Journal of the Korean Data & Information Science Society
2011, **22**(6), 1233–1240

한국데이터정보과학회지

Geometric ergodicity for the augmented asymmetric power GARCH model†

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Received 4 October 2011, revised 28 October 2011, accepted 5 November 2011

Abstract

An augmented asymmetric power GARCH(p, q) process is considered and conditions for stationarity, geometric ergodicity and β-mixing property with exponential decay rate are obtained.

Keywords: Asymmetric power GARCH (p,q) process, β-mixing, drift condition, geometric ergodicity, irreducibility, stationarity, uniform countable additivity condition.

1. Introduction

Since the seminal works of Engle (1982) and Bollerslev (1986), the ARCH (autoregressive conditional heteroscedasticity) model and generalized ARCH (GARCH) model have been widely used to analyze financial time series. The GARCH model captures so-called stylized facts such as jumps, time varying volatility and heavy tailedness successfully, however, empirical studies show that further extensions need to be developed to explain asymmetry and long range dependence phenomena which cannot be represented by the classical GARCH model.

In this paper, we consider the augmented asymmetric power GARCH (p, q) (APARCH(p, q)) model which is introduced by Ding *et al.* (1993) to feature both asymmetry and long range dependence :

$$\varepsilon_t = h_t e_t \tag{1.1}$$

$$h_t^\delta = \alpha_0 + \sum_{i=1}^{p} \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{q} \beta_j h_{t-j}^\delta \tag{1.2}$$

† This paper was supported by grant 2009 URP.

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where $\alpha_0 > 0,\ \delta > 0,\ \alpha_i \geq 0,\ \beta_j \geq 0,\ |\gamma_i| < 1\ (i = 1, ..., p,\ j = 1, ..., q)$, and $\{e_t\}_{t=0}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $E|e_t|^{\delta} < \infty$. If $\delta = 2, \gamma_i = 0, \forall i$, then the process (1.1)-(1.2) reduces to the classical GARCH(p, q) model.

Ling and McAleer (2002) provide the necessary and sufficient condition for the strict stationarity, ergodicity and existence of higher order moments of the transformed GARCH(p, q) model with $\gamma_i = \gamma$ for all $i = 1, \cdots, p$. Geometric ergodicity and existence of moments conditions for the ARCH-type model ($q = 0$) are given in Hwang and Kim (2004). Lee and Shin (2004) shows the geometric ergodicity and β-mixing property of the APARCH(p, q) model with $\gamma_i = \gamma, \forall i$ based on a polynomial matrix transformation. Bellini and Bottolo (2007) give stationarity domains for δ-power GARCH model. There is no literature that we know of, in which the geometric ergodicity of the model (1.1)-(1.2) is obtained. Geometric ergodicity and mixing properties are useful for numerous applications including asymptotic statistics.

Section 2 gives some definitions and preliminary results. In Section 3, we prove our main results that are concerned with the geometric ergodicity and β-mixing property. A higher order moment condition is also given.

2. Preliminaries

Let $\{X_t\}$ be a Markov process with state space S and t-step transition probability function $P^t(x, G)$ with $x \in S,\ G \in \mathcal{B}(S)$.

A Markov chain $\{X_t\}$ is ϕ-irreducible if, for some nontrivial measure ϕ on $(S, \mathcal{B})$, $\sum_{t>0} P^t(x, G) > 0$ for all $x \in S$ whenever $\phi(G) > 0$. $\{X_t\}$ is called a Feller chain, if for each bounded continuous f on S, $E[f(X_t)|X_{t-1} = x]$ is a continuous function of x.

A Markov chain $\{X_t\}$ is said to satisfy the drift condition if there are a positive function V on S, a compact set K and some real numbers $M < \infty,\ \epsilon > 0,\ 0 < \rho < 1$ such that:

(1) $E[V(X_{t+1})|X_t = x] \leq \rho V(x) - \epsilon$ if $x \in K^c$;

(2) $E[V(X_{t+1})|X_t = x] \leq M$ if $x \in K$.

$\{X_t\}$ is said to be geometrically ergodic if there exist a probability measure π and a constant $\rho,\ 0 < \rho < 1$, such that

$$\rho^{-t}\|P^t(x, \cdot) - \pi(\cdot)\| \to 0 \ \ as \ \ t \to \infty$$

for each $x \in S$.

Theorem 2.1 (Meyn and Tweedie, 1993) Suppose that a Markov chain $\{X_t\}$ has the Feller property. If $\{X_t\}$ satisfies the drift condition for a compact set K, then there exists an invariant probability measure. In addition, if the process is ϕ-irreducible and aperiodic, then the given process is geometrically ergodic.

