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GLOBAL EXISTENCE FOR 3D NAVIER-STOKES EQUATIONS IN A THIN PERIODIC DOMAIN

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ABSTRACT. We consider the global existence of strong solutions of the 3D incompressible Navier-Stokes equations in a thin periodic domain. We present a simple proof that a strong solution exists globally in time when the initial velocity in H^1 and the forcing function in $L^p(0,\infty;L^2)$, $2 \le p \le \infty$ satisfy certain condition. This condition is basically similar to that by Iftimie and Raugel[7], which covers larger and larger initial data and forcing functions as the thickness of the domain ϵ tends to zero.

1. INTRODUCTION

We consider the incompressible Navier-Stokes equations,

$$u_t - \nu \triangle u + (u \cdot \nabla)u + \nabla p = f, \tag{1.1}$$
$$\nabla \cdot u = 0, \tag{1.2}$$

in a thin periodic domain $\Omega = T^3 = [0,l_1] \times [0,l_2] \times [0,\epsilon]$, $0 < \epsilon << l_1, l_2$. Here u denotes the velocity of a homogeneous, viscous incompressible fluid, f is the density of force per unit volume, p denotes the pressure, and ν is the kinematic viscosity. We require that the forcing function f and the initial data u_0 satisfy

$$\nabla \cdot f = \nabla \cdot u_0 = 0.$$

We assume in addition that

$$\int_\Omega f dx = \int_\Omega u dx = 0, \tag{1.3}$$

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which could be achieved by the Galilean transformation with suitable vectors $c(t)$ and e,

$$u(x,t) \to u(x+c(t)+et,t) - \frac{dc}{dt} - e.$$

Indeed, we can take

$$c(t) = \int_0^t \int_0^r \int f(x,s)dxdsdr, \quad e = \int u_0 dx.$$

By the classical results of Leray and Hopf ([12], [5]), there exists a global weak solution of the Navier-Stokes equations in a three dimensional torus. It is also known that the solution becomes necessarily strong(regular) for all regular data in a two dimensional domain. But in a three dimensional domain, global strong solutions have only been guaranteed for small initial data(See, for example, [3], [4], [15], [16] and the references therein).

In [14], Raugel and Sell treated the problem on thin periodic domain and they obtained a significant existence result on global regular solutions. The main idea is that if the thickness of the domain is small enough, the solution of the Navier-Stokes equations is close to the 2D Navier-Stokes equations. They proved that there are large sets $R(\epsilon) \subset H^1(\Omega)$ and $S(\epsilon) \subset L^\infty((0,\infty), L^2(\Omega))$ such that if $u(0) = u_0 \in R(\epsilon)$ and $f \in S(\epsilon)$, then there exists a strong solution $u(t)$ that remains in $H^1(\Omega)$ for all $t \geq 0$. The sets $R(\epsilon)$ and $S(\epsilon)$ get larger and larger as $\epsilon \to 0$.

Since then, there have been many improvements on the estimates of the size of these sets $R(\epsilon)$ and $S(\epsilon)$ under various boundary conditions(see [2], [6], [13], [7], [8], [9], [10], [17] and the references therein). Roughly, under various boundary conditions except the periodic boundary condition, it has been shown that if

$$\|u_0\|_{H^1} \leq C\epsilon^{-1/2} \quad \text{and} \quad \|f\|_{L^\infty((0,\infty),L^2)} \leq C\epsilon^{-1/2} \tag{1.4}$$

for some constant $C = C(\nu)$, then the corresponding global strong solution exists(see [2], [17]). We note that the above condition can cover very large initial data and forcing functions if $\epsilon > 0$ is small enough. However, under the periodic boundary condition, it is not known whether (1.4) implies the existence of global strong solutions. Under the periodic boundary condition, it is shown in [11] that, when $f = 0$, the existence of the global strong solution is guaranteed under the condition

$$\|u_0\|_{H^1} \leq C\epsilon^{-1/2}|\log\epsilon|^{1/2},$$

and in [7] under the following condition

$$\|(Mu_0)_3\| \leq C\nu\epsilon^{1/2}, \ \|Mf\|_{L^\infty(0,\infty;L^2)} \leq C\nu^2\epsilon^{1/2},$$
$$\|\nabla u_0\| \leq C\nu\epsilon^{-1/2}, \ \|f\|_{L^\infty(0,\infty;L^2)} \leq C\nu^2\epsilon^{-1/2}.$$

Here, M is the average operator with respect to the thin direction. We note that the first two conditions in the above are not so restrictive since Mu_0 and Mf are independent of the third variable and so they are in fact ϵ independent conditions.

