Optimal Control for Cash Management with Investment and Retrieval

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투자와 회수를 고려한 기업의 최적 현금 관리

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We develop a cash management model in which firms face randomly occurred investment projects and retrieve investments upon the maturity of these projects. Using the Markov Decision Problem approach, we examine a control policy which dynamically adjusts the cash balance under the discounted cost criterion. The existence of an optimal policy is shown under some conditions. The optimal solution procedure is developed to find the optimal points and the optimal sizes for adjusting the cash balance. In numerical experiment, we investigate important structural properties of the optimal cash management policy.

Keyword: cash management, investment, retrieval, markov decision problem, optimal control

1. Introduction

The problem of managing the operating cash to meet demand for cash at minimum cost is known as the cash balance or cash management problem (Bensoussan *et al.*, 2009). Managing a firm's cash is of vital importance for its operational and financial success. Dittmar and Mahrt-Smith (2007) state that in 2003, the sum of all cash and marketable securities represented more than 13% of the sum of all assets for large publicly traded US firms. According to Hugonniery *et al.* (2011), firms has accumulated enormous piles of cash over the past decades in response to the increase in idiosyncratic volatility, with an average cash-to-assets ratio for U.S. industrial firms that has increased from 10 : 5% in 1980 to 23 : 2% in 2006.

By keeping excess cash in the cash account, the company will lose the potential interest (an opportunity cost) generated by investing the money in securities. On the other hand, if cash in the cash account is insufficient, penalty costs can be incurred as a result of delay in meeting demands for cash or loss of the potential interest generated by investing the money in securities. This implies that firms must maintain a balance between the amounts of cash sitting in the cash account and invested in securities. The difficulty of the cash management problem arise first from the interdependence of the periodical investment/refinancing decisions and second from the uncertain future evolvement of the cash flows and of the returns on the financial markets (Schmid, 2011).

Although some version of the cash balance problem is faced by manufacturing firm, it is also of vital importance to financial institutions such as trust funds, pension funds, mutual funds, insurance com-

panies, banks, etc (Eppen and Fama, 1969). Indeed, the cash balance problem captures the essence of the dynamic portfolio problems of mutual funds and other financial institutions (Black and Telmer, 1999). For example, the earning assets of a mutual fund comprise a portfolio of securities. The goal of the fund is to maximize the return on this portfolio subject to certain self-imposed restrictions on riskiness. However, the fund is subject to stochastic inflows and outflows of cash. Thus, it is necessary to operate a policy which decides when to shift from cash to securities and vice versa. Through their apparent importance, cash balance problems have received a considerable amount of attention in the literature of economics, finance, and management sciences (Feinburg and Lewis, 2005).

The cash management problem in the literature was first analyzed by Baumol (1952). He recognized the similarities between cash and inventory management because the cash balance is also an inventory level of the product, and applied inventory control analysis to the management of cash balance under the two-asset setting. He extended the economic order quantity (EOQ) model to examine its implications to cash management. The Baumol model assumes the cash manager invests excess funds in interest bearing securities and liquidates them to meet the firm's demand for cash. As investment returns increase, the opportunity cost of holding cash increases and the cash manager decreases cash balances. As transaction costs (cost of liquidating short-term investments) increase, the cash manager decreases the number of times of liquidating securities, leading to higher cash balances. He formulated a simple inventory model to compute the average cash balance of a firm, assuming that the demand for cash is deterministic and occurs at a constant rate per period.

Miller and Orr (1966) studied the model with the cash balance changing randomly and proposed the simple policy of managing cash under a two asset setting. When the cash balance gradually grows and eventually reaches the upper control limit set by the firm, the excessive cash holdings are used to buy securities. When the cash balance will gradually fall off and eventually the lower control limit will be reached, the firm will sell some of the liquid assets. Every time the cash balance reaches the upper or the lower control limit, the cash balance is assumed to be restored to a level which is called the return point and the process restarts. The Miller-Orr model is based on the assumptions that (i) the daily rate of in-

terest earned on the portfolio is a constant, (ii) the transaction cost is a constant and is independent of the size, (iii) the lead time in buying and selling securities is negligible, (iv) the cash balance will increase or decrease by m dollars during any hour and the cash inflow and outflow are determined by a sequence of independent Bernoulli trials with equal probability. Apart from the cash management problem, the Miller-Orr model has variety of non financial applications such as controlling the trajectory of missiles in the presence of various forces that make the trajectory to be deviated from the target, controlling the water level of a dam within control limits, etc (Premachandra, 2004).

Girgis (1968) and Neave (1970) formulated the cash management problem as a multi period inventory problem where the financial manager is allowed to change the cash level in any direction at the beginning of each period.

There has been several papers which apply the optimal control technique of impulse control to the cash management problem. Constantinides and Richard (1978) first showed the existence of a simple optimal impulse policy of a control band type in continuous time under the assumption that linear holding and penalty costs. Harrison et al. (1983) proved a similar result assuming a non-negative constraint for the cash balance which implies that the lowest barrier is equal to zero, and presented a simple numerical procedure to compute the three remaining optimal barriers. Baccarin (2002) also showed that there exists an optimal policy for the model of quadratic holding and penalty costs. Premachandra (2004) improved Miller and Orr's model using a diffusion approximation model by making no assumptions about the probability distribution of the cash balance, assuming different transaction costs for buying and selling securities, and assuming a small non-zero lead time interval.

