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On connected dominating set games†

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Abstract

Many authors studied cooperative games that arise from variants of dominating set games on graphs. In wireless networks, the connected dominating set is used to reduce routing table size and communication cost. In this paper, we introduce a connected dominating set game to model the cost allocation problem arising from a connected dominating set on a given graph and study its core. In addition, we give a polynomial time algorithm for determining the balancedness of the game on a tree, for finding an element of the core.

Keywords: Balancedness, connected dominating set, cooperative games, core.

1. Introduction

Given a finite set V, a V-indexed vector is a vector z in $\mathbb{R}^{|V|}$ such that each coordinate corresponds to exactly one element of V and vice versa. We denote z_v by the value of the coordinate corresponding to an element v of V in a vector z. We denote $\mathbb{R}^V$ by the set of all V-indexed vectors. Given a finite set V, a vector $z \in \mathbb{R}^V$, and $S \subseteq V$, we let $z(S) = \sum_{v \in S} z_v$.

A *transferable utility game* (or game for short) is a triple $(V, \mathcal{S}, c)$ of a player set V, a coalitional structure $\mathcal{S} \subset 2^V$ with $\{\emptyset, V\} \subset \mathcal{S}$, and a characteristic function $c : \mathcal{S} \to \mathbb{R}$ with $c(\emptyset) = 0$. For each $S \in \mathcal{S}$, S is called a *coalition*, and $c(S)$ represents the cost which the players in S achieve together. Given a game $(V, \mathcal{S}, c)$, the main issue is how to distribute fairly the total cost $c(V)$ among all the players. By imposing different requirements for fairness of distributions, there are many concepts of fair distributions. The core is one of these distribution concepts, which is considered as an important distribution for $c(V)$. Given a game $(V, \mathcal{S}, c)$, the *core* of $(V, \mathcal{S}, c)$ is a set of vectors in $\mathbb{R}^n$ defined by

$$\mathcal{C}(V, \mathcal{S}, c) = \{z \in \mathbb{R}^n \mid z(V) = c(V) \text{ and } \forall S \in \mathcal{S},\ z(S) \le c(S)\}.$$

The constraint imposed on $\mathcal{C}(V, \mathcal{S}, c)$, which is called group rationality, ensures that no coalition in $\mathcal{S}$ would have an incentive to split from the grand coalition V, and do better on its own. We say a game is *balanced* if the core of a game is nonempty. Since the core of a

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game is one of important distribution method, characterization of the cores of games and determination of the balancedness for games have been considered as important research questions. In this paper, we introduce cooperative cost games that arises from connected dominating set problems on graphs and investigate the cores of those games.

Games arising from problems on graph structures have been studied actively, since Velzen (2004) studied cooperative games to model the cost allocation problem arising from domination set on graphs. After then, many authors studied variants of dominating games on graphs (Deng *et al.*, 1999; Bilbao, 2000; Kim and Fang, 2006; Liu and Fang, 2007; Kim and Park, 2009; Park *et al.*, 2011). In wireless networks, the connected dominating set is used to reduce routing table size and communication cost (Na and Kwon, 2009; Bae, 2010).

We mainly consider a connected dominating set game with restricted coalitional structures and study its core. In addition, we give a polynomial time algorithm for determining the balancedness of the game on a tree, for finding an element of the core.

2. Connected dominating set game and its core

Throughout this paper, we assume that a graph means a simple connected graph with no isolated vertices. An edge of G with endpoints u and v is denoted by uv. A weighted graph is a graph G with a vertex weight function $\omega : V(G) \to \mathbb{R}_+$ ($\mathbb{R}_+$ is the set of positive real numbers). For a graph G and a set $S \subseteq V(G)$, we denote $G[S]$ by the subgraph of G induced by S.

A *dominating set* of a graph G is a vertex set $D \subset V(G)$ such that for each vertex $v \in V(G) \setminus D$, there exists a vertex $w \in D$ which is adjacent to v. For a graph G, a set $D \subset V(G)$ is called a *connected dominating set* of G if D is a dominating set of G and $G[D]$ is connected. Let G be a weighted graph G with an vertex weight function ω. For $D \subseteq V(G)$, we let $\omega(D) = \sum_{v\in D} \omega(v)$. The *connected dominating number* of G is defined by

$$\gamma^c(G) = \min\{\omega(D) \mid D \text{ is a connected dominating set of } G\}.$$

The connected dominating problem of G is to find the connected dominating number of G. The following is a simple observation of the connected dominating number of a graph. For a tree T, the *essential tree* of T is a tree obtained by deleting pendent vertices of T.