In this paper, we consider the global existence of strong solutions in a thin periodic domains and improve the result in [7] in a simple way for forcing function $f \in L^p([0,\infty); L^2)$, $2 \le p \le \infty$. Concretely, we show in a simple way that the global regularity is guaranteed if

$$\|(Mu_0)_3\| \le C\nu\epsilon^{1/2},\ \|Mf\|_{L^p(0,\infty;L^2)} \le C\nu^{(2p-1)/p}\epsilon^{1/2}, \tag{1.5}$$

$$\|\nabla u_0\| \le C\nu\epsilon^{-1/2},\ \|f\|_{L^p(0,\infty;L^2)} \le C\nu^{(2p-1)/p}\epsilon^{-1/2} \tag{1.6}$$

for some $2 \le p \le \infty$. The above result generalizes the result in [7] to the case $p \in [2,\infty]$.

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2. Preliminary estimates

From now on, Ω is assumed to be a three dimensional thin torus, $[0,l_1]\times[0,l_2]\times[0,\epsilon]$, $0<\epsilon<1$ and $l_1, l_2 > 0$ are fixed. Also, $\tilde{\Omega} = (0,l_1)\times(0,l_2)$. We denote

$$H = \{u \in L^2(\Omega) | \nabla\cdot u = 0,\ \int_\Omega u = 0\}$$

and $V = H \cap W^{1,2}(\Omega)$. It is well known that $\|\nabla u\|_{L^2}$ is an equivalent norm for V due to the Poincaré inequality. For convenience's sake, we also denote

$$\|\cdot\|_{L^p} = \|\cdot\|_p, \quad \|\cdot\|_2 = \|\cdot\|, \quad \|\cdot\|_{L^p(0,\infty;L^q(\Omega))} = \|\cdot\|_{p,q},$$

the Leray projection on $L^2(\Omega)$ into H by $\mathbb{P}$, and the Stokes operator by $A = \mathbb{P}(-\Delta)$. We define the bilinear form $B(u,v) = \mathbb{P}(u\cdot\nabla)v$ and the trilinear form $b(u,v,w)$ by

$$b(u,v,w) =< B(u,v), w >= \int_\Omega B(u,v)\cdot w dx.$$

We now define an orthogonal projection M on $L^2(\Omega)$ by

$$Mu = \frac{1}{\epsilon}\int_0^\epsilon u(x_1,x_2,s)ds \tag{2.1}$$

and denote $v = Mu$ and $w = (I-M)u$ for simplicity. We recall that the following Poincaré inequality holds for w since $Mw = 0$:

$$\|w\|^2 \le \frac{\epsilon^2}{4\pi^2}\|\nabla w\|^2. \tag{2.2}$$

Lemma 2.1. *For $w \in V$ with $Mw = 0$, there exist absolute constants K_1 and K_2 independent of ϵ such that*

$$\|w\|_3 \le K_1\|w\|^{1/2}\|\nabla w\|^{1/2}, \tag{2.3}$$

and

$$\|\nabla v\|_{L^q} \le K_2\epsilon^{\frac{2-q}{2q}}\|\nabla^2 v\|^{\frac{q-2}{q}}\|\nabla v\|^{2/q}, \quad 2 \le \forall q \le 4. \tag{2.4}$$

Proof. Since $w(\cdot, x_3)$ is average zero on $[0, \epsilon]$ for any $(x_1, x_2) \in \tilde{\Omega}$, there exists $a \in [0, \epsilon]$ with $w(x_1, x_2, a) = 0$. Then,

$$|w|^3(\cdot, x_3) = \left| \int_a^{x_3} dx_3 \partial_3 w^2 \right|^{1/2} |w|^2(\cdot, x_3) \leq |w|^2(x) \|w\|_{L^2(0,\epsilon)}^{1/2} \|\nabla w\|_{L^2(0,\epsilon)}^{1/2}.$$

Then, integrating the above on Ω,

$$\begin{aligned} \int |w|^3 &\leq \int_{\tilde{\Omega}} \|w\|_{L^2(0,\epsilon)}^{1/2} \|\nabla w\|_{L^2(0,\epsilon)}^{1/2} \int dx_3 |w|^2 \\ &\leq \left(\int_{\tilde{\Omega}} \|w\|_{L^2(0,\epsilon)}^{10/3} dx \right)^{3/4} \|\nabla w\|^{1/2}. \end{aligned}$$

