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Noninformative priors for the common mean in log-normal distributions[†]

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Abstract

In this paper, we develop noninformative priors for the log-normal distributions when the parameter of interest is the common mean. We developed Jeffreys' prior, the reference priors and the first order matching priors. It turns out that the reference priors and Jeffreys' prior do not satisfy a first order matching criterion, and Jeffreys' prior, the reference prior and the first order matching prior are different. Some simulation study is performed and a real example is given.

Keywords: Common mean, log-normal distribution, matching prior, reference prior.

1. Introduction

The log-normal distribution is used in a wide range of applications, when the multiplicative scale is appropriate and the log-transformation removes the skew and brings about symmetry of the data distribution (Limpert *et al.*, 2001). Normality is the preferred distributional assumption in many contexts, and logarithm is often the most commonly used transformation that an analyst considers to promote it. However, there are instances when moments, and the expectation in particular, are of interest on the original scale. For example, the log-normal distribution is frequently applied to variables in environmental science (Parkhurst, 1998), occupational health research (Rappaport and Selvin, 1987), health expenditure (Zhou *et al.*, 1997), monetary units (Zabel, 1999; Longford and Pittau, 2006), etc. The population mean of such a variable may be a much more relevant target for inference than the population mean of its logarithm.

The present paper focuses on developing noninformative priors for the common mean of the log-normal distributions. In the absence of sources of information or past data, Bayesian methods rely on the objective priors or the noninformative priors.

We consider Bayesian priors such that the resulting credible intervals for the common log-normal mean have coverage probabilities equivalent to their frequentist counterparts. Although this matching can be justified only asymptotically, our simulation results indicate

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that this is indeed achieved for small or moderate sample sizes as well. This matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), Datta and Ghosh (1995, 1996), Mukerjee and Ghosh (1997).

On the other hand, Bernardo (1979) introduced the reference priors which maximizes the Kullback-Leibler divergence between the prior and the posterior. Ghosh and Mukerjee (1992) and Berger and Bernardo (1989,1992) gave a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems (Kim *et al.*, 2009a; Kang *et al.*, 2011). Quite often reference priors satisfy the matching criterion described earlier (Kang *et al.*, 2008; Kim *et al.*, 2009b).

For testing the equality of the two independent log-normal means, Zhou *et al.* (1997) have proposed a Z-score test and a nonparametric bootstrap approach. In their study, the Z-score test is the best among all five tests considered in their paper. But the Z-score test does not perform well in a range of small sample settings. So Wu *et al.* (2002) proposed two methods which are based on the signed log-likelihood ratio statistic and the modified likelihood signed log-likelihood ratio statistic. Wu *et al.* (2002) showed that the method based on the modified likelihood signed log-likelihood ratio statistic gives essentially exact coverage probabilities, and the Z-score test does not perform well in a range of small sample settings from the simulation results.

For the common log-normal mean, Gupta and Li (2006) derived the maximum likelihood estimator, and the confidence interval based on the large sample approach. In their simulation results, the coverage probabilities of the confidence interval of the common mean are lower than the nominal level. And the coverage probabilities closer to the nominal level as the sample sizes get larger. Tian and Wu (2007) developed an approach for the confidence interval estimation and hypothesis testing using the concept of generalized confidence interval and generalized p-values. Tian and Wu (2007) showed that the generalized confidence interval estimates tend to be slightly conservative as sample sizes are small and become closer to the nominal level as sample sizes increase, and the generalized approach provides much better confidence interval estimates than the large sample approach.

The outline of the remaining sections is as follows. In Section 2, we develop the first order probability matching priors for the common mean. Next we derive Fisher information matrix, and also derive the reference priors for the common mean. It turns out that Jeffreys' prior, the reference priors and the first order matching priors are different. In Section 3, We provide that the propriety of the posterior distribution for a general class of prior distributions which include Jeffreys' prior, the reference prior as well as first order matching prior. In Section 4, simulated frequentist coverage probabilities under the proposed priors are given. A real example is given.