There are several papers that studies the cash management problem in problem settings other than Baumol and Miller and Orr's models. Ogden and Sundaram (1998) extended the Baumol model to include meeting cash disbursements by borrowing from a line of credit rather than liquidating short-term securities. They explored the tradeoffs between transactions costs (converting securities to cash) versus lost investment income (from the short-term investments portfolio) and interest expense (from credit line borrowing). Sato and Sawaki (2009) dealt with the cash management model in which the two types of funds with different transaction costs are available when-

ever the manager adjusts the cash level, and showed that there exists an optimal policy for infinite-horizon by using the impulse control. The policy is based on band policy $d < D < U < u$; when the cash level falls to d (rise to u), then it is adjusted up to level D (down to U). Furthermore, each values of band changes depending on the amount of short-term debt outstanding at intervention time. Bensoussan *et al.*(2009) considered a cash management problem involving two types of assets, namely deposits in a bank account and investments in stock and studied the problem of maximizing the terminal value of the total assets when the rate of return on the stock is modeled by a diffusion process.

In this paper, we consider a firm that manages two types of asset accounts; the firm's cash balance and a portfolio of marketable securities, and deal with the cash management model depicted in <Figure 1>. A firm faces investment projects which causes the cash outflow. When an investment comes to maturity, it is completely retrieved and increases the firm's cash balance. From time to time the firm has to transfer money from one account to the other by selling or buying securities in order to bring the cash balance to a suitable level. These transactions incur transaction costs and therefore the firm is interested in determining the optimal values for the lower and upper control limits which minimize the overall cost of managing the cash.

Compared to the references cited above, the most distinct feature of our model comes form the fact that the cash inflow and outflow generated by investment projects are correlated because investments are retrieved when projects are matured. In the cash management literature, it has been generally assumed that these two flows are independent of each other. In our model, however, the decisions on cash management are affected by the size of investment projects under operation as well as the cash balance at the present time, which makes the analysis complicated.

This paper addresses the following research questions : (1) When should the firm increase the cash balance from selling securities? (2) When should the firm decrease the cash balance from buying securities? (3) What are the optimal sizes of selling and buying securities? (4) If the system parameters are changed, what are their impacts on the cash balance decision?

To provide answers to the questions above, we will model the dynamics of the cash balance using the Markov Decision Problem (MDP) framework. Based on this, we show the existence of an optimal policy under some conditions. We develop the optimal solution procedure which finds the optimal points and the optimal sizes for adjusting the cash balance. Through numerical experiment, we investigate important structural properties of the optimal cash management policy. In this paper, we restrict our attention to the case that investment periods as an exponential random variable. The exponential investment period is appropriate when the period is random with a larger variance-to-mean ratio. In particular, if the investment period is flexible and extendable, it is known that the exponential distribution approximation is reasonable. Of course most investment projects of a company are largely predictable and they often follow a seasonal pattern. However, there are always some randomness which either cannot be perfectly predicted or it would take long to foresee. Our model is intended to provide the researchers/managers with insights into the effective cash management when the occurrence of investment project cannot be predicted or would be too expensive to foresee.

The rest of the paper is organized as follows. We present major model assumptions in the next section. In Section 3, we provide a formulation of our model. Section 4 characterizes the optimal cash management policy under a special case. In Section 5, we present a numerical procedure that finds the optimal control parameters. Section 6 discusses the results of numerical experiment and the last section states conclusions.
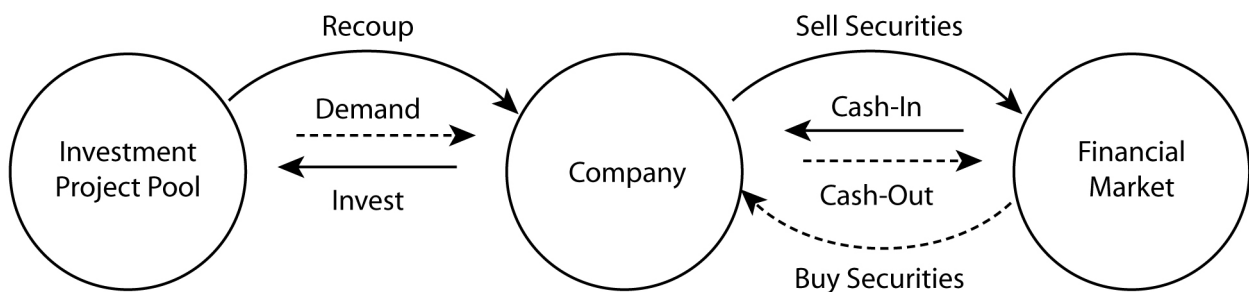

Figure 1. Dynamic cash flow management model with investment and retrieval

2. Model Assumptions

The major assumptions made in the model are as follows :

Investment projects occur according to a Poisson distribution with rate λ.

A project requires m_i amounts of cash with a probability p_i, $i = 1, \cdots, I$, $\sum_{i=1}^{I} p_i = 1$. Therefore, the occurrence rate of projects with m_i cash requirement is $p_i \lambda \equiv \lambda_i$. Upon occurrence of a project with m_i cash requirement, if the cash balance is not sufficient, it is lost and charged at c_L^i.

The maximum number of projects the company can manage is limited to F. Therefore, when the company is investing the money to F projects, the action of selling securities to increase the cash balance is not necessary.

Denote by $n_i(t)$ the number of projects under investment with m_i cash requirement at time t.

The investment durations of projects are exponentially random variables with mean μ^{-1}.

We assume that all projects never fail, that is, the investments are completely retrieved at the end of their period.