Theorem 2.1 shows that the crucial step to prove the geometric ergodicity of a process is to show that the given Markov chain is $\phi-$irreducible and holds the drift condition. In many cases, however, proving irreducibility of a Markov process is an awkward task. Consulting the following Theorem 2.2, irreducibility of the process can be derived from connections between ϕ-irreducibility and the uniform countable additivity condition. A Markov chain

$\{X_t\}$ is said to hold the uniform countable additivity condition, if its transition probability function satisfies that for any sequence of compact sets $G_n \downarrow \emptyset$,

$$\lim_{G_n \downarrow \emptyset} \sup_{x \in K} P(x, G_n) = 0 \text{ for every compact set } K \in \mathcal{B}.$$

Theorem 2.2 (Tweedie, 2001) Suppose that the drift condition holds with a test set K and the uniform countable additivity condition holds for the same set K. Then there is a unique invariant measure for X_t if and only if X_t is ϕ-irreducible.

For more information on Markov chain theory, we refer to Meyn and Tweedie (1993).

3. Main results

Consider the APGARCH(p, q) model given by (1.1) and (1.2). Let

$$e_t^+ = max\{0, e_t\},\ e_t^- = max\{0, -e_t\},\ e_t^{+\delta} = (e_t^+)^\delta,\ e_t^{-\delta} = (e_t^-)^\delta.$$

Multiplying both sides of (1.2) by $e_t^{+\delta}$ and $e_t^{-\delta}$ respectively, and denoting $y_t^+ = h_t^\delta e_t^{+\delta}$, $y_t^- = h_t^\delta e_t^{-\delta}$, we obtain that

$$y_t^+ = \alpha_0 e_t^{+\delta} + \sum_{i=1}^p \alpha_i (1 - \gamma_i)^\delta e_t^{+\delta} y_{t-i}^+ + \sum_{i=1}^p \alpha_i (1 + \gamma_i)^\delta e_t^{+\delta} y_{t-i}^- + \sum_{j=1}^q \beta_j e_t^{+\delta} h_{t-j}^\delta$$

and

$$y_t^- = \alpha_0 e_t^{-\delta} + \sum_{i=1}^p \alpha_i (1 - \gamma_i)^\delta e_t^{-\delta} y_{t-i}^+ + \sum_{i=1}^p \alpha_i (1 + \gamma_i)^\delta e_t^{-\delta} y_{t-i}^- + \sum_{j=1}^q \beta_j e_t^{-\delta} h_{t-j}^\delta.$$

Then the equation (1.1) and (1.2) can be rewritten in the following Markovian representation:

$$Y_t = A(e_t) Y_{t-1} + B(e_t), \qquad t \geq 0 \tag{3.1}$$

where

$$Y_t = (y_t^+, \cdots, y_{t-p+1}^+, y_t^-, \cdots, y_{t-p+1}^-, h_t^\delta, \cdots, h_{t-q+1}^\delta)',$$

$A_t = A(e_t) =$

$$\left(\begin{array}{ccc|ccc|ccc}
a_1 e_t^{+\delta} & \cdots & a_p e_t^{+\delta} & b_1 e_t^{+\delta} & \cdots & b_p e_t^{+\delta} & \beta_1 e_t^{+\delta} & \cdots & \beta_q e_t^{+\delta} \\
 & J_1 & & & O_{(p-1)\times p} & & & O_{(p-1)\times p} & \\
\hline
a_1 e_t^{-\delta} & \cdots & a_p e_t^{-\delta} & b_1 e_t^{-\delta} & \cdots & b_p e_t^{-\delta} & \beta_1 e_t^{-\delta} & \cdots & \beta_q e_t^{-\delta} \\
 & O_{(p-1)\times p} & & & J_1 & & & O_{(p-1)\times p} & \\
\hline
a_1 & \cdots & a_p & b_1 & \cdots & b_p & \beta_1 & \cdots & \beta_q \\
 & O_{(p-1)\times p} & & & O_{(p-1)\times p} & & & J_2 &
\end{array}\right)$$

$$J_1 = (I_{(p-1)\times(p-1)} O_{(p-1)\times 1}), \quad J_2 = (I_{(q-1)\times(q-1)} O_{(q-1)\times 1}),$$
$$a_i = \alpha_i (1 - \gamma_i)^\delta, \quad b_i = \alpha_i (1 + \gamma_i)^\delta,$$

and $B_t = B(e_t) = (\alpha_0 e_t^{+\delta}, 0, \cdots, 0, \alpha_0 e_t^{-\delta}, 0, \cdots, 0, \alpha_0, 0, \cdots, 0)'$. $\{A_t\}_{t=0}^{\infty}$ is a sequence of i.i.d. nonnegative $(2p+q) \times (2p+q)$ random matrices and $\{B_t\}_{t=0}^{\infty}$ a sequence of i.i.d and nonnegative $(2p+q) \times 1$ random vectors. Note that if $s < t$, then (A_t, B_t) and Y_s are independent.

