta_x = o$ and $g\beta_x = x$ for all $g \in G - \{o\}$.

Note that $\alpha_o \in M_0(G)$, but α_x is not in $M_0(G)$ for $x \neq o$. The following result ties this in with Example 1.1.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $(a+b)\theta = a\theta + b\theta$, (ii) $(ab)\theta = a\theta b\theta$. Obviously, $R\theta < S$ and $T\theta^{-1} < R$ for any $T < S$.

We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as in ring theory [1]. If θ from R to S is a near-ring isomorphism, then we say that R *is isomorphic to* S, and denoted it by $R \cong S$.

Proposition 1.3. *Let G be an additive group, and let $x \in G$. Then*

$$\gamma\alpha_x = \alpha_x, \forall \gamma \in M(G).$$

Also, $\{\alpha_x | x \in G\} < M(G)$ and isomorphic to the trivial constant near-ring on G, and $\{\beta_x | x \in G\} < M_0(G)$ and isomorphic to the trivial zero symmetric near-ring on G

Proof. Because of the definition of α_x, for any $t \in G$,

$$t\gamma\alpha_x = x = t\alpha_x, \forall \gamma \in M(G),$$

we see that $\gamma\alpha_x = \alpha_x$. So the first part is easily proved.

Next, since for any α_x and α_y in the given set, $\alpha_x - \alpha_y = \alpha_{x-y}$ and $\alpha_x\alpha_y = \alpha_y$, from the 1st part. Hence by the criterion of subnear-ring, $\{\alpha_x | x \in G\} < M(G)$. Similarly, we can prove that $\{\beta_x | x \in G\} < M_0(G)$.

Finally, the two maps which are given by $\alpha_x \to x,\ \beta_x, \to x$, are clearly isomorphisms. □

We can check that $\{\alpha_x | x \in G\}$ is the unique maximal constant subnear-ring of $M(G)$, so we see that $\{\alpha_x | x \in G\} = M_c(G)$.

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G. We denote it by G_R.

We may write that xr (as a scalar product in G) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (if R has a unity 1), for all $x \in G$ and $a,\ b \in R$.

Naturally, we can define a new concept of R-group: An R-group G is called *distributive*, in case $(x+y)a = xa + ya$, for all $x, y \in G$ and for each $a \in R$. For example, every distributive near-ring R is a distributive R-group.

Evidently, every near-ring R can be given the structure of an R-group (unitary, if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

Let R be a near-ring, and let G and H be two R-groups. Then a mapping θ from G to H is called an *R-group homomorphism, or R-homomorphism* if

(i) $(x+y)\theta = x\theta + y\theta$, (ii) $(xa)\theta = (x\theta)a$.

We can replace R-homomorphism by R-monomorphism, R-epimorphism, R-isomorphism, R-endomorphism and R-automorphism, if these terms have their usual meanings as in module theory [1]. If θ from G to H is an isomorphism, then we say that *G is R-isomorphic to H*, and denoted it by $G \cong_R H$.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G.

For the remainder concepts and results on near-rings, we refer to G. Pilz [6].

2. Some Properties of (R, S)-groups

Distributive near-rings, which are near-rings satisfying both distributive laws, are very close to rings. In the mean time, we consider a larger class of near-rings which has a lot of distributivity built in.

In the period 1958-1962, A. Frohlich published some papers on distributively generated near-rings [3], [4], [5]. These mark the real beginning of these subjects.

A near-ring R is called a *distributively generated near-ring*, denoted by *d.g. near-ring*, if $(R, +)$ is generated as a group by a semigroup $(S, \cdot)$ of distributive elements.

A d.g. near-ring R which is generated by a semigroup S is denoted by (R, S).

Rings are special cases of d.g. near-rings, because of all elements of a ring are distributive.

Example 2.1. Let G be an additive group.

(1) The trivial near-ring on a group G has all its elements distributive.
(2) Consider the set $EndG = \{\alpha | \alpha \textit{ is an endomorphism of } \mathrm{G}\}$. Then $EndG \subseteq M_0(G)$, and $EndG$ consists of distributive elements of $M_0(G)$.

Lemma 2.2. *Let R be an near-ring. The set R_d of all distributive elements of R forms a multiplicative semigroup.*

Proof. Let $x,\ y \in R$ and $a,\ b \in R_d$. Then

$$(x+y)ab = (xa+ya)b = (xa)b + (ya)b = x(ab) + y(ab).$$

□

The proof of the following result is very similar to that of ring theorestic result.

Lemma 2.3. *Let R be a near-ring (not necessarily zero symmetric), and let $x \in R_d$. Then*

$$0x = 0,\ (-y)x = -yx\ \forall y \in R.$$

Proposition 2.4. *Let G be an additive group. Then*

$$M_0(G)_d = EndG.$$

Proof. We see that $EndG \subseteq M_0(G)_d$ is immediate from the definition of near-ring homomorphism. So consider that $f \in M_0(G)_d$. Since f is distributive, using Example 1.2, for any $x,\ y \in G$,

$$(\beta_x + \beta_y)f = \beta_x f + \beta_y f.$$

On the other hand, applying this equation to any non-zero element t in G yields the equation

$$(x+y)f = xf + yf,\ \forall x,\ y \in G,$$

that is $f \in EndG$. Hence $M_0(G)_d \subseteq EndG$. □

Proposition 2.5. *Let R be a near-ring and S a semigroup of distributive elements of R. Then the d.g. near-ring generated by S is precisely the d.g. near-ring generated, as a near-ring, by S.*

