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Objective Bayesian testing for the location parameters in the half-normal distributions

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Abstract

This article deals with the problem of testing the equality of the location parameters in the half-normal distributions. We propose Bayesian hypothesis testing procedures for the equality of the location parameters under the noninformative prior. The non-informative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to arbitrary constants. This problem can be dealt with the use of the fractional Bayes factor or intrinsic Bayes factor. So we propose the default Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors. Simulation study and an example are provided.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, half-normal distribution, location parameter, reference prior.

1. Introduction

The half-normal distribution has been used as a model for truncated data from application areas as diverse as fibre buckling (Haberle, 1991), blowfly dispersion (Dobzhansky and Wright, 1943), sports science physiology (Pewsey, 2002, 2004) and stochastic frontier modeling (Aigner *et al.*, 1977; Meeusen and van den Broeck, 1977).

In spite of the usefulness of this distribution, its statistical inference has been developed recently. Likelihood based inference for the half-normal distribution has been considered by Pewsey (2002, 2004). Wiper *et al.* (2008) gave Bayesian inference for the half-normal using conjugate prior. They show that a generalized version of the normal-gamma distribution is conjugate to the half-normal likelihood.

Even though the comparison for two location parameters in half-normal distribution is as important as the comparison for two means in normal distribution, but it has not been

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studied enough. Therefore, there is a necessity for developing Bayesian hypothesis testing procedure.

Consider X and Y are independently distributed random variables according to the half-normal distribution $\mathcal{HN}(\xi_1, \eta_1)$ with the location parameter ξ_1 and the scale parameter η_1, and the half-normal distribution $\mathcal{HN}(\xi_2, \eta_2)$ with the location parameter ξ_2 and the scale parameter η_2. Then the half-normal distributions of X and Y are given by

$$f(x|\xi_1, \eta_1) = \sqrt{\frac{2}{\pi}} \frac{1}{\eta_1} \exp\left\{-\frac{1}{2\eta_1^2}(x-\xi_1)^2\right\}, x \geq \xi_1, -\infty < \xi_1 < \infty, \eta_1 > 0, \quad (1.1)$$

and

$$f(y|\xi_2, \eta_2) = \sqrt{\frac{2}{\pi}} \frac{1}{\eta_2} \exp\left\{-\frac{1}{2\eta_2^2}(x-\xi_2)^2\right\}, y \geq \xi_2, -\infty < \xi_2 < \infty, \eta_2 > 0, \quad (1.2)$$

respectively. The present paper focuses on testing the equality of the location parameters in the half-normal distributions.

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training sample in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang *et al.*, 2006, 2008). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

In this paper, we propose the objective Bayesian hypothesis testing procedures for the equality of the location parameters in half-normal distributions based on the Bayes factors. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference prior, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors. In Section 4, simulation study and an example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses H_1,H_2 ,$\cdots$,H_q are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i. The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis H_i,$i = 1, 2, \cdots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be used as training sample and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \cdots, q. \tag{2.3}$$

In view (2.3), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor $B_{ji}(l)$ with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training samples $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all the training sample Bayes factors.

Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.5) and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction,b ,of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.7}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. For details, see O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995).

3. Bayesian hypothesis testing procedures

Let $X_i, i = 1, \cdots, n_1$ denote observations from the half-normal distribution $\mathcal{HN}(\xi_1, \eta_1)$, and $Y_i, i = 1, \cdots, n_2$ denote observations from the half-normal distribution $\mathcal{HN}(\xi_2, \eta_2)$. Then likelihood function is given by

$$f(\mathbf{x}, \mathbf{y}|\xi_1, \xi_2, \eta_1, \eta_2) = \left(\frac{2}{\pi}\right)^{\frac{n_1+n_2}{2}} \eta_1^{-n_1}\eta_2^{-n_2} \exp\left\{-\frac{\sum_{i=1}^{n_1}(x_i-\xi_1)^2}{2\eta_1^2} - \frac{\sum_{i=1}^{n_2}(y_i-\xi_2)^2}{2\eta_2^2}\right\}, \tag{3.1}$$

where $\mathbf{x} = (x_1, \cdots, x_{n_1})$, $\mathbf{y} = (y_1, \cdots, y_{n_2})$, $-\infty < \xi_1 < \infty$, $-\infty < \xi_2 < \infty$, $\eta_1 > 0$ and $\eta_2 > 0$. We are interested in testing the hypotheses $H_1 : \xi_1 = \xi_2$ versus $H_2 : \xi_1 \neq \xi_2$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing procedure based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \xi_1 = \xi_2 \equiv \xi$ is