Lemma 2.1 For a connected graph G whose vertex weight function is ω,

$$\gamma^c(G) = \min\{w(V(T)) \mid T \text{ is the essential tree of a spanning tree in } G\}.$$

Proof: Let $r = \min\{w(V(T)) \mid T$ is the essential tree of a spanning tree in $G\}$ for convenience. For any spanning tree in G, its essential tree is connected and the vertex set of the essential tree is a dominating set of G, and so it follows that $\gamma^c(G) \le r$.

Let D be a minimum connected domination set of G. Then $w(D) = \gamma^c(G)$. Since $G[D]$ is connected, there exits a spanning tree T' of $G[D]$. Since D is a dominating set of G, there exists a function $f : V(G) \setminus D \to D$ such that for each $v \in V(G) \setminus D$, v and $f(v)$ are adjacent in G. Then T obtained by adding edges in $\{vf(v) \mid v \in V(G) \setminus D\}$ to T' is a spanning tree of G. All vertices $v \in V(G) \setminus D$ are pendent vertices of T, and vice versa. Therefore, T' is the essential tree of T. By the definition of r, $r \le w(V(T'))$. Since $V(T') = D$, $w(V(T')) = w(D) = \gamma^c(G)$. Thus $r \le \gamma^c(G)$. □

Let G be a weighted graph with a vertex weight function ω and

$$\mathcal{S}(G) = \{S \subset V(G) \mid G[S] \text{ is connected}\}$$

For $S \in \mathcal{S}(G)$, $D \subset S$ is called a *connected dominating set for* S if D is a connected dominating set of the graph $G[S]$. We introduce a connected dominating set game by considering the cost allocation problem arising from connected dominating set problems on graphs.

Definition. Given a weighted graph G with a vertex weight function ω, a game $(V, \mathcal{S}, c)$ is called the *connected dominating set game associated with* G if $V = V(G)$, $\mathcal{S} = \mathcal{S}(G)$, and $c : \mathcal{S} \to \mathbb{R}$ is a function such that $c(\emptyset) = 0$ and for $S \in \mathcal{S}(G) \setminus \{\emptyset\}$,

$$c(S) = \min\{\omega(D) \mid D \text{ is a connected dominating set for } S\}.$$

Now we investigate the core of the connected dominating set game associated with a graph.

Given a vertex v of G, the set of vertices that are adjacent to v is denoted by $N_G(v)$. For a vertex v, a v-*star* is a subset of $\{v\} \cup N_G(v)$ containing $\{v\}$. A subset of $V(G)$ is called a *star* if it is a v-star for some vertex v. We denote $\mathcal{T}_v(G)$ by the set of all v-stars of G, and let $\mathcal{T}(G) = \cup_{v \in V(G)} \mathcal{T}_v(G)$.

Theorem 2.1 is a characterization for the core of a connected dominating set game.

Theorem 2.1 Let $(V, \mathcal{S}, c)$ be the connected dominating set game associated with a weighted graph G whose vertex weight function is ω. Then $z \in \mathcal{C}(V, \mathcal{S}, c)$ if and only if it holds that

(a) $z(V) = c(V)$

(b) for a vertex v, for any $T \in \mathcal{T}_v(G)$, $z(T) \leq \omega(v)$.

Proof: The 'only if' part is easy. Let z be an element of $\mathcal{C}(V, \mathcal{S}, c)$. Then by the definition of the core, (a) follows immediately. To show (b), take any $T \in \mathcal{T}_v(G)$. Then $z(T) \leq c(T)$ by the definition of the core. Let $D = \{v\}$. Then D is a connected dominating set for T. Therefore $c(T) \leq \omega(v)$, and so (b) holds.

Now we will show the 'if' part. Suppose that z satisfies (a) and (b). It is sufficient to show that $z(S) \leq c(S)$ for any $S \in \mathcal{S}$. Take $S \in \mathcal{S}$. Then there is a connected dominating set D for S such that $c(S) = w(D)$. Let $D = \{v_1, v_2, \ldots, v_k\}$ and $G[D]$ is connected. Then there exists a function $f : S \setminus D \to D$ such that $f(v) \in N_{G[S]}(v)$ for any $v \in S \setminus D$. For each $1 \leq i \leq k$, let

$$T_i = \{v_i\} \cup f^{-1}(\{v_i\}).$$

Then T_i is v_i-star. In addition, $\{T_1, T_2, \ldots, T_k\}$ is a partition of S, and so $z(S) = \sum_{i=1}^{k} z(T_i)$. By the assumption (b), we have $z(T_i) \leq \omega(v_i)$. Therefore,

$$z(S) \leq \sum_{i=1}^{k} \omega(v_i) = \sum_{v \in D} \omega(v) = w(D).$$

Hence $z(S) \leq c(S)$. □

3. The core of the connected dominating set game on a tree

In this section, we study the core of a connected dominating set game on a tree. The following gives a sufficient and necessary condition for the core of the connected dominating set game associated with a tree being nonempty:

Theorem 3.1 Let $(V, \mathcal{S}, c)$ be the connected dominating set game associated with a weighted tree G with at least three vertices whose vertex weight function is ω. Then $\mathcal{C}(V, \mathcal{S}, c) \neq \emptyset$ if and only if for each vertex $v \in V(G)$ which is not a pendent vertex,

(i) v is a neighbor of a pendent vertex in G;

(ii) the sum of weights all pendent neighbors of v is greater than or equal to $\omega(v)$.