A holding cost c_h is incurred per unit of time per unit cash during which cashes are not used in investment. This cost is primarily the opportunity cost of holding cash rather than investing the wealth in interest-bearing bonds.

Whenever company sells securities to increase the cash balance by q^s (buys securities to decrease the cash balance by q^B), a transaction cost $c_S(q^S)$ ($c_B(q^B)$) is incurred. These costs can be thought of as primarily brokerage fees which, in financial markets, are very often approximately proportional to the amount of funds transferred.

Lead times required for selling or buying securities are assumed to be negligible. Given the speed of transfer in financial markets, this does not seem to be an unrealistic assumption. The lead time for most of the firms is short enough to assume that no cash inflow or outflow occurs during this time period.

Denote by $B(t)$ the cash balance at any time t.

3. Problem Formulation

A state at time t is described by the vector $(N(t),$ $B(t))$ where $N(t) \equiv (n_1(t), \cdots, n_I(t))$. Note that $\sum_{i=1}^{I} n_i(t) \leq F$. We denote the state space by Γ. Let $v(N(t), B(t))$ be the optimal expected discounted cost over an infinite horizon when the initial state is given by $(N(t), B(t))$. In state $(N(t), B(t))$, there are three mutually exclusive sets of possible decisions; Increase the cash balance by q^s, Decrease the cash balance q^B, Do nothing.

Since the state $(N(t), B(t))$ is a continuous time stochastic process and the inter-arrival times of projects and the investment durations of projects follow the exponential distribution, the original problem is a continuous time Markov decision problem (MDP) where the sum of the transition rates at every state is bounded by $\gamma \equiv \lambda + Nm$. In other words, the time intervals between two consecutive decision epochs are exponentially distributed and their mean is at most $1/\gamma$. Let the cost at time $w \in R^+$ be discounted with a factor $e^{-\beta w}$ where β is an interest rate. If we follow the uniformization process (Lipmann, 1975), a continuous time MDP can be formulated with an equivalent discrete time MDP with a transition rate γ. The essence of this uniformization process is to allow fictitious self-loop transitions for the states that have a smaller transition rate than γ. By doing so, each state has the same transition rate equal to γ and the expected transition time is constant and equals $1/\gamma$. Furthermore, the discount factor between two consecutive decision epochs becomes $\gamma/(\beta + \gamma)$. The set of decision epochs is the set of project arrival epoch and project maturity epoch. Without any loss of optimality, the class of admissible strategies is taken to be the set of non-anticipative, stationary, non-randomized, Markov policies that are based on perfect observations of the queue length processes. For the discrete time MDP, we denote the system state by (N, B).

The following state transitions can be considered in state (N, B) :

- Occurrence of a project with m_i cash requirement : n_i increases by one and B decreases by m_i when $B \geq m_i$, otherwise, n_i remains the same and the penalty cost c_L^j occurs.
- Maturity of a project with m_i cash requirement : n_i decreases by one and B increases by m_i.
- Selling securities : B increases by q^S
- Buying securities : B decreases by q^B

The goal of this paper is to find a control policy that minimizes the long-run expected discounted cost of managing the cash balance. Let $v(N, B)$ be the optimal expected discounted cost over an infinite horizon when the initial state is given by (N, B). We first define the one stage expected discounted cost in state (N, B) which is given by

$$g(N, B) = c_h B + \sum_{i=1}^{I} \lambda_i c_L^i 1(B < m_i)$$

where the indicator function $1(B < m_i)$ is 1 if $B < m_i$ is true, otherwise, 0. Note that γ is the discount factor for the discrete time MDP and the expected transition time is $1/\gamma$. When $B < m_i$ upon occurrence of a project with m_i cash requirement, a cost of losing the investment opportunity is incurred at a rate of $\lambda_i c_L$ during $1/\gamma$. Denote e_k is the kth unit vector of dimension I. Since the expected discounted cost during $1/\gamma$ is bounded, the optimal total discounted cost function v can be shown to satisfy the following optimality equation (Bellman's Equation) (see chapter 6 in Puterman, 2005) :

$$v(N, B) = Tv(N, B) \qquad (1)$$
$$= \min\{T_U v(N, B), T_S v(N, B),$$
$$T_B v(N, B)\}$$

where
$$T_U v(N, B) = 1/(\beta + \gamma)[g(N, B) +$$
$$+ \sum_{i=1}^{I} \lambda_i \left\{ \begin{array}{l} v(N + e_i, B - m_i) 1(B \geq m_i) + \\ v(N, B) 1(B < m_i) \end{array} \right\} \quad (2)$$
$$+ \sum_{i=1}^{I} n_i \mu v(N - e_i, B + m_i) + (F - \sum_{i=1}^{I} n_i)$$
$$\mu v(N, B)],$$
$$T_S v(N, B) = c_S(q^S) + T_U v(N, B + q^S) \qquad (3)$$
$$T_B v(N, B) = c_B(q^B) + T_U v(N, B - q^B) \qquad (4)$$

The operator T is the value iteration operator while T_U, T_S, and T_B are operators corresponding to Do nothing, Sell securities, and Buy securities, respectively.

4. Optimal Cash Management Policy Under a Special Case

In this section, we characterize the optimal properties of the cash management policy under a special case when the amounts of cash required by the proj-

ects are equal, say m, and $q^S = q^B = q$. The assumption of the equal amounts of cash expenditure is also found in Miller and Orr (1966). Indeed, this assumption can be reasonable when the variance in the amounts of cash the projects require is not large.