$$L_1(\xi, \eta_1, \eta_2|\mathbf{x}, \mathbf{y}) = \left(\frac{2}{\pi}\right)^{\frac{n_1+n_2}{2}} \eta_1^{-n_1}\eta_2^{-n_2} \exp\left\{-\frac{\sum_{i=1}^{n_1}(x_i-\xi)^2}{2\eta_1^2} - \frac{\sum_{i=1}^{n_2}(y_i-\xi)^2}{2\eta_2^2}\right\}. \tag{3.2}$$

And under the hypothesis H_1, the reference prior for (ξ, η_1, η_2) derived by Kang *et al.* (2010) and is

$$\pi_1^N(\xi, \eta_1, \eta_2) \propto \eta_1^{-1}\eta_2^{-1}. \tag{3.3}$$

Then from the likelihood (3.2) and the reference prior (3.3), the element $m_1^b(\mathbf{x},\mathbf{y})$ of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x},\mathbf{y}) &= \int_{-\infty}^{z_{(1)}} \int_0^\infty \int_0^\infty L_1^b(\xi,\eta_1,\eta_2|\mathbf{x},\mathbf{y}) \pi_1^N(\xi,\eta_1,\eta_2) d\eta_1 d\eta_2 d\xi \\ &= \frac{1}{4}\left(\frac{2}{\pi}\right)^{\frac{b(n_1+n_2)}{2}} \Gamma\left[\frac{bn_1}{2}\right]\Gamma\left[\frac{bn_2}{2}\right] \\ &\times \int_{-\infty}^{z} \left[\frac{b(s_1^2+n_1(\bar{x}-\xi)^2)}{2}\right]^{-\frac{bn_1}{2}} \left[\frac{b(s_2^2+n_2(\bar{y}-\xi)^2)}{2}\right]^{-\frac{bn_2}{2}} d\xi, \end{aligned} \tag{3.4}$$

where $z_{(1)} = \min\{x_1,\cdots,x_{n_1},y_1,\cdots,y_{n_2}\}$, $\bar{x} = \sum_{i=1}^{n_1} x_i/n_1$, $s_1^2 = \sum_{i=1}^{n_1}(x_i-\bar{x})^2$, $\bar{y} = \sum_{i=1}^{n_2} y_i/n_2$ and $s_2^2 = \sum_{i=1}^{n_2}(y_i-\bar{y})^2$. For the hypothesis $H_2: \xi_1 \neq \xi_2$, the reference prior for $(\xi_1,\xi_2,\eta_1,\eta_2)$ is

$$\pi^N(\xi_1,\xi_2,\eta_1,\eta_2) \propto \eta_1^{-1}\eta_2^{-1} \tag{3.5}$$

and can be easily derived following Kang *et al.* (2010). The likelihood function under the hypothesis H_2 is

$$L_2(\xi_1,\xi_2,\eta_1,\eta_2|\mathbf{x},\mathbf{y}) = \left(\frac{2}{\pi}\right)^{\frac{n_1+n_2}{2}} \eta_1^{-n_1}\eta_2^{-n_2} \exp\left\{-\frac{\sum_{i=1}^{n_1}(x_i-\xi_1)^2}{2\eta_1^2} - \frac{\sum_{i=1}^{n_2}(y_i-\xi_2)^2}{2\eta_2^2}\right\}. \tag{3.6}$$

Thus from the likelihood (3.6) and the reference prior (3.5), the element $m_2^b(\mathbf{x},\mathbf{y})$ of FBF under H_2 is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x},\mathbf{y}) &= \int_{-\infty}^{y_{(1)}} \int_{-\infty}^{x_{(1)}} \int_0^\infty \int_0^\infty L_2^b(\xi_1,\xi_2,\eta_1,\eta_2|\mathbf{x},\mathbf{y}) \pi_2^N(\xi_1,\xi_2,\eta_1,\eta_2) d\eta_1 d\eta_2 d\xi_1 d\xi_2 \\ &= \frac{1}{4}\left(\frac{2}{\pi}\right)^{\frac{b(n_1+n_2)}{2}} \Gamma\left[\frac{bn_1}{2}\right]\Gamma\left[\frac{bn_2}{2}\right] \\ &\times \int_{-\infty}^{y_{(1)}} \int_{-\infty}^{x_{(1)}} \left[\frac{b(s_1^2+n_1(\bar{x}-\xi_1)^2)}{2}\right]^{-\frac{bn_1}{2}} \left[\frac{b(s_2^2+n_2(\bar{y}-\xi_2)^2)}{2}\right]^{-\frac{bn_2}{2}} d\xi_1 d\xi_2, \end{aligned} \tag{3.7}$$