The first condition (i) of Theorem 3.1 is a structural constraint of a tree whose connected dominating set game has nonempty core, and the condition (ii) is a constraint of the weight function of such tree.

Proof: Let T^* be the essential tree of G. For each $v \in V(T^*)$, let

$$N^*_G(v) := N_G(v) \setminus N_{T^*}(v)$$
$$N^*_G[v] := \{v\} \cup N^*_G(v).$$

Note that $\{N^*_G[v] \mid v \in V(T^*)\}$ is a partition of $V(G)$.

To show the 'only if' part, suppose that $\mathcal{C}(V, \mathcal{S}, c) \neq \emptyset$. Then there exists a vector $z \in \mathcal{C}(V, \mathcal{S}, c)$. First, we will show that $z(N^*_G[v]) = \omega(v)$ for each $v \in V(T^*)$. Since $\{N^*_G[v] \mid v \in V(T^*)\}$ is a partition of V, $z(V) = \sum_{v \in V(T^*)} z(N^*_G[v])$. Since each $N^*_G[v]$ is a v-star and $z \in \mathcal{C}(V, \mathcal{S}, c)$, $z(N^*_G[v]) \leq \omega(v)$ by (b) of Theorem 2.1. Therefore it holds that

$$z(V) = \sum_{v \in V(T^*)} z(N^*_G[v]) \leq \sum_{v \in V(T^*)} \omega(v) = \omega(T^*),$$

and so $z(V) \leq \omega(T^*)$. By Lemma 2.1, we have $z(V) = \omega(T^*)$. Thus it holds that $z(N^*_G[v]) = \omega(v)$ for each $v \in V(T^*)$.

Suppose that there exists a vertex v which is not a pendent vertex of G, such that any neighbor of v in G is not a pendent vertex. Then $v \in V(T^*)$ and $V(T^*) \neq \{v\}$ and so there exists a neighbor u of v in T^*. Then $N^*_G[u] \cup \{v\}$ is a u-star of G and so $z(T(u) \cup \{v\}) \leq \omega(u)$ by (b) of Theorem 2.1. Since $z(N^*_G[u] \cup \{v\}) = z(N^*_G[u]) + z(N^*_G[v])$, we have $z(N^*_G[u] \cup \{v\}) = \omega(u) + \omega(v)$. Then it holds that $\omega(v) \leq 0$, which is a contradiction. Therefore (i) holds.

For a vertex $v \in V(T^*)$, for a pendent neighbor u of v,

$$\omega(v) = z(N^*_G[v]) = z(\{u, v\}) + \sum_{x \in N^*_G(v) \setminus \{u\}} z(x) \leq \omega(u) + \sum_{x \in N^*_G(v) \setminus \{u\}} \omega(x) = \omega(N^*_G[v]),$$

and so $\omega(v) \leq \omega(N^*_G[v])$. Therefore (ii) holds. Hence the 'only if' part holds.

To show the 'if' part, suppose that (i) and (ii) hold. Let z be a vector defined by

$$z_v = \begin{cases} \omega(v) & \text{if } v \in V(G) \setminus V(T^*) \\ \omega(v) - \omega(N^*_G(v)) & \text{if } v \in V(T^*) \end{cases}$$

It is easy to see that

$$\begin{aligned} z(V) &= \sum_{v\in V(T^*)} z(N_G^*[v]) \\ &= \sum_{v\in V(T^*)} (z(N_G^*(v)) + z_v) \\ &= \sum_{v\in V(T^*)} (\omega(N_G^*(v) + \omega(v) - \omega(N_G^*(v))) \\ &= \sum_{v\in V(T^*)} \omega(v) \\ &= \omega(V(T^*)). \end{aligned}$$

By Lemma 2.1, $\omega(V(T^*)) = \gamma^c(G) = c(V)$. Take any v-star S of G. Then $z(S) \leq z(S \setminus V(T^*))$ since $z_u \leq 0$ for any $u \in V(T^*)$. If v is a pendent of G, then $S = \{uv\}$ for some $u \in V(T^*)$ and so $z_u + z_v \leq z_v = \omega(v)$. Suppose that v is not a pendent of G. Then $v \in V(T^*)$ and

$$z(S) \leq z(S \setminus V(T^*)) = \omega(S \setminus V(T^*)) \leq \omega(N_G^*(v)) \leq \omega(v).$$