Let $x (\leq F)$ be the number of projects under investment. Denote by (x, B) the system state. Under this special case, the operator T_U of the optimality equation becomes

$$T_U v(x, B) = 1/(\beta + \gamma)[g(x, B) + \lambda \{v(x+1, B-m)$$
$$1(B \geq m) + v(x, B) 1(B < m)\} +$$
$$x \mu v(x-1, B+m) + (N-x) \mu v(x, B)] \quad (5)$$

and the one stage expected discounted cost in state (x, B) becomes $g(x, B) = c_h B + \lambda c_L 1(B < m)$. Note that to denote the number of projects under investment, we use x instead of N because we focus on a single class project. Without loss of optimality, we assume that q is an integer multiple of m.

We start with introducing the following lemma. Its proof directly follows from the definition of value function T_U, T_S, and T_B. The first property of Lemma 1 says if it is optimal to sell securities to increase q amounts of cash in state (x, B), then it is not optimal to buy securities to decrease q amounts of cash in state $(x, B+q)$. Similarly, the second property states that if it is optimal to buy securities to decrease q amounts of cash in state (x, B), then it is not optimal to sell securities to increase q amounts of cash in state $(x, B-q)$.

Lemma 1 (i) If $\quad v(x, B) = T_S v(x, B)$,
$\qquad$ then $v(x, B+q) = T_U v(x, B+q)$.
$\qquad$ (ii) If $\quad v(x, B) = T_B v(x, B)$,
$\qquad$ then $v(x, B-q) = T_U v(x, B-q)$.

In order to establish the structural properties of the optimal cash management policy, it is sufficient to show that certain properties of the functions defined on state space Γ are preserved under the operator T (Porteus, 1982). Let G be the set of all functions defined on Γ such that if $f \in G$, then

$$-c_S(q) \leq f(x, B+q) - f(x, B) \leq c_B(q) \qquad (6)$$

$$f(x+1, B+q) - f(x, B+q+m) \leq \qquad (7)$$
$$f(x+1, B) - f(x, B+m)$$

Before stating the main result, we first present two preliminary results in the following consecutive lem-

mas. Lemma 2 states that it is not optimal to sell securities when the cash level is larger than or equal to m. This result is straightforward because we assume that the lead time is negligible.

Lemma 2 If $f \in G$,
$$T_U f(x, B) < T_S f(x, B),$$
$$x < F, B \geq m. \tag{8}$$

Proof : See the Appendix.

The following result establishes that the optimal control of buying securities can be defined by a threshold function :

Lemma 3 If $f \in G$,
$$T_U f(x+1, B) - T_U f(x, B+m) \leq,$$
$$T_B f(x+1, B) - T_B f(x, B+m)$$
$$B > q \tag{9}$$

Proof : See the Appendix.

Equation (9) provides greater incentive to buy securities as the cash balance increases. To see this, suppose that $T_U f(x+1, B) - T_S f(x+1, B) > 0$. Hence, it is optimal to buy securities to decrease the cash level in state $(x+1, B)$. By (9),

$$T_U f(x, B+m) - T_B f(x, B+m) \geq,$$
$$T_U f(x+1, B) - T_B f(x+1, B) > 0$$

which means if it is optimal to buy securities in $(x+1, B)$, then it is also optimal to buy securities in $(x, B+m)$.

Now we identify the structure of the optimal control of buying securities as follows :

Theorem 1 Let
$$\Theta^U(x) := \min\{B: T_B f(x, B) = \min\{T_U v(x, B),$$
$$T_S v(x, B), T_B v(x, B)\}\}. \tag{10}$$

Then, in state (x, B), it is optimal to buy securities to decrease the cash balance whenever $B \geq \Theta^U(x)$.

Proof : See the Appendix.

5. Optimal Solution Procedure

Now consider a VI algorithm (see ch. 8 in Puterman,

2005) to solve for (1) :

$$v^{k+1}(N, B) = Tv^k(N, B) = \min\{T_U v^k(N, B),$$
$$T_S v^k(N, B), T_B v^k(N, B)\} \tag{11}$$

where T_U, T_S, T_B are defined as in (2), (3), and (4), respectively, and $v^0(N, B) = 0$ for every state (N, B). Here $v^k(N, B)$ is the optimal expected discounted cost given the initial state (N, B) when the problem is terminated after the kth iteration of VI algorithm. VI algorithm continues until component-wide deviations of Tv^k and v^k converge to a termination criterion. If VI algorithm stops at the $(l+1)^{th}$ iteration, $v(N, B)$ is approximated by $v^l(N, B)$ for every state (N, B).

In the following, we present a numerical procedure which jointly finds the optimal cost function v and the optimal buying and selling points $\Theta^U(N)$ and $\Theta^L(N)$. If $\Theta^U(N)$ and $\Theta^L(N)$ and found, then the optimal cash management policy is given as follows :

In state (N, B), it is optimal to buy securities to decrease the cash balance by q^B whenever $B \geq \Theta^U(N)$ while it is optimal to sell securities to increase the cash balance by q^S whenever $B \leq \Theta^L(N)$.

Let ϵ be the termination criterion of the numerical procedure. Let Q^S and Q^B be the sets of possible amounts of cash the company can expect to increase by selling or decrease by buying securities, respectively.

1. Start with $q^S \in Q^S$ and $q^B \in Q^B$.
2. Implementation of Value Iteration

   (a) **Initialization** :
   Set $k = 0$, and for each state (N, B),
   pick the value function $v^0(N, B) = 0$.