2. The noninformative priors

2.1. The probability matching priors

Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random sample of size n from a log-normal population with parameters μ_1 and σ_1^2, and let $\mathbf{Y} = (Y_1, \cdots, Y_m)$ be a random sample of size m from a log-normal population with parameters μ_2 and σ_2^2. That is, $\log X_i$ is normally distributed with mean μ_1 and variance σ_1^2, and $\log Y_i$ is normally distributed with mean μ_2 and variance σ_2^2. Then common mean is $\eta = \exp\left(\mu_i + \sigma_i^2/2\right), i = 1, 2$, and this common mean is of interest. The problem for this common mean in developing of the noninformative priors is equivalent to $\mu = \mu_i + \sigma_i^2/2, i = 1, 2$.

For a prior π, let $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ denote the $(1-\alpha)$th percentile of the posterior distribution of θ_1, that is,

$$P^{\pi}[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})|\mathbf{X}] = 1 - \alpha, \tag{2.1}$$

where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_t)^T$ and θ_1 is the parameter of interest. We want to find priors π for which

$$P[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})|\boldsymbol{\theta}] = 1 - \alpha + o(n^{-r}). \tag{2.2}$$

for some $r > 0$, as n goes to infinity. Priors π satisfying (2.2) are called matching priors. If $r = 1/2$, then π is referred to as a first order matching prior, while if $r = 1$, π is referred to as a second order matching prior.

In order to find such matching priors π, let

$$\theta_1 = \mu, \theta_2 = \mu - \frac{1}{2}\sigma_1^2 - 2\log\sigma_1 \text{ and } \theta_3 = \mu - \frac{1}{2}\sigma_2^2 - 2\log\sigma_2.$$

The Jacobian matrix of this transformation is

$$\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\mu, \sigma_1, \sigma_2)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{2}{\sigma_1} - \sigma_1 & 0 \\ 1 & 0 & -\frac{2}{\sigma_2} - \sigma_2 \end{pmatrix}. \tag{2.3}$$

Therefore the inverse of the expected Fisher information matrix can be written as

$$\begin{aligned} I^{-1}(\theta_1, \theta_2, \theta_3) &= \left(\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\mu, \sigma_1, \sigma_2)}\right) I^{-1}(\mu, \sigma_1, \sigma_2) \left(\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\mu, \sigma_1, \sigma_2)}\right)^t \\ &= \begin{pmatrix} \frac{\sigma_1^2\sigma_2^2(2+\sigma_1^2)(2+\sigma_2^2)}{2[m\sigma_1^2(2+\sigma_1^2)+n\sigma_2^2(2+\sigma_2^2)]} & 0 & \\ 0 & \frac{2+\sigma_1^2}{n} & 0 \\ 0 & 0 & \frac{2+\sigma_2^2}{m} \end{pmatrix}. \end{aligned} \tag{2.4}$$

By (2.4), the Fisher information matrix is

$$I(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \frac{2[m\sigma_1^2(2+\sigma_1^2)+n\sigma_2^2(2+\sigma_2^2)]}{\sigma_1^2\sigma_2^2(2+\sigma_1^2)(2+\sigma_2^2)} & 0 & \\ 0 & \frac{n}{2+\sigma_1^2} & 0 \\ 0 & 0 & \frac{m}{2+\sigma_2^2} \end{pmatrix}. \tag{2.5}$$

Thus θ_1 is orthogonal to θ_2 and θ_3 in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of first order probability matching prior is characterized by

$$\pi_m(\theta_1, \theta_2, \theta_3) \propto \frac{[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^{\frac{1}{2}}}{\sigma_1\sigma_2(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}} g(\theta_2, \theta_3), \tag{2.6}$$

where $g(\theta_2, \theta_3) > 0$ is an arbitrary function differentiable in its argument. We may also note that the matching prior prior in the original parametrization $(\mu, \sigma_1, \sigma_2)$ is given by

$$\pi_m(\mu, \sigma_1, \sigma_2) \propto \sigma_1^{-2}\sigma_2^{-2}(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}[m\sigma_1^2(2+\sigma_1^2)+n\sigma_2^2(2+\sigma_2^2)]^{\frac{1}{2}} \times g\left(\mu - \frac{1}{2}\sigma_1^2 - 2\log\sigma_1, \mu - \frac{1}{2}\sigma_2^2 - 2\log\sigma_2\right). \tag{2.7}$$

2.2. The reference priors

The likelihood function of parameters μ, σ_1, σ_2 is given by

$$L(\mu, \sigma_1, \sigma_2) \propto \sigma_1^{-n}\sigma_2^{-m} \exp\{-\sum_{i=1}^{n}\frac{(\log x_i - \mu + \frac{\sigma_1^2}{2})^2}{2\sigma_1^2}\} \exp\{-\sum_{i=1}^{m}\frac{(\log y_i - \mu + \frac{\sigma_2^2}{2})^2}{2\sigma_2^2}\}. \tag{2.8}$$