We make the assumptions:
Assumption A1. $\sum_{i=1}^{q} \beta_i + \sum_{i=1}^{p} \alpha_i E(|e_{t-i}| - \gamma_i e_{t-i})^{\delta} < 1$.
Assumption A2. e_t has a probability density function g with respect to a Lebesgue measure μ on R and g is bounded on compacts.

For notational convenience, we assume that $p = q$. If $p > q$, then take $\beta_{q+1} = \cdots = \beta_p = 0$.
Let $A = E(A_t)$, $E(e_t^{+\delta}) = \mu_1$, $E(e_t^{-\delta}) = \mu_2$. $\rho(A)$ denotes the spectral radius of a matrix A.

Lemma 3.1 $\rho(A) < 1$ if and only if the Assumption A1 holds.

Proof. A simple calculation leads to the relation: $det(A - \lambda I) = (-1)^{3p} \lambda^{2p} \{\lambda^p - (\mu_1 a_1 + \mu_2 b_1 + \beta_1)\lambda^{p-1} - (\mu_1 a_2 + \mu_2 b_2 + \beta_2)\lambda^{p-2} - \cdots - (\mu_1 a_p + \mu_2 b_p + \beta_p)\}$. Therefore, by Lemma 2.3 in Ling(1999), $\rho(A) < 1$ if and only if the Assumption A1 holds.

Lemma 3.2 If the Assumption A1 holds, then there exists a unique strictly stationary solution of (3.1).

Proof. Recall that the top Lyapunov exponent γ associated with a sequence $\{A_t, t \in Z\}$ of i.i.d. random matrices is given by

$$\gamma = \lim_{t \to \infty} \frac{1}{t} \log \|A_0 A_{-1} \cdots A_{-t}\|.$$

Since A_t is nonnegative, $\gamma < \log \rho(A)$ (Kesten and Spitzer (1984)) and hence $\rho(A) < 1$ implies that $\gamma < 0$. Applying Theorem 2.5 in Bougerol and Picard (1992) and above Lemma 3.1, the existence of a unique strictly stationary solution of the equation (3.1) is obtained.

Lemma 3.3 $\{Y_t\}$ has the Feller property.

Proof. For any bounded and continuous function f on R^{2p+q}, $f(A_t x_n + B_t)$ converges to $f(A_t x + B_t)$ if $x_n \to x$ as n goes to ∞. Apply the Lebesgue dominated convergence theorem to get that $E[f(A_t x_n + B_t)]$ converges to $E[f(A_t x + B_t)]$ as $n \to \infty$.

Lemma 3.4 Under the Assumption A1, the drift condition holds.

Proof. Define a test function $V : R^{3p} \to R^+$ by

$$V(x_1, \cdots, x_{3p}) = \sum_{i=1}^{p} (c_i|x_i| + d_i|x_{p+i}| + f_i|x_{2p+i}|),$$

where nonnegative constants $c_i, d_i, f_i (1 \le i \le p)$ are to be defined later.

For $Y_{t-1} = y = (x_1, \cdots, x_p, z_1, \cdots, z_p, w_1, \cdots, w_p) \in R^{+3p}$, we have that

$$E[V(Y_t)|Y_{t-1} = y]$$
$$= c_1 E[y_t^+|Y_{t-1} = y] + d_1 E[y_t^-|Y_{t-1} = y] + f_1 h_t^\delta + \sum_{i=2}^{p}(c_i x_{i-1} + d_i z_{i-1} + f_i w_{i-1})$$
$$= \sum_{i=1}^{p}[(ma_i + c_{i+1})x_i + (mb_i + d_{i+1})z_i + (m\beta_i + f_{i+1})w_i] + \alpha_0 m, \quad (3.2)$$

where $m = c_1\mu_1 + d_1\mu_2 + f_1$. (assume $c_{p+1} = d_{p+1} = f_{p+1} = 0$.)