   (b) **Value iteration step** :
   Implement a VI on the current value function estimate v^k :
   $$Tv^k(N, B) =$$
   $$\min\{T_U v^k(N, B), T_S v^k(N, B), T_B v^k(N, B)\}$$

where
$$T_U v^k(N, B) = 1/(\beta+\gamma)[g(N, B)$$
$$+ \sum_{i=1}^{I} \lambda_i \left\{ \begin{array}{l} v^k(N+e_i, B-m_i) 1(B \geq m_i) + \\ v^k(N, B) 1(B < m) \end{array} \right\}$$
$$+ \sum_{i=1}^{I} n_i \mu v^k(N-e_i, B+m_i) + (N - \sum_{i=1}^{I} n_i)$$
$$\mu v^k(N, B)],$$
$$T_S v^k(N, B) = c_S(q^S) + T_U v^k(N, B+q^S),$$
$$T_B v^k(N, B) = c_B(q^B) + T_U v^k(N, B-q^B).$$

(c) **Termination test :**
    Perform the following convergence test :
$$b_k \equiv \min_{(N,\,B)}\{Tv^k(N,\,B) - v^k(N,\,B)\}$$
$$\overline{b_k} \equiv \max_{(N,\,B)}\{Tv^k(N,\,B) - v^k(N,\,B)\}$$
    If $(\overline{b_k} - b_k) \geq \epsilon$, for every state $(N,\,B)$,
        let $v^{k+1}(N,\,B) = Tv^k(N,\,B)$,
    increase k by one, and go to **Value iteration step.**
Otherwise, go to **Evaluation step.**

(d) **Evaluation step :**
    Approximate $v(N,\,B) \approx Tv^k(N,\,B)$. Let
$\Theta^U(N) := \min\{B : T_B v(N,\,B) = \min$
$\{T_U v(N,\,B), T_S v(N,\,B), T_B v(N,\,B)\}\}$ and
$\Theta^L(N) := \max\{B : T_S v(N,\,B) = \min$
$\{T_U v(N,\,B), T_S v(N,\,B), T_B v(N,\,B)\}\}$.
Stop the procedure.

The optimal values of q^S and q^B can be set by iterating the optimal solution procedure over Q^S and Q^B, and by finding q^{S^*} and q^{B^*} which achieves the best performance.

6. Numerical Experiment

In this section, we explain the structure of the optimal cash management policy using an example in which the company invests two types of projects, say type 1 and type 2. Problem parameters of the example are as follows. The projects occur according to a Poisson distribution with rate $\lambda = 0.6$ and their investment durations are exponentially distributed with mean $\mu^{-1} = 0.05^{-1}$. Each of type 1 projects, occurring with a probability $p_1 = 0.8$, requires $m_1 = \$100,000$ amounts of cash while each of type 2 projects, occurring with a probability $p_2 = 0.2$, requires $m_2 = \$200,000$ amounts of cash. Penalty costs of not meeting the cash requirement of type 1 and type 2 projects due to the insufficient cash balance are $c_L^1 = \$10,000$ and $c_L^2 = \$50,000$, respectively. $c_h = \$100$ is incurred per unit of time per $\$100,000$ during which they are not used in investment. Whenever the company sells or buys securities, a transaction cost $c_S = c_B = \$2,000$ is incurred. The maximum number of projects the company can manage is $F = 15$. The sets of possible amounts of cash the company can increase by selling or decrease by buying securities $Q^S = Q^B = \{\$200,000, \$300,000, \$400,000, \$500,000, \$600,000, \$700,000\}$. The interest rate is $\beta = 0.06$. The termination criterion of the numerical procedure is set to $\epsilon = 0.0001$.

We applied the optimal solution procedure in Section 5 to the example above and the computational results are summarized as follows. The optimal size of selling securities is $q^{S^*} = \$400,000$ and the optimal size of buying securities is $q^{B^*} = \$600,000$. The optimal policy allows the company to sell securities only when $B = \$100,000$ and $n_1 + n_2 < F$, which is straightforward because we assume that lead times required for selling or buying securities are negligible. The optimal buying point, $\Theta^U(n_1, n_2)$, which controls when to buy securities in state (n_1, n_2) is presented in <Table 1>.

As shown in the table, the company's optimal buying points vary depending on state (n_1, n_2). For example, $\Theta^U(1,1) = 1,300,000$, which means that in state $(1, 1)$ it is optimal to buy securities only when $B \geq \$1,300,000$. <Table 1> also show that the company should not buy securities to decrease the cash balance when $B \leq \$900,000$ and it should buy securities when $B \geq \$1,300,000$, regardless of the system state (n_1, n_2).

An interesting behavior of the optimal control of buying securities comes when $B = \$1,000,000, \$1,100,000, \$1,200,000$, which are shown in <Figure 2>. In each of these cases, the state space $\{(n_1, n_2, B)\}$ is divided into two regions corresponding that the actions of Buying securities and Do nothing are optimal, respectively. In other words, given B, the company should decrease the cash balance by $\$600,000$ or not, depending on the values of n_1 and n_2. For example, when $n_1 = 6$ and $n_2 = 7$, it is optimal to buy securities if $B = \$1,100,000$ or $\$1,200,0000$ while the company should not buy securities if $B = \$1,000,000$. When $n_1 = 5$ and $n_2 = 5$, the company should buy securities if $B = \$1,200,000$ while it is optimal to not buy securities if $B = \$1,000,000$ or $\$1,100,0000$. When $n_1 = 7$ and $n_2 = 8$, it is optimal to buy securities for each of B. The intuition behind this observation can be explained as follows. Since $n_1 + n_2 = 15$, the chance that the cash balance increases due to the maturity of the projects becomes very high at the next state transition (note that the maturity rate of projects is $(n_1 + n_2)\mu$). Therefore, the possibility that the company will keep excessively large amounts of cashes becomes also high.