Based on (2.8), the Fisher information matrix is given by

$$I(\mu, \sigma_1, \sigma_2) = \begin{pmatrix} \frac{n}{\sigma_1^2} + \frac{m}{\sigma_2^2} & -\frac{n}{\sigma_1} & -\frac{m}{\sigma_2} \\ -\frac{n}{\sigma_1} & n\left(1+\frac{2}{\sigma_1^2}\right) & 0 \\ -\frac{m}{\sigma_2} & 0 & m\left(1+\frac{2}{\sigma_2^2}\right) \end{pmatrix}. \tag{2.9}$$

We firstly derived the two group reference prior for the parameter grouping $\{\mu, (\sigma_1, \sigma_2)\}$ where μ is the parameter of interest, and σ_1 and σ_2 are treated as nuisance parameters. The reference prior algorithm is described by Berger and Bernardo (1992).

The compact subsets were taken to be Cartesian products of sets of the form

$$\mu \in [a_1, b_1], \sigma_1 \in [a_2, b_2], \sigma_2 \in [a_3, b_3].$$

In the limit a_1 will tend to $-\infty$, a_2 and a_3 will tend to 0, and $b_i, i = 1, 2, 3$ will tend to ∞. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (2.9):

$$h_1 = \frac{2[m\sigma_1^2(2+\sigma_1^2)+n\sigma_2^2(2+\sigma_2^2)]}{\sigma_1^2\sigma_2^2(2+\sigma_1^2)(2+\sigma_2^2)} \text{ and } h_2 = \frac{nm(2+\sigma_1^2)(2+\sigma_2^2)}{\sigma_1^2\sigma_2^2}.$$

Here, and below, a subscripted Q denotes a function that is constant and does not depend on any parameters but any Q may depend on the ranges of the parameters.
Step 1. Note that

$$\int_{a_3}^{b_3}\int_{a_2}^{b_2} h_2^{1/2} d\sigma_1 d\sigma_2 = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\left[\frac{nm(2+\sigma_1^2)(2+\sigma_2^2)}{\sigma_1^2\sigma_2^2}\right]^{1/2} d\sigma_1 d\sigma_2 = (nm)^{\frac{1}{2}} Q_1.$$

It follows that

$$\pi_2^l(\sigma_1, \sigma_2|\mu) = Q_1^{-1}\frac{(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}}{\sigma_1\sigma_2}.$$

Step 2. Now

$$E^l\{\log h_1|\mu\} = \int_{a_3}^{b_3}\int_{a_2}^{b_2} \frac{(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}}{Q_1\sigma_1\sigma_2} \log\left[\frac{2[m\sigma_1^2(2+\sigma_1^2)+n\sigma_2^2(2+\sigma_2^2)]}{\sigma_1^2\sigma_2^2(2+\sigma_1^2)(2+\sigma_2^2)}\right] d\sigma_1 d\sigma_2$$
$$= Q_{21}.$$

It follows that

$$\int_{a_1}^{b_1} \exp[E^l\{\log h_1|\mu\}/2]d\mu = Q_2\exp\{Q_{21}/2\}, Q_2 = b_1 - a_1.$$

Hence

$$\pi_1^l(\mu,\sigma_1,\sigma_2) = Q_1^{-1}Q_2^{-1}\frac{(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}}{\sigma_1\sigma_2}.$$

Therefore the two group reference prior is

$$\pi_{TR}(\mu,\sigma_1,\sigma_2) \propto \frac{(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}}{\sigma_1\sigma_2}. \tag{2.10}$$

Next we derived the one-at-a-time reference prior for the parameter grouping $\{\mu,\sigma_1,\sigma_2\}$. For the derivation of the reference prior, we obtain the following quantities from the Fisher information (2.9):

$$h_1 = \frac{2[m\sigma_1^2(2+\sigma_1^2)+n\sigma_2^2(2+\sigma_2^2)]}{\sigma_1^2\sigma_2^2(2+\sigma_1^2)(2+\sigma_2^2)}, h_2 = n\left(1+\frac{2}{\sigma_1^2}\right) \text{ and } h_3 = m\left(1+\frac{2}{\sigma_2^2}\right).$$