where $x_{(1)} = \min\{x_1,\cdots,x_{n_1}\}$ and $y_{(1)} = \min\{y_1,\cdots,y_{n_2}\}$. Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{S_2(\mathbf{x},\mathbf{y})}{S_1(\mathbf{x},\mathbf{y})}, \tag{3.8}$$

where

$$S_1(\mathbf{x},\mathbf{y}) = \int_{-\infty}^{z_{(1)}} \left[s_1^2+n_1(\bar{x}-\xi)^2\right]^{-\frac{n_1}{2}} \left[s_2^2+n_2(\bar{y}-\xi)^2\right]^{-\frac{n_2}{2}} d\xi$$

and

$$S_2(\mathbf{x},\mathbf{y}) = \int_{-\infty}^{y_{(1)}} \int_{-\infty}^{x_{(1)}} \left[s_1^2+n_1(\bar{x}-\xi_1)^2\right]^{-\frac{n_1}{2}} \left[s_2^2+n_2(\bar{y}-\xi_2)^2\right]^{-\frac{n_2}{2}} d\xi_1 d\xi_2.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x},\mathbf{y})}{m_2^b(\mathbf{x},\mathbf{y})} = \frac{S_1(\mathbf{x},\mathbf{y};b)}{S_2(\mathbf{x},\mathbf{y};b)}, \tag{3.9}$$

where

$$S_1(\mathbf{x},\mathbf{y};b) = \int_{-\infty}^{z_{(1)}} \left[s_1^2 + n_1(\bar{x}-\xi)^2\right]^{-\frac{bn_1}{2}} \left[s_2^2 + n_2(\bar{y}-\xi)^2\right]^{-\frac{bn_2}{2}} d\xi$$

and

$$S_2(\mathbf{x},\mathbf{y};b) = \int_{-\infty}^{y_{(1)}} \int_{-\infty}^{x_{(1)}} \left[s_1^2 + n_1(\bar{x}-\xi_1)^2\right]^{-\frac{bn_1}{2}} \left[s_2^2 + n_2(\bar{y}-\xi_2)^2\right]^{-\frac{bn_2}{2}} d\xi_1 d\xi_2.$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_2(\mathbf{x},\mathbf{y})}{S_1(\mathbf{x},\mathbf{y})} \cdot \frac{S_1(\mathbf{x},\mathbf{y};b)}{S_2(\mathbf{x},\mathbf{y};b)}. \tag{3.10}$$

Note that the calculations of the FBF of H_2 versus H_1 requires only one dimensional integration.

3.2. Bayesian hypothesis testing procedure based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal density of (X_{j_1}, X_{j_2}) and (Y_{k_1}, Y_{k_2}) is finite for all $1 \le j_1 < j_2 \le n$ and $1 \le k_1 < k_2 \le m$ under each hypothesis. Thus we conclude that any training sample of size 4 is a minimal training sample.

The marginal density $m_1^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})$ under H_1 is given by

$$\begin{aligned} & m_1^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) \\ &= \int_{-\infty}^{z_{(j_1)}} \int_0^\infty \int_0^\infty f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}|\xi, \eta_1, \eta_2)\pi_1^N(\xi,\eta_1,\eta_2) d\eta_1 d\eta_2 d\xi \\ &= \int_{-\infty}^{z_{(j_1)}} \left[\frac{(x_{j_1}-x_{j_2})^2}{2} + \frac{(x_{j_1}+x_{j_2}-2\xi)^2}{2}\right]^{-1} \left[\frac{(y_{k_1}-y_{k_2})^2}{2} + \frac{(y_{k_1}+y_{k_2}-2\xi)^2}{2}\right]^{-1} d\xi, \end{aligned}$$

where $z_{(j_1)} = \min\{x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}\}$, And the marginal density $m_2^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})$ under H_2 is given by