Note that the optimal control of buying securities in these three cases is governed by a monotonic threshold function, say $f(n_1/B)$, which is decreasing in n_1. More importantly, <Figure 2> shows that this function is likely to decrease in B. Indeed, in extensive numerical experiments with varying the val-

Table 1. The optimal buying points, $\Theta^U(n_1, n_2)$

(Unit : $100,000)

n_1 \ n_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	13	13	12	12	12	12	12	12	12	12	12	11	11	10	10	10
1	13	13	12	12	12	12	12	12	12	12	11	11	10	10	10	
2	13	12	12	12	12	12	12	12	12	11	11	10	10	10		
3	13	12	12	12	12	12	12	12	12	11	10	10	10			
4	12	12	12	12	12	12	12	12	11	11	10	10				
5	12	12	12	12	12	12	12	11	10	10	10					
6	12	12	12	12	12	12	11	11	10	10						
7	12	12	12	12	12	11	11	10	10							
8	12	12	12	12	11	11	10	10								
9	12	12	12	11	11	10	10									
10	12	12	11	11	10	10										
11	12	11	11	10	10											
12	11	11	10	10												
13	11	10	10													
14	10	10														
15	10															

ues of problem parameters, we observed that these functional properties of the optimal control of buying securities hold without any exceptions.
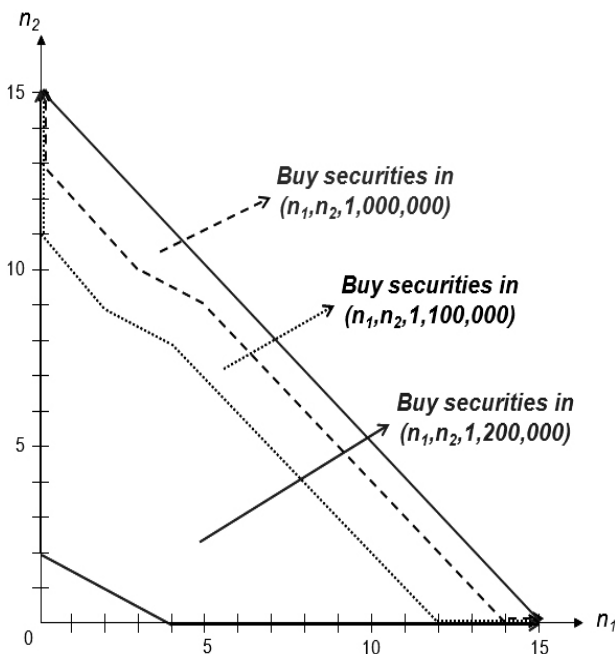

Figure 2. The optimal control of buying securities in states $(n_1, n_2, 1,000,000)$, $(n_1, n_2, 1,100,000)$, and $(n_1, n_2, 1,200,000)$

Numerical investigation also reveals that $\Theta^U(n_1, n_2)$ and $f(n_1/B)$ are sensitive to the change in problem parameter values. <Table 2> and <Figure 3> display how $\Theta^U(n_1, n_2)$ and $f(n_1/B)$ are impacted when the transaction costs change from $c_S = c_B = \$2,000$ to $c_S = c_B = \$1,000$. We observed that $\Theta^U(n_1, n_2)$ under $c_S = c_B = \$1,000$ is equal to or less than $\Theta^U(n_1, n_2)$ under $c_S = c_B = \$2,000$ for each (n_1, n_2). When $c_S = c_B = \$1,000$, the optimal control (i) never buys securities if $B \leq \$900,000$, (ii) always buys securities if $B \geq \$1,100,000$, and (iii) buys securities depending on (n_1, n_2) if $B = \$1,000,000$. We also observed that when $c_S = c_B = \$1,000$, $f(n_1/B = 1,000,000)$ under $c_S = c_B = \$1,000$ is equal to or less than $f(n_1/B = 1,000,000)$ under $c_S = c_B = \$2,000$. The optimal sizes of selling and buying securities are also impacted by the change in the transaction costs. When $c_S = c_B = \$1,000$, we found that $q^{S^*} = \$300,000$ and $q^{B^*} = \$500,000$. Compared to the case under $c_S = c_B = \$2,000$, the optimal sizes of selling securities decreases from $\$400,000$ to $\$300,000$ and the optimal sizes of buying securities decreases from $\$600,000$ to $\$500,000$. This phenomenon can be easily explained by the theory of the economic order quantity.