Step 1. Note that

$$\int_{a_3}^{b_3} h_3^{1/2} d\sigma_2 = \int_{a_3}^{b_3} m^{\frac{1}{2}}\left(1+\frac{2}{\sigma_2^2}\right)^{1/2} d\sigma_2 = (m)^{\frac{1}{2}}Q_1.$$

It follows that

$$\pi_3^l(\sigma_2|\mu,\sigma_1) = Q_1^{-1}\left(1+\frac{2}{\sigma_2^2}\right)^{1/2}.$$

Step 2. Now

$$E^l\{\log h_2|\mu,\sigma_1\} = \int_{a_3}^{b_3} Q_1^{-1}\left(1+\frac{2}{\sigma_2^2}\right)^{1/2}\log\left[n\left(1+\frac{2}{\sigma_1^2}\right)\right]d\sigma_2$$
$$= \log\left[n\left(1+\frac{2}{\sigma_1^2}\right)\right].$$

It follows that

$$\int_{a_2}^{b_2} \exp[E^l\{\log h_2|\mu,\sigma_1\}/2]d\sigma_1 = n^{\frac{1}{2}}Q_2.$$

Hence

$$\pi_2^l(\sigma_1, \sigma_2|\mu) = Q_1^{-1} Q_2^{-1} \left(1 + \frac{2}{\sigma_1^2}\right)^{1/2} \left(1 + \frac{2}{\sigma_2^2}\right)^{1/2}.$$

Step 3. Now

$$\begin{aligned} E^l\{\log h_1|\mu\} &= \int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_1^{-1} Q_2^{-1} \left(1 + \frac{2}{\sigma_1^2}\right)^{1/2} \left(1 + \frac{2}{\sigma_2^2}\right)^{1/2} \\ &\quad \times \log\left[\frac{2[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]}{\sigma_1^2\sigma_2^2(2+\sigma_1^2)(2+\sigma_2^2)}\right] d\sigma_2 d\sigma_1 \\ &= Q_{31}. \end{aligned}$$

It follows that

$$\int_{a_1}^{b_1} \exp[E^l\{\log h_1|\mu\}/2] d\mu = Q_3 \exp\{Q_{31}/2\}, Q_3 = b_1 - a_1.$$

Hence

$$\pi_1^l(\mu, \sigma_1, \sigma_2) = Q_1^{-1} Q_2^{-1} Q_3^{-1} \left(1 + \frac{2}{\sigma_1^2}\right)^{1/2} \left(1 + \frac{2}{\sigma_2^2}\right)^{1/2}.$$

Thus the one-at-a-time reference prior is

$$\pi_{OR}(\mu, \sigma_1, \sigma_2) \propto \sigma_1^{-1}\sigma_2^{-1}(\sigma_1^2 + 2)^{1/2}(\sigma_2^2 + 2)^{1/2}. \tag{2.11}$$

Note that the two group reference prior and the one-at-a-time reference prior are the same. Also from the Fisher information (2.9), Jeffreys' prior is

$$\pi_J(\mu, \sigma_1, \sigma_2) \propto \sigma_1^{-2}\sigma_2^{-2}[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^{\frac{1}{2}}. \tag{2.12}$$

Notice that the matching priors (2.7) include many different matching priors because of the arbitrary selection of the function g. And for some functions, there does not seem to be any improvement in the coverage probabilities with these posteriors. So we consider a particular first order matching prior where g is a constant in matching priors (2.7). This prior is given by

$$\pi_m(\mu, \sigma_1, \sigma_2) \propto \sigma_1^{-2}\sigma_2^{-2}(2+\sigma_1^2)^{\frac{1}{2}}(2+\sigma_2^2)^{\frac{1}{2}}[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^{\frac{1}{2}}. \tag{2.13}$$

Remark 2.1 Note that Jeffrey's prior, the first order matching priors and the reference priors are different each other.

3. Propriety of the posterior distribution

We investigate the propriety of posteriors for a general class of priors which include Jeffreys' prior (2.12), the reference prior (2.11) and the first order matching prior (2.13). We consider the class of priors

$$\pi(\mu, \sigma_1, \sigma_2) \propto [m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^a \frac{(2+\sigma_1^2)^c(2+\sigma_2^2)^c}{\sigma_1^b\sigma_2^b}, \tag{3.1}$$

where $a \geq 0, b > 0$ and $c \geq 0$. The following general theorem can be proved.