$$\begin{aligned} & m_2^N(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) \\ &= \int_{-\infty}^{y_{(k_1)}} \int_{-\infty}^{x_{(j_1)}} \int_0^\infty \int_0^\infty f(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}|\xi_1, \xi_2, \eta_1, \eta_2)\pi_2^N(\xi_1, \xi_2, \eta_1, \eta_2) d\eta_1 d\eta_2 d\xi_1 d\xi_2 \\ &= \int_{-\infty}^{y_{(k_1)}} \int_{-\infty}^{x_{(j_1)}} \left[\frac{(x_{j_1}-x_{j_2})^2}{2} + \frac{(x_{j_1}+x_{j_2}-2\xi_1)^2}{2}\right]^{-1} \\ &\times \left[\frac{(y_{k_1}-y_{k_2})^2}{2} + \frac{(y_{k_1}+y_{k_2}-2\xi_2)^2}{2}\right]^{-1} d\xi_1 d\xi_2, \end{aligned}$$

where $x_{(j_1)} = \min\{x_{j_1}, x_{j_2}\}$ and $y_{(k_1)} = \min\{y_{k_1}, y_{k_2}\}$. Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\mathbf{x},\mathbf{y})}{S_1(\mathbf{x},\mathbf{y})} \left[\frac{1}{L} \sum_{j_1,j_2}^{n} \sum_{k_1,k_2}^{m} \frac{T_1(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})}{T_2(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})}\right], \tag{3.11}$$

where $L = [n_1 n_2(n_1-1)(n_2-1)]/4$,

$$T_1(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) = \int_{-\infty}^{z_{(j_1)}} \left[(x_{j_1}-x_{j_2})^2 + (x_{j_1}+x_{j_2}-2\xi)^2\right]^{-1} \times \left[(y_{k_1}-y_{k_2})^2 + (y_{k_1}+y_{k_2}-2\xi)^2\right]^{-1} d\xi$$

and

$$T_2(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2}) = \int_{-\infty}^{y_{(k_1)}} \int_{-\infty}^{x_{(j_1)}} \left[(x_{j_1}-x_{j_2})^2 + (x_{j_1}+x_{j_2}-2\xi_1)^2\right]^{-1} \times \left[(y_{k_1}-y_{k_2})^2 + (y_{k_1}+y_{k_2}-2\xi_2)^2\right]^{-1} d\xi_1 d\xi_2.$$

Also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x},\mathbf{y})}{S_1(\mathbf{x},\mathbf{y})} ME\left[\frac{T_1(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})}{T_2(x_{j_1}, x_{j_2}, y_{k_1}, y_{k_2})}\right]. \tag{3.12}$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require only one dimensional integration.

4. Numerical studies

In order to assess our approaches, we evaluate the posterior probability for several configurations of (ξ_1,η_1), (ξ_2,η_2) and (n_1,n_2). In particular, for fixed (ξ_1,η_1) and (ξ_2,η_2), we take 500 independent random samples of X_i and Y_i with sample size n_1 and n_2 from the models (1.1) and (1.2), respectively. We want to test the hypotheses $H_1 : \xi_1 = \xi_2$ versus $H_2 : \xi_1 \neq \xi_2$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. Table 4.1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities. In Table 4.1, $P^F(\cdot)$,$P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, AIBF and MIBF, respectively. We take the fraction b of FBF as $4/n$. From Table 4.1, the FBF, the AIBF and the MIBF give fairly reasonable answers for all configurations. Also the FBF, the AIBF and the MIBF give a similar behavior for all sample sizes. However for the large values of η_2, the AIBF and the MIBF slightly favor the hypothesis H_1 than the FBF.

Example 4.1 This example is the artificial example. We take random sample of X_i with sample size 15 from half-normal $\mathcal{HN}(1,1)$, and also take random sample of Y_i with sample size 15 form half-normal $\mathcal{HN}(2,3)$. The generated data sets are given by

Group 1: 1.36, 1.30, 2.85, 1.46, 1.85, 1.60, 2.31, 1.82, 2.53, 2.46, 1.46, 1.75, 1.88, 1.09, 1.02.
Group 2: 7.08, 4.41, 6.57, 2.03, 2.96, 3.02, 4.01, 4.58, 5.69, 5.60, 4.02, 2.26, 2.67, 6.21, 3.09.