Table 2. $\Theta^U(n_1, n_2)$ when $c_S = c_B = \$1,000$

(Unit : $100,000)

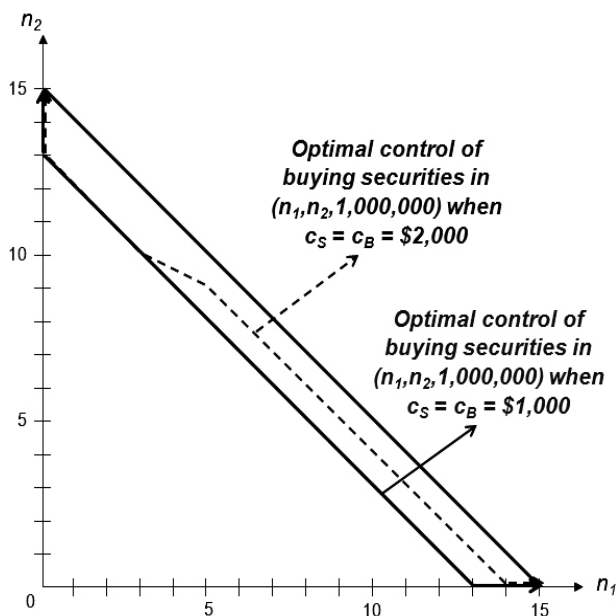| n_1 \ n_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	11	11	11	11	11	11	11	11	11	11	11	11	11	10	10	10
1	11	11	11	11	11	11	11	11	11	11	11	11	10	10	10	
2	11	11	11	11	11	11	11	11	11	11	11	10	10	10		
3	11	11	11	11	11	11	11	11	11	11	10	10	10			
4	11	11	11	11	11	11	11	11	11	10	10	10				
5	11	11	11	11	11	11	11	11	10	10	10					
6	11	11	11	11	11	11	11	10	10	10						
7	11	11	11	11	11	11	10	10	10							
8	11	11	11	11	11	10	10	10								
9	11	11	11	11	10	10	10									
10	11	11	11	10	10	10										
11	11	11	10	10	10											
12	11	10	10	10												
13	10	10	10													
14	10	10														
15	10	10														

Figure 3. Impact of the change in transaction cost on $f(n_1/B)$

7. Concluding Remarks

In this paper, we have formulated the cash management model with investment and retrieval. Using the Markov Decision Problem approach, we examined a control policy which dynamically decides when to increase and decrease the cash balance and how much to increase and decrease the cash balance. Compared to the existing literature in the area of cash management, the most distinct feature of our model comes from the fact that the cash return process completely depends on the cash demand process. Under some restrictive conditions, we showd that the optimal control of buying securities can be characterized as a threshold function.

Based on value iteration method, we developed the optimal solution procedure that finds the optimal selling and buying points and the optimal sizes of selling and buying securities. In a numerical experiment with two types of investment projects, we examined important structural properties of the optimal cash management policy. Much of the empirical literature on firms' cash holdings tries to identify a target cash-inventory by weighing the costs and benefits of holding cash. The implicit idea is that this target level helps determine when a firm should increase its cash savings and when it should dissave. Our numerical analysis, however, shows that the cash management policy for the problem presented in this

paper can be much more complicated even for the simple model, which is mainly caused by the dependency between cash demand and return processes. It would be important to extend the results of the paper in several directions. One promising extension would be to allow the optimal sizes of selling and buying securities to change depending the system state. Introducing the profitability on investment project opportunities would also modify the analysis in an important way. In particular, rationing investment projects according to the profitability they can generate must await for future work. Finally, it would be crucial to investigate how to determine the optimal target cash balance as well as how to decide the optimal selling and buying points. However, the problem is analytically intractable, the numerical investigation based on the simulation study could be challenging.

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<Appendix>

Proof of Lemma 2 :
$$T_U f(x, B) - T_S f(x, B) = -c_h q - c_S(q)$$
$$+ \lambda[f(x+1, B-m) - f(x+1, B+q-m)]$$
$$+ x\mu[f(x-1, B+m) - f(x-1, B+q+m)]$$
$$+ (N-x)\mu[f(x, B) - f(x, B+q)] < -c_h q - c_S(q)$$
$$+ \gamma c_S(q) \text{ (by (6))} < 0 \text{ since } \gamma < 1. \ \square$$

Proof of Lemma 3 :
$$T_U f(x+1, B) - T_U f(x, B+m) - (T_B f(x+1, B)$$
$$- T_B f(x, B+m)) = \lambda[f(x+2, B-m)$$
$$- f(x+1, B) - (f(x+2, B-q-m)$$
$$- f(x+1, b-q)] + x\mu[f(x, B+m)$$
$$- f(x-1, B+2m) - (f(x, B-q+m)$$
$$- f(x-1, B-q+2m))] + (N-x-1)\mu[f(x+1, B)$$
$$- f(x, B+m) - (f(x+1, B-q)$$
$$- f(x, B-q+m))] \leq 0$$

From Lemmas 1 and 2, $B > q$. The inequality of λ, $x\mu$, and $(N-x-1)\mu$ terms follows by (7). $\square$

Proof of Theorem 1 :
In the following, we first show that Equations (6)-(7) are preserved under T, that is, if $f \in G$, $Tf \in G$. Denote by $(0/1/2)$ the optimal action in (x, B) where 0, 1, and 2 respectively represent Do nothing, Sell securities, and Buy securities in (x, B).

(i) Let $\Delta^q \equiv Tf(x, B+q) - Tf(x, B)$. We focus on the combination of actions admissible in $(x, B+q)$ and (x, B). Since $q > m$ by assumption,

cases $(1, \cdot)$ are excluded by Lemma 1. For case $(2, 0)$, $\Delta^q = c_B(q) + T_U f(x, B) - T_U f(x, B) = c_B(q) \geq -c_S(q) \Delta^q = c_B(q) + T_U f(x, B) - T_U f(x, B) = c_B(q) \geq -c_S(q)$.