From the Assumption A1, we may choose $0 < \rho < 1$, such that

$$\sum_{i=1}^{p}\beta_i + \mu_1\sum_{i=1}^{p}a_i + \mu_2\sum_{i=1}^{p}b_i < \rho^p < \rho < 1. \quad (3.3)$$

Let $\tilde{A} = \sum_{i=1}^{p}\rho^{p-i}a_i,\ \tilde{B} = \sum_{i=1}^{p}\rho^{p-i}b_i,\ \tilde{C} = \sum_{i=1}^{p}\rho^{p-i}\beta_i$.

Choose $c_1 > 0$ arbitrarily but fixed and then take

$$f_1 = \frac{\tilde{C}}{\tilde{A}}c_1, \quad d_1 = \frac{(c_1\mu_1 + f_1)\tilde{B}}{\rho^p - \mu_2\tilde{B}}. \quad (3.4)$$

Here $d_1 > 0$, since $\tilde{C} + \mu_1\tilde{A} + \mu_2\tilde{B} \le \rho^p < \rho < 1$.

Now define, for $i = 2, \cdots, p$,

$$c_i = \frac{1}{\rho}(ma_i + c_{i+1}),\ d_i = \frac{1}{\rho}(mb_i + d_{i+1}),\ f_i = \frac{1}{\rho}(m\beta_i + f_{i+1}). \quad (3.5)$$

Then simple calculation using (3.3)-(3.5) yields that for $1 \le i \le p$,

$$a_i m + c_{i+1} \le \rho c_i, \quad b_i m + d_{i+1} \le \rho d_i, \quad \beta_i m + f_{i+1} \le \rho f_i. \quad (3.6)$$

Combine (3.2) and (3.6) to derive that

$$E[V(Y_t)|Y_{t-1} = y] \le \rho V(y) + \alpha_0 m.$$

Since $V(y) \to \infty$ as $\|y\| \to \infty$, there are some constants $\rho', \rho < \rho' < 1$, $\epsilon > 0$, $k > 0$ and $M_K < \infty$ such that for the compact set $K = \{x : \|x\| \le k\}$, the following two inequalities hold:

$$E[V(Y_t)|Y_{t-1} = y] \le \rho' V(y) - \epsilon, \quad y \in K^c, \quad (3.7)$$

and

$$\sup_{y \in K} E[V(Y_t)|Y_{t-1} = y] \le M_K. \quad (3.8)$$

Thus Y_t satisfies the drift condition.

Lemma 3.5 Suppose the Assumption A1 and A2 hold. Then the process Y_t given in equation (3.1) is ϕ-irreducible and aperiodic.

Proof. For given $Y_{t-1} = y = (x_1, \cdots, x_p, z_1, \cdots, z_p, w_1, \cdots, w_p) \in R^+_{3p}$, $k(y) := \alpha_0 + \sum_{i=1}^p (a_i x_i + b_i z_i + \beta_i w_i) > 0$, we have that

$$\begin{aligned}
&P(y, G)\\
&\quad = P(Y_t \in G | Y_{t-1} = y)\\
&\quad = \alpha P((k(y)e_t^{+\delta}, x_1, \cdots, x_{p-1}, 0, z_1, \cdots, z_{p-1}, k(y), w_1, \cdots, w_{p-1}) \in G | e_t > 0)\\
&\quad + (1-\alpha) P((0, x_1, \cdots, x_{p-1}, k(y)e_t^{-\delta}, z_1, \cdots, z_{p-1}, k(y), w_1, \cdots, w_{p-1}) \in G | e_t < 0)\\
&\quad = \alpha \int_G f_1(y, u_1) d\lambda_1(u) + (1-\alpha) \int_G f_2(y, u_{p+1}) d\lambda_2(u) \qquad (3.9)
\end{aligned}$$

where $\alpha = P(e_t > 0)$, $u = (u_1, \cdots, u_{3p})$,

$$f_1(y, w) = \frac{1}{\alpha}\, g\Big(\big(\frac{w}{k(y)}\big)^{1/\delta}\Big) \frac{w^{(1-\delta)/\delta}}{\delta (k(y))^{1/\delta}} I_{\{w>0\}},$$

$$f_2(y, w) = \frac{1}{1-\alpha}\, g\Big(-\big(\frac{w}{k(y)}\big)^{1/\delta}\Big) \frac{w^{(1-\delta)/\delta}}{\delta (k(y))^{1/\delta}} I_{\{w>0\}},$$

$$\lambda_1 = \mu(u_1) \prod_{i=1}^{p-1} \delta_{x_i}(u_{i+1}) \delta_0(u_{p+1}) \prod_{i=1}^{p-1} \delta_{z_i}(u_{p+i+1}) \delta_{k(y)}(u_{2p+1}) \prod_{i=1}^{p-1} \delta_{w_i}(u_{2p+i+1}),$$