Thus z satisfies (a) and (b) of Theorem 2.1. □

If the vertex weight function of a graph is a constant function then the condition (i) implies the condition (ii) in Theorem 3.1, and so the following holds:

Corollary 3.1 Let $(V, \mathcal{S}, c)$ be the connected dominating set game associated with a weighted tree G with at least three whose vertex weight function is a constant function. Then $\mathcal{C}(V, \mathcal{S}, c) \neq \emptyset$ if and only if for each vertex $v \in V(G)$ which is not a pendent vertex, v is a neighbor of a pendent vertex in G.

Corollary 3.1 says that it is sufficient to check the structure of a tree for determining if the core of associated connected dominating set game is nonempty.

From the proof of Theorem 3.1, we can find a polynomial time algorithm determining if $\mathcal{C}(V, \mathcal{S}, c) \neq \emptyset$ or not, and then finding elements in $\mathcal{C}(V, \mathcal{S}, c)$ if $\mathcal{C}(V, \mathcal{S}, c) \neq \emptyset$ where $(V, \mathcal{S}, c)$ is the connected dominating set game associated with a weighted tree G with at least three. It might be easy to check the following Algorithm A gives a correct answer by Theorem 3.1. We follow the notation of Haynes *et al.* (1998).

Algorithm A

Input *A tree T with at least three vertices whose vertices are $v_1, v_2, \ldots, v_n$ and the pendent vertices are v_1, v_2, $\ldots$, v_r $(r < n)$, and a vertex weight function ω of T.*

Output *A set $C = \mathcal{C}(V, \mathcal{S}, c)$.*

Step 1 *Let $i = r + 1$.*

Step 2 *Let $N_i = \{j \mid j \leq r, v_j$ is adjacent to $v_i\}$.*

Step 3 *If* $N_i = \emptyset$

then let $C = \emptyset$ *and go to Step 7.*

else

go to Step 4.

Step 4 *If* $\sum_{j \in N_i} \omega(v_j) > \omega(v_i)$

then let $C = \emptyset$ *and go to Step 7.*

else

let

$$C_i = \{z \in \mathbb{R}^{V(T)} \mid \sum_{v \in N_i \cup \{v_i\}} z_v = \omega(v_i), z_v = 0 \text{ if } v \notin N_i \cup \{v_i\}\}$$

and let $i = i + 1$.

Step 5 *If* $i \leq n$

then go to Step 2

else

go to Step 6.

Step 6 *Let* $C = \bigcup_{k=r+1}^{n} C_k$.

Step 7 *Print* C.

From Theorem 3.1, it could be more easier to find an element of the core of connected dominating set game associated with a graph. We start with the following observation. Let G be a weighted graph G whose vertex weight function is ω and $\mathcal{C}(G)$ be the set of spanning trees G' of G satisfying that $\gamma^c(G) = \gamma^c(G')$.

Remark 3.1 Let $(V, \mathcal{S}, c)$ be the connected dominating set game associated with a weighted graph G whose vertex weight function is ω, and let $(V, \mathcal{S}_{G'}, c_{G'})$ be the connected dominating set game associated with a spanning tree G' of G in $\mathcal{C}(G)$. Then

$$\mathcal{C}(V, \mathcal{S}, c) \subset \bigcap_{G' \in \mathcal{C}(G)} \mathcal{C}(V, \mathcal{S}_{G'}, c_{G'}).$$

Proof: Let $C = \bigcap_{G' \in \mathcal{C}(G)} \mathcal{C}(V, \mathcal{S}_{G'}, c_{G'})$ for convenience. Take an element $z \in \mathcal{C}(V, \mathcal{S}, c)$. Since $z(V) = c(V) = \gamma^c(G) = \gamma^c(G') = c_{G'}(V)$ for any $G' \in \mathcal{C}(G)$, it is sufficient to check (b) of Theorem 2.1 for checking if z is in the core. Take any $G' \in \mathcal{C}(G)$. For a vertex v, for any $T \in \mathcal{T}_v(G')$, since $\mathcal{T}_v(G') \subset \mathcal{T}_v(G)$, it holds that $T \in \mathcal{T}_v(G)$ and so $z(T) \leq \omega(v)$. Therefore $z \in \mathcal{C}(V, \mathcal{S}_{G'}, c_{G'})$. Thus $z \in C$. □

By Theorem 3.1 and Remark 3.1, we may find graph classes whose core in the dominating set game is empty. For example, any graph having path of length at least 4 has the empty core in the connected dominating set game associated with the graph, since any spanning tree does not satisfy the condition (i) of Theorem 3.1.

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