For this data sets, the maximum likelihood estimates of ξ_1 and η_1 in group 1 are 1.02 and 0.93, and for group 2, the maximum likelihood estimates of ξ_2 and η_2 are 2.03 and 2.75.

We want to test the hypotheses $H_1 : \xi_1 = \xi_2$ versus $H_2 : \xi_1 \neq \xi_2$. The values of the Bayes factors and the posterior probabilities of H_1 are given in Table 4.2. From Table 4.2, the posterior probabilities based on various Bayes factors give the same answer, and select the hypothesis H_2. The AIBF has smaller posterior probability of H_1 than any other posterior probabilities based on the FBF and the MIBF, but the values of three Bayes factors are almost the same.

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

ξ_1	ξ_2	(n_1, n_2)	$P^F(H_1\|\mathbf{x},\mathbf{y})$	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
			$\eta_1 = 1.0, \eta_2 = 2.0$		
0.0	0.0	5,5	0.651(0.149)	0.662 (0.180)	0.672 (0.180)
		5,10	0.742(0.154)	0.738 (0.171)	0.750 (0.170)
		10,10	0.777(0.151)	0.788 (0.165)	0.798 (0.164)
		10,20	0.827(0.148)	0.825 (0.158)	0.834 (0.155)
	0.5	5,5	0.561(0.156)	0.541 (0.188)	0.553 (0.189)
		5,10	0.568(0.194)	0.533 (0.218)	0.546 (0.219)
		10,10	0.524(0.193)	0.509 (0.207)	0.524 (0.209)
		10,20	0.330(0.206)	0.296 (0.209)	0.309 (0.212)
	1.5	5,5	0.323(0.142)	0.283 (0.153)	0.294 (0.157)
		5,10	0.131(0.101)	0.096 (0.090)	0.101 (0.093)
		10,10	0.089(0.074)	0.080 (0.073)	0.084 (0.076)
		10,20	0.004(0.006)	0.003 (0.005)	0.003 (0.005)
	3.5	5,5	0.104(0.072)	0.082 (0.072)	0.084 (0.073)
		5,10	0.006(0.007)	0.003 (0.005)	0.003 (0.005)
		10,10	0.003(0.005)	0.003 (0.005)	0.003 (0.005)
		10,20	0.000(0.000)	0.000 (0.000)	0.000 (0.000)
	5.5	5,5	0.045(0.034)	0.033 (0.031)	0.034 (0.032)
		5,10	0.000(0.001)	0.000 (0.000)	0.000 (0.000)
		10,10	0.000(0.000)	0.000 (0.000)	0.000 (0.000)
		10,20	0.000(0.000)	0.000 (0.000)	0.000 (0.000)
			$\eta_1 = 1.0, \eta_2 = 3.0$		
0.0	0.0	5,5	0.638(0.145)	0.647 (0.174)	0.663 (0.177)
		5,10	0.732(0.161)	0.734 (0.179)	0.749 (0.178)
		10,10	0.754(0.164)	0.766 (0.176)	0.779 (0.173)
		10,20	0.819(0.143)	0.820 (0.154)	0.832 (0.150)
	0.5	5,5	0.576(0.151)	0.569 (0.176)	0.585 (0.178)
		5,10	0.601(0.187)	0.583 (0.208)	0.599 (0.210)
		10,10	0.585(0.188)	0.587 (0.199)	0.604 (0.201)
		10,20	0.501(0.192)	0.481 (0.199)	0.501 (0.201)
	1.5	5,5	0.389(0.149)	0.372 (0.168)	0.382 (0.172)
		5,10	0.235(0.146)	0.209 (0.151)	0.217 (0.156)
		10,10	0.196(0.123)	0.196 (0.133)	0.206 (0.140)
		10,20	0.027(0.032)	0.024 (0.030)	0.025 (0.032)
	3.5	5,5	0.177(0.096)	0.164 (0.108)	0.168 (0.111)
		5,10	0.024(0.025)	0.018 (0.022)	0.018 (0.022)
		10,10	0.018(0.018)	0.019 (0.020)	0.019 (0.020)
		10,20	0.000(0.000)	0.000 (0.000)	0.000 (0.000)
	5.5	5,5	0.091(0.062)	0.082 (0.070)	0.085 (0.073)
		5,10	0.003(0.004)	0.002 (0.003)	0.002 (0.003)
		10,10	0.002(0.003)	0.002 (0.003)	0.002 (0.003)