For case $(0, 1)$, $\Delta^q = T_U f(x, B+q) - (c_S(q) + T_U f(x, B+q)) = -c_S(q) \leq c_B(q)$. For case $(0, 0)$, $\Delta^q \leq T_B f(x, B+q) - T_U f(x, B) = c_B(q)$ and $\Delta^q \geq T_U f(x, B+q) - T_S f(x, B) = -c_S(q)$.

For case $(1, 1)$, $\Delta^q = c_S(q) + T_U f(x, B+2q) - (c_S(q) + T_U f(x, B+q)) = T_U f(x, B+2q) - T_U f(x, B+q) = 1/(\beta + \gamma)[\lambda(f(x+1, B+2q-m) - f(x+1, B+q-m)) + x\mu(f(x-1, B+2q+m) - f(x-1, B+q+m)) + (N-x)\mu(f(x, B+2q) - f(x, B+q))]$.

Hence, $\Delta^q \leq 1/(\beta+\gamma)(\gamma c_B(q)) \leq c_B(q)$ and $\Delta^q \geq 1/(\beta+\gamma)(-\gamma c_S(q)) \geq -c_S(q)$.

For case $(2, 2)$, $\Delta^q = c_B(q) + T_U f(x, B+q) - (c_B(q) + T_U f(x, B)) = T_U f(x, B+q) - T_U f(x, B)$ and thus its proof is similar to case $(1, 1)$.

For case $(0, 2)$, the first inequality follows because $\Delta^q = T_U f(x, B+q) - T_B f(x, B) \geq T_U f(x, B+q) - T_S f(x, B) = T_U f(x, B+q) - (c_S(q) + T_U f(x, B+q)) = -c_S(q)$.
$\Delta^q = T_U f(x, B+q) - T_B f(x, B) \leq T_B f(x, B+q) - T_B f(x, B) = T_U f(x, B) - T_U f(x, B-q))$ and thus $\Delta^q \leq c_B(q)$ can be shown using the same approach as case $(1, 1)$.

(ii) $\Delta^{1,m} \equiv Tf(x+1, B+q) - Tf(x, B+q+m) - (Tf(x+1, B) - Tf(x, B+m))$. Suppose $B > m$. We focus on the combination of actions admissible in $(x+1, B+q)$, $(x, B+q+m)$, $(x+1, B)$, and $(x, B+m)$. By Lemma 2, the action of buying securities in these four states cannot be optimal. Case $(0, 0, 0, 0)$ and $(2, 2, 2, 2)$ follow from (6).

For case $(2, 2, 0, 0)$, $\Delta^{1,m} = c_B(q) + T_U f(x+1, B) - (c_B(q) + T_U f(x, B+m)) - (T_U f(x+1, B) - T_U f(x, B+m)) = 0$. For cases $(0, 2, 0, 0)$ and $(2, 2, 0, 2)$, $\Delta^{1,m} \leq c_B(q) + T_U f(x+1, B) - (c_B(q) + T_U f(x, B+m)) - (T_U f(x+1, B) - T_U f(x, B+m)) = 0$. Cases $(2, 0, \cdot, \cdot)$ and $(\cdot, \cdot, 2, 0)$ are excluded by Lemma 3. Suppose $B = m$. Then, the optimal action in state $(x+1, m)$ is Sell securities. By Lemma 2, the action of selling securities in states $(x+1, m+q)$, $(x, 2m+q)$, and $(x, 2m)$ cannot be optimal. From Lemmas 1, the optimal actions in states $(x+1, m+q)$ is *Do nothing*. By assumption, $m < q$ and thus the optimal action in state $(x, 2m)$ is also *Do nothing*. If the optimal action in $(x, 2m+q)$ is *Do nothing*,

$$\begin{aligned} \Delta^{1,m} &= T_U f(x+1, m+q) - T_U f(x, 2m+q) - \\ &\quad (T_S f(x+1, m) - T_U f(x, 2m)) \leq \\ &\quad T_U f(x+1, m+q) - T_U f(x, 2m+q) - \\ &\quad (T_S f(x+1, m) - T_S f(x, 2m)) \\ &\quad T_U f(x+1, m+q) - T_U f(x, 2m+q) - \\ &\quad (c_S(q) + T_U f(x+1, m+q) - \\ &\quad (c_S(q) + T_U f(x, 2m)) = 0 \end{aligned}$$

If the optimal action in $(x, 2m+q)$ is *Buying securities*,

$$\begin{aligned} \Delta^{1,m} &= T_U f(x+1, m+q) - T_B f(x, 2m+q) - \\ &\quad (T_S f(x+1, m) - T_U f(x, 2m)) = T_U f \\ &\quad (x+1, m+q) - (c_B(q) + T_U f(x, 2m)) - \\ &\quad (c_S(q) + T_U f(x+1, m+q) - T_U f(x, 2m)) \\ &\quad = -c_B(q) - c_S(q) \leq 0. \end{aligned}$$

Now we prove the result given in Theorem 1 using contradiction. Suppose that it is not optimal to buy securities in state $(x-1, \Theta^U(x) + m)$, i.e.,
$T_U f(x-1, \Theta^U(x) + m) < T_S f(x-1, \Theta^U(x) + m)$.
From the definition of $\Theta^U(x)$, we have $T_U f(x, \Theta^U(x)) > T_S f(x, \Theta^U(x))$ By subtracting the first inequality from the second one, $T_U f(x, \Theta^U(x)) - T_U f(x-1, \Theta^U(x) + m) > T_B f(x+1, \Theta^U(x)) - T_B f(x, \Theta^U(x) + m)$. This is a contradiction by Lemma 3. $\square$

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