Theorem 3.1 The posterior distribution of $(\mu, \sigma_1, \sigma_2)$ under the prior π, (3.1), is proper if $n - 4a + b - 2c - 2 > 0$ and $m - 4a + b - 2c - 2 > 0$.

Proof. Note that the joint posterior for μ, σ_1 and σ_2 given $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}\pi(\mu, \sigma_1, \sigma_2|\mathbf{x}, \mathbf{y}) &\propto \sigma_1^{-n-b}\sigma_2^{-m-b}[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^a(2+\sigma_1^2)^c(2+\sigma_2^2)^c \\ &\quad \times \exp\left\{-\sum_{i=1}^n \frac{(\log x_i - \mu + \frac{\sigma_1^2}{2})^2}{2\sigma_1^2} - \sum_{i=1}^m \frac{(\log y_i - \mu + \frac{\sigma_2^2}{2})^2}{2\sigma_2^2}\right\}. \qquad (3.2)\end{aligned}$$

Firstly, we integrate with respect to μ from (3.2). Then

$$\begin{aligned}\pi(\sigma_1, \sigma_2|\mathbf{x}, \mathbf{y}) &\propto \sigma_1^{-n-b}\sigma_2^{-m-b}[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^a(2+\sigma_1^2)^c(2+\sigma_2^2)^c \\ &\quad \times \left(\frac{n}{\sigma_1^2} + \frac{m}{\sigma_2^2}\right)^{-\frac{1}{2}} \exp\left\{-\frac{S_1^2}{2\sigma_1^2} - \frac{S_2^2}{2\sigma_2^2}\right\} \exp\left\{-\frac{nm(\sigma_1^2 + 2\bar{x} - \sigma_2^2 - 2\bar{y})^2}{8(m\sigma_1^2 + n\sigma_2^2)}\right\} \\ &\leq \sigma_1^{-n-b}\sigma_2^{-m-b}[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^a(2+\sigma_1^2)^c(2+\sigma_2^2)^c \\ &\quad \times \left(\frac{n}{\sigma_1^2} + \frac{m}{\sigma_2^2}\right)^{-\frac{1}{2}} \exp\left\{-\frac{S_1^2}{2\sigma_1^2} - \frac{S_2^2}{2\sigma_2^2}\right\} \equiv \pi'(\sigma_1, \sigma_2|\mathbf{x}, \mathbf{y}), \qquad (3.3)\end{aligned}$$

where $S_1^2 = \sum_{i=1}^n (\log x_i - \bar{x})^2$, $\bar{x} = \sum_{i=1}^n \log x_i/n$, $S_2^2 = \sum_{i=1}^m (\log y_i - \bar{y})^2$ and $\bar{y} = \sum_{i=1}^m \log y_i/m$. If $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$ then

$$\pi'(\sigma_1, \sigma_2|\mathbf{x}, \mathbf{y}) \leq k_1 \sigma_1^{-n-b}\sigma_2^{-m-b} \exp\left\{-\frac{S_1^2}{2\sigma_1^2} - \frac{S_2^2}{2\sigma_2^2}\right\}. \qquad (3.4)$$

Therefore the (3.4) is proper, if $n + b - 1 > 0$ and $m + b - 1 > 0$. Here k_1 is a constant. If $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$ then

$$\pi'(\sigma_1, \sigma_2|\mathbf{x}, \mathbf{y}) d\sigma_1 d\sigma_2 \leq k_2 \sigma_1^{-n+4a-b+2c+1}\sigma_2^{-m+4a-b+2c+1} \exp\left\{-\frac{S_1^2}{2\sigma_1^2} - \frac{S_2^2}{2\sigma_2^2}\right\}. \quad (3.5)$$

Then the (3.5) is proper, if $n - 4a + b - 2c - 2 > 0$ and $m - 4a + b - 2c - 2 > 0$. Here k_2 is a constant. If $0 < \sigma_1 < 1$ and $\sigma_2 \geq 1$ then

$$\pi'(\sigma_1, \sigma_2|\mathbf{x}, \mathbf{y}) \leq k_3 \sigma_1^{-n-b}\sigma_2^{-m+4a-b+2c} \exp\left\{-\frac{S_1^2}{2\sigma_1^2} - \frac{S_2^2}{2\sigma_2^2}\right\}. \qquad (3.6)$$