Proof. Let (T, S) be the d.g. near-ring generated by S. It suffices to show that T is closed under products. Thus let $\epsilon_1 s_1 + \cdots + \epsilon_n s_n$, and $\eta_1 t_1 + \cdots + \eta_m t_m$ in T, where $\epsilon_i = \pm 1,\ \eta_j = \pm 1,\ s_i, t_j \in S\ \forall i, j,\ 1 \leq i \leq n, 1 \leq j \leq m$. Then

$$(\epsilon_1 s_1 + \cdots + \epsilon_n s_n)(\eta_1 t_1 + \cdots + \eta_m t_m) =$$

$$(\epsilon_1 s_1 + \cdots + \epsilon_n s_n)\eta_1 t_1 + \cdots + (\epsilon_1 s_1 + \cdots + \epsilon_n s_n)\eta_m t_m =$$

$$\eta_1(\epsilon_1 s_1 t_1 + \cdots + \epsilon_n s_n t_1) + \cdots + \eta_m(\epsilon_1 s_1 t_m + \cdots + \epsilon_n s_n t_m).$$

Since $s_i t_j \in S\ \forall i, j,\ 1 \leq i \leq n, 1 \leq j \leq m$, this proves our result. □

Example 2.6. Let G be an additive group and let $S \subseteq EndG$ a semigroup of endomorphisms of G. Then S is a semigroup of distributive elements of the near-ring $M_0(G)$. In this case, near-ring R generated by S is a d.g. near-ring (R, S). There are three kinds of d.g. near-rings which arise in this way.

(1) $S = EndG$, we denote the d.g. near-ring by $(E(G), EndG)$
(2) $S = AutG$, the group of all automorphisms of G, we denote the d.g. near-ring by $(A(G), AutG)$
(3) $S = InnG$, the group of all inner automorphisms of G, we denote the d.g. near-ring by $(I(G), InnG)$.

Note that $I(G) \subseteq A(G) \subseteq E(G)$, and $E(G) = EndG$, in case G is abelian.

Let G be an R-group and K a nonempty subset of G. Define $Ann(K) = \{a \in R | Ka = 0\}$ which is called the *annihilator* of K in R. We say that G is a faithful R-group if $Ann(G) = \{0\}$.

Let (R, S) be the d.g. near-ring generated by S and let G be an R-group. Let θ be the representation which defines G as an R-group. We call G an *(R, S)-group* if

$$S\theta \subseteq EndG.$$

Also, let (R, S) be the d.g. near-ring. A representation θ of R is called a d.g. representation if $\theta : R \to M(G)$ satisfies $S\theta \subseteq EndG$, where G is an R-group associated with the representation θ.

Note that a d.g. representation is a representation associated with an (R, S)-group. In other words, G is an (R, S)-group if the elements of S induces endomorphisms on G. Examples 2.6 gives us some (R, S)-groups.

Some more examples of (R, S)-groups arise as following.

Lemma 2.7. *Let (R, S) be a d.g. near-ring generated by S. Then all R-subgroups and R-homomorphic images of an (R, S)-group are (R, S)-groups.*

Proof. These can be verified very easily from the definition of (R, S)-group. □

Proposition 2.8. *Let R be a near-ring and G an R-group. Then we have the following statements:*

(1) *$Ann(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/Ann(G)$-group.*
(2) *For any $x \in G$, we get $xR \cong R/Ann(x)$ as R-groups.*

Proof. (1) Obviously, $Ann(G)$ is a two-sided ideal of R. We now make G an $R/Ann(G)$-group by defining, for $x \in G, r + Ann(G) \in R/Ann(G)$, the action

$x(r + Ann(G)) = xr$. If $r + Ann(G) = r' + Ann(G)$, then $-r' + r \in Ann(G)$ hence $x(-r' + r) = 0$ for all x in G, that is to say, $xr = xr'$. This implies that

$$x(r + Ann(G)) = xr = xr' = x(r' + Ann(G));$$

thus the action of $R/Ann(G)$ on G is well defined. The verification of the structure of an $R/Ann(G)$-group is a routine fact.

Finally, to see that G is a faithful $R/Ann(G)$-group, we note that if $x(r + Ann(G)) = 0$ for all $x \in G$, then by the definition of $R/Ann(G)$-group structure, we have $xr = 0$. Hence $r \in Ann(G)$. This says that only the zero element of $R/Ann(G)$ annihilates all of G. Thus G is a faithful $R/Ann(G)$-group.
(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi_x : R \to xR$ defined by $r\phi_x = xr$ is an R-epimorphism, so that from the isomorphism theorem in near-ring theory and the kernel of ϕ_x is $Ann(x)$, we can induce that

$$xR \cong R/Ann(x)$$

as R-groups. □

Proposition 2.9. *If (R, S) is a d.g. near-ring generated by S, then every monogenic R-group is an (R, S)-group.*

Proof. Let G be a monogenic R-group with x as a generator. Then $G = xR$ and the map $\phi_x : R_R \to G_R$ defined by $a\phi_x = xa$ is an R-epimorphism from R to G as R-groups. We see that by the Proposition 2.8,

$$G \cong R/Ann(x),$$

where $Ann(x) = Ker\phi_x$. From the Lemma 2.7, we see that G is an $(R,\ S)$-group. □

Corollary 2.10. *Let G be a monogenic R-group with x as a generator. Then we have the following isomorphic relation.*

$$G \cong R/Ann(x).$$

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