$$\lambda_2 = \delta_0(u_1) \prod_{i=1}^{p-1} \delta_{x_i}(u_{i+1}) \mu(u_{p+1}) \prod_{i=1}^{p-1} \delta_{z_i}(u_{p+i+1}) \delta_{k(y)}(u_{2p+1}) \prod_{i=1}^{p-1} \delta_{w_i}(u_{2p+i+1}),$$

and δ_x denotes the degenerate measure on x. Here $f_1(y, w)$ and $f_2(y, w)$ are conditional probability density functions of $k(y)e_t^{+\delta}$ given $e_t > 0$ and $k(y)e_t^{-\delta}$ given $e_t < 0$, respectively.

Choose $\varepsilon > 0$ arbitrary and fix. Let K be a compact set in the equation (3.7) and (3.8). From the fact that $0 < \alpha_0 \le k(y) \le b < \infty$, for some constant b if $y \in K$, there exists a compact set K_1 satisfying $K \subset K_1$ and

$$\sup_{y \in K} P(y, K_1^c) \le \varepsilon. \qquad (3.10)$$

Moreover, we have that

$$\sup_{y \in K} P(k(y)e_t^{+\delta} \in (0, a)) \le P(e_t^{+\delta} \in (0, a/\alpha_0))$$

and then we may choose $a > 0$ so that

$$\sup_{y \in K} P(k(y)e_t^{+\delta} \in (0, a)) \le \varepsilon, \quad \sup_{y \in K} P(k(y)e_t^{-\delta} \in (0, a)) \le \varepsilon.$$

Take $\bar{C} = \{u | 0 < u_1 < a, 0 < u_{p+1} < a\}, C = \{u | u_1 \ge a, u_{p+1} \ge a\}$. Then

$$\sup_{y \in K} P(y, \bar{C}) \le \varepsilon \qquad (3.11)$$

and for some constant $m < \infty$,

$$\max\{f_1(y,w), f_2(y,w)\} \leq m, \quad y \in K, w \in K_1 \cap C. \tag{3.12}$$

Now, for any decreasing sequence of compact sets $G_n, G_n \downarrow \emptyset$, we have that

$$\begin{aligned} \sup_{y\in K} P(y, G_n) &\leq \sup_{y\in K}\{P(y, G_n \cap K_1 \cap C) + P(y, \bar{C}) + P(y, K_1^c)\} \\ &\leq 2\varepsilon + \sup_{y\in K} P(y, G_n \cap K_1 \cap C) \\ &\leq 2\varepsilon + m\mu(G_n) \\ &\leq 3\varepsilon, \end{aligned} \tag{3.13}$$

for sufficient large n. The 2nd inequality in (3.13) follows from (3.10) and (3.11) and the 3rd inequality is true because of (3.9) and (3.12).

Therefore $\lim_{G_n\downarrow\emptyset} \sup_{y\in K} P(y, G_n) = 0$ holds and by applying Lemma 3.2, Lemma 3.4 and Theorem 2.2, the proof of irreducibility of Y_t is completed.

On the other hand, Lemma 3.2 ensures the irreducible Markov chain Y_t in (3.1) has a unique strictly stationary solution and then $P^t(x, dy)$ converges weakly to a distribution $\pi(dy)$, say as $t \to \infty$ and π is independent on a starting state x. Hence we may choose a compact set C and a large n such that $\pi(C) > 0$ and

$$P^n(x, C) > 0 \quad \text{and} \quad P^{n+1}(x, C) > 0, \quad \forall x \in C, \tag{3.14}$$

which implies that Y_t is aperiodic. The proof is accomplished.

Theorem 3.1 Let the Assumption A1 and A2 hold. Then h_t^δ and ε_t are geometrically ergodic.

Proof. Combining Lemma 3.3-3.5 and Theorem 2.1 yields that Y_t is a geometric ergodic process and so are h_t^δ and ε_t.

Due to its importance, we state the following theorem in which the higher order moments condition is given. Note that $A^{\otimes m} = A \otimes A \otimes \cdots \otimes A($ m factors) where $\otimes$ denotes the Kronecker product.

Theorem 3.2 The necessary and sufficient condition for $E(|\varepsilon_t|^{m\delta}) < \infty$ is $\rho[E(A_t^{\otimes m})] < 1$.

Proof. Since all elements of A_t, B_t and Y_t are nonnegative, proof can be derived by adopting the same manner as that of Theorem 2.1 in Ling and McAleer (2002) and is omitted.

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