Therefore the (3.6) is proper, if $n + b - 1 > 0$ and $m - 4a + b - 2c - 1 > 0$. Here k_3 is a constant. This completes the proof. □

Theorem 3.2 Under the prior (3.1), the marginal posterior density of μ is given by

$$\begin{aligned}\pi(\mu|\mathbf{x}, \mathbf{y}) &\propto \int_0^\infty \int_0^\infty \sigma_1^{-n-b}\sigma_2^{-m-b}[m\sigma_1^2(2+\sigma_1^2) + n\sigma_2^2(2+\sigma_2^2)]^a(2+\sigma_1^2)^c(2+\sigma_2^2)^c \\ &\quad \times \exp\left\{-\sum_{i=1}^n \frac{(\log x_i - \mu + \frac{\sigma_1^2}{2})^2}{2\sigma_1^2} - \sum_{i=1}^m \frac{(\log y_i - \mu + \frac{\sigma_2^2}{2})^2}{2\sigma_2^2}\right\} d\sigma_1 d\sigma_2. \qquad (3.7)\end{aligned}$$

Note that actually, normalizing constant for the marginal density of μ required two dimensional integration. Therefore we have the marginal posterior density of μ, and so it is easy to compute the marginal moment of μ. In Section 4, we investigate the frequentist coverage probabilities for the π_m, π_J and π_{OR}, respectively.

4. Numerical studies

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posteriors density of μ under the noninformative prior π given in (3.1) for several configurations μ, σ_1, σ_2 and (n, m). That is to say, the frequentist coverage of a $(1-\alpha)th$ posterior quantile should be close to $1-\alpha$. This is done numerically. Table 1 gives numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the our prior. The computation of these numerical values is based on the following algorithm for any fixed true $(\mu, \sigma_1, \sigma_2)$ and any prespecified probability value α. Here α is 0.05 (0.95). Let $\mu^\pi(\alpha|\mathbf{X}, \mathbf{Y})$ be the posterior α-quantile of μ given $(\mathbf{X}, \mathbf{Y})$. That is, $F(\mu^\pi(\alpha|\mathbf{X}, \mathbf{Y})|\mathbf{X}, \mathbf{Y}) = \alpha$, where $F(\cdot|\mathbf{X}, \mathbf{Y})$ is the marginal posterior distribution of μ. Then the frequentist coverage probability of this one sided credible interval of μ is

$$P_{(\mu,\sigma_1,\sigma_2)}(\alpha; \mu) = P_{(\mu,\sigma_1,\sigma_2)}(0 < \mu \le \mu^\pi(\alpha|\mathbf{X}, \mathbf{Y})). \tag{4.1}$$

The computed $P_{(\mu,\sigma_1,\sigma_2)}(\alpha; \mu)$ when $\alpha = 0.05(0.95)$ is shown in Table 4.1. In particular, for fixed (n, m) and $(\mu, \sigma_1, \sigma_2)$, we take $10,000$ independent random samples of $\mathbf{X}$ and $\mathbf{Y}$ from the log-normal populations.

In Table 4.1, we can observe that the matching prior π_m and the reference prior π_{OR} meet well the target coverage probabilities than Jeffreys' prior π_J. Also note that the results of table are not much sensitive to the change of the values of $(\mu, \sigma_1, \sigma_2)$. Thus we recommend to use the matching prior and the reference prior.

Example 4.1 This example is a bioavailability study in which a randomized, parallel-group experiment (Wu *et al.*, 2002) was conducted with 20 subjects to compare a new test formulation with a reference formulation of a drug product with a long half-life. Among other statistical analyses, testing the equality of the means of the two formulations is of great importance in determining if the two formulations have different bioavailability. The QQ plots for the original data ($C_{\max}$ data) and log-transformed data are given in Wu *et al.* (2002). As they reported, the Shapiro-Wilk tests for normality on the log-transformed data give a p-value of 0.595 for the test formulation group and a p-value of 0.983 for the reference formulation group. Therefore the log transformation normalizes the data.