		10,20	0.000(0.000)	0.000 (0.000)	0.000 (0.000)
			$\eta_1 = 1.0, \eta_2 = 5.0$		
0.0	0.0	5,5	0.626(0.141)	0.641 (0.162)	0.661 (0.167)
		5,10	0.719(0.145)	0.733 (0.161)	0.751 (0.162)
		10,10	0.730(0.169)	0.752 (0.175)	0.768 (0.176)
		10,20	0.819(0.147)	0.834 (0.150)	0.847 (0.147)
	0.5	5,5	0.563(0.151)	0.575 (0.168)	0.592 (0.177)
		5,10	0.643(0.161)	0.652 (0.175)	0.668 (0.179)
		10,10	0.621(0.186)	0.643 (0.191)	0.661 (0.196)
		10,20	0.614(0.196)	0.624 (0.203)	0.644 (0.205)
	1.5	5,5	0.447(0.148)	0.462 (0.166)	0.473 (0.174)
		5,10	0.376(0.166)	0.378 (0.182)	0.389 (0.188)
		10,10	0.347(0.160)	0.376 (0.173)	0.387 (0.179)
		10,20	0.132(0.092)	0.137 (0.100)	0.144 (0.105)
	3.5	5,5	0.272(0.119)	0.289 (0.145)	0.297 (0.153)
		5,10	0.098(0.071)	0.095 (0.078)	0.098 (0.081)
		10,10	0.075(0.059)	0.090 (0.074)	0.093 (0.076)
		10,20	0.002(0.003)	0.002 (0.003)	0.002 (0.003)
	5.5	5,5	0.162(0.093)	0.172 (0.114)	0.177 (0.119)
		5,10	0.022(0.021)	0.020 (0.022)	0.020 (0.023)
		10,10	0.015(0.015)	0.019 (0.020)	0.019 (0.021)
		10,20	0.000(0.000)	0.000 (0.000)	0.000 (0.000)
			$\eta_1 = 1.0, \eta_2 = 10.0$		
0.0	0.0	5,5	0.561(0.148)	0.602 (0.156)	0.623 (0.170)
		5,10	0.669(0.161)	0.712 (0.168)	0.729 (0.175)
		10,10	0.665(0.174)	0.714 (0.172)	0.730 (0.177)
		10,20	0.778(0.163)	0.813 (0.159)	0.828 (0.159)
	0.5	5,5	0.546(0.139)	0.588 (0.148)	0.610 (0.164)
		5,10	0.632(0.160)	0.674 (0.166)	0.690 (0.171)
		10,10	0.623(0.184)	0.674 (0.183)	0.691 (0.190)
		10,20	0.683(0.182)	0.727 (0.179)	0.744 (0.181)
	1.5	5,5	0.485(0.134)	0.535 (0.146)	0.551 (0.160)
		5,10	0.496(0.161)	0.545 (0.175)	0.557 (0.181)
		10,10	0.472(0.173)	0.540 (0.182)	0.551 (0.189)
		10,20	0.372(0.167)	0.430 (0.184)	0.447 (0.192)
	3.5	5,5	0.356(0.129)	0.410 (0.154)	0.420 (0.163)
		5,10	0.257(0.128)	0.298 (0.154)	0.306 (0.159)
		10,10	0.220(0.113)	0.288 (0.143)	0.298 (0.151)
		10,20	0.059(0.045)	0.078 (0.062)	0.083 (0.066)
	5.5	5,5	0.274(0.114)	0.328 (0.145)	0.340 (0.157)
		5,10	0.120(0.075)	0.140 (0.095)	0.146 (0.099)
		10,10	0.102(0.068)	0.146 (0.096)	0.152 (0.100)
		10,20	0.007(0.008)	0.010 (0.012)	0.011 (0.013)

Table 4.2 Bayes factor and posterior probabilities of $H_1 : \xi_1 = \xi_2$

B_{21}^F	$P^F(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{AI}	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
2.754	0.266	2.767	0.265	2.642	0.275

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the equality of the location parameters in half-normal distributions under the reference priors. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. However the AIBF and the MIBF slightly favors the hypothesis H_1 than the AIBF for the large values of η_2. From our simulation and example, we recommend the use of the FBF than the AIBF and MIBF for practical application in view of its simplicity and ease of implementation.

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