For testing equal means of $C_{\max}$ between two formulations, the Z-score test (Zhou *et al.*, 1997) and r^*-test (Wu *et al.*, 2002) give a two-sided p-value of 0.203 and 0.173, respectively. Therefore we conclude that two means are equal. The Bayes estimates for μ are 6.62799, 6.63230 and 6.64791 for Jeffreys' prior, the matching prior and the reference prior, respectively. Also the 95% Bayesian credible intervals for μ are (6.37304, 6.96219), (6.37175, 6.98078) and (6.37637, 7.01571) for Jeffreys' prior, the matching prior and the reference prior, respectively. All methods yield almost the same lower bounds, but slightly different upper bounds. For comparison, the maximum likelihood estimate and the confidence interval by large sample approach are 6.60751 and (6.35888, 6.85614).

Table 4.1 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of μ

μ	σ_1	σ_2	n	π_J	π_{OR}	π_m
1	1	2	5,5	0.058 (0.904)	0.076 (0.958)	0.071 (0.933)
			5,10	0.056 (0.908)	0.073 (0.955)	0.069 (0.933)
			10,10	0.046 (0.929)	0.061 (0.957)	0.056 (0.943)
			15,20	0.048 (0.935)	0.059 (0.955)	0.056 (0.946)
	1	5	5,5	0.064 (0.914)	0.081 (0.965)	0.071 (0.929)
			5,10	0.056 (0.912)	0.073 (0.966)	0.064 (0.929)
			10,10	0.054 (0.937)	0.066 (0.962)	0.058 (0.944)
			15,20	0.048 (0.937)	0.058 (0.957)	0.051 (0.943)
	1	10	5,5	0.067 (0.921)	0.089 (0.969)	0.072 (0.933)
			5,10	0.062 (0.919)	0.085 (0.969)	0.069 (0.933)
			10,10	0.058 (0.939)	0.071 (0.963)	0.062 (0.946)
			15,20	0.049 (0.937)	0.061 (0.959)	0.051 (0.944)
10	1	2	5,5	0.061 (0.905)	0.078 (0.955)	0.074 (0.931)
			5,10	0.055 (0.910)	0.070 (0.953)	0.067 (0.932)
			10,10	0.051 (0.926)	0.066 (0.956)	0.062 (0.943)
			15,20	0.047 (0.933)	0.058 (0.955)	0.053 (0.945)
	1	5	5,5	0.059 (0.919)	0.078 (0.971)	0.067 (0.935)
			5,10	0.061 (0.918)	0.083 (0.968)	0.071 (0.935)
			10,10	0.047 (0.932)	0.059 (0.963)	0.051 (0.942)
			15,20	0.050 (0.938)	0.061 (0.960)	0.055 (0.946)
	1	10	5,5	0.063 (0.919)	0.086 (0.970)	0.069 (0.932)
			5,10	0.061 (0.922)	0.081 (0.972)	0.067 (0.934)
			10,10	0.052 (0.934)	0.065 (0.964)	0.055 (0.942)
			15,20	0.053 (0.940)	0.065 (0.961)	0.057 (0.945)
100	1	2	5,5	0.059 (0.906)	0.075 (0.955)	0.071 (0.930)
			5,10	0.051 (0.907)	0.066 (0.949)	0.063 (0.927)
			10,10	0.052 (0.925)	0.066 (0.957)	0.061 (0.942)
			15,20	0.051 (0.936)	0.063 (0.954)	0.059 (0.946)
	1	5	5,5	0.060 (0.914)	0.078 (0.967)	0.068 (0.931)
			5,10	0.056 (0.914)	0.072 (0.965)	0.064 (0.929)
			10,10	0.053 (0.934)	0.066 (0.962)	0.057 (0.943)
			15,20	0.051 (0.940)	0.062 (0.959)	0.055 (0.945)
	1	10	5,5	0.061 (0.918)	0.082 (0.967)	0.067 (0.931)
			5,10	0.064 (0.923)	0.085 (0.974)	0.070 (0.939)
			10,10	0.051 (0.934)	0.066 (0.963)	0.056 (0.942)
			15,20	0.052 (0.946)	0.065 (0.966)	0.055 (0.952)

5. Concluding remarks

In the log-normal models, we have found a prior which is a first order matching prior and reference prior for the common mean. We revealed the two group reference prior and the one-at-a-time reference prior are the same. It turns out that Jeffrey's prior, the first order matching prior and the reference prior are different each other. As illustrated in our numerical study, the matching prior and the reference prior seem to be the best appropriate results than Jeffreys' prior in the sense of asymptotic frequentist coverage property.

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