Denoting

$$Q(x_1, x_2) \equiv \int dx_3 |w|^2,$$

and applying the Hölder and Sobolev inequality to Q in $\tilde{\Omega}$, we have

$$\begin{aligned} \int_{\tilde{\Omega}} Q^{5/3} dx &\leq \left(\int_{\tilde{\Omega}} Q \right)^{1/3} \left(\int_{\tilde{\Omega}} Q^2 \right)^{2/3} \\ &\leq C \|w\|^{2/3} \left(\|Q\|_{L^1(\tilde{\Omega})} + \|\nabla Q\|_{L^1(\tilde{\Omega})} \right)^{4/3} \\ &\leq C \|w\|^{2/3} \left(\|w\|^2 + \int_{\tilde{\Omega}} \int_0^\epsilon |\nabla w||w| \right)^{4/3} \\ &\leq C \|w\|^{2/3} \left(\|w\|^2 + \|\nabla w\| \|w\| \right)^{4/3}. \end{aligned}$$

Thus, plugging the above into the previous inequality and using (2.2), we obtain (2.3).

Next, we show (2.4). Since v is two dimensional and average zero on $\tilde{\Omega}$, v satisfies the following two dimensional Gargliardo-Nirenberg inequality.

$$\|\nabla v\|_{L^q(\tilde{\Omega})}^q \leq C \|\nabla^2 v\|_{L^2(\tilde{\Omega})}^{q-2} \|\nabla v\|_{L^2(\tilde{\Omega})}^2.$$

Integrating the above with respect to x_3, we then have

$$\int dx_3 \int_{\tilde{\Omega}} |\nabla v|^q \leq C \epsilon^{\frac{1-(q-1)}{2}} \left(\int_\Omega |\nabla^2 v|^2 \right)^{\frac{q-2}{2}} \|v\|^2,$$

which gives (2.4). □

We now present estimates for the trilinear form $b(\cdot, \cdot, \cdot)$.

Lemma 2.2. *Given $u \in V$, let $v = Mu$ and $w = (I - M)u$. We have*

$$|b(w, w_3, v_3)| \leq C \epsilon^{5/2} \|\nabla u\| \|Au\|^2, \tag{2.5}$$

$$|b(v, w, Aw)|, \ |b(w, v, Aw)|, \ |b(w, w, Av)| \leq C \epsilon^{1/2} \|\nabla u\| \|Au\|^2, \tag{2.6}$$

$$|b(v, v, Av)| \le C\epsilon^{-1/2}\|\nabla v_3\|\|\nabla v\|\|Av\|, \tag{2.7}$$

$$|b(w, w, Aw)| \le C\epsilon^{1/2}\|\nabla w\|\|Aw\|^2. \tag{2.8}$$

Here, all $C = C(K_1, K_2)$'s are independent of ϵ.

Proof. By the Hölder inequality, (2.3), and (2.4),

$$|b(w, w_3, v_3)| = \left|\int (w \cdot \nabla)w_3 \cdot v_3\right| = \left|\int (w \cdot \nabla)v_3 \cdot w_3\right|$$
$$\le \|w\|\|w\|_3\|\nabla v_3\|_6 \le C\epsilon^{-1/3}\|w\|^{3/2}\|\nabla w\|^{1/2}\|\nabla v\|^{1/3}\|Av\|^{2/3}.$$

Further, by (2.2),

$$|b(w, w_3, v_3)| \le C\epsilon^{5/2}\|\nabla w\|^{2/3}\|Aw\|^{4/3}\|\nabla v\|^{1/3}\|Av\|^{2/3}.$$

This verifies (2.5). Next, for $b(v, w, Aw)$, we use sequentially integration by parts, and divergence theorem to have

$$b(v, w, Aw) = -\int (v \cdot \nabla)w \cdot \Delta w = \int (\nabla_j v \cdot \nabla)w \cdot \nabla_j w + v \cdot \nabla(\nabla_j w)\nabla_j w$$
$$= \int (\nabla_j v \cdot \nabla)w \cdot \nabla_j w.$$

Then, since $M\nabla w = 0$, applying (2.3) to ∇w and using (2.4), and (2.2),

$$|b(v, w, Aw)| \le \|\nabla v\|_6\|\nabla w\|_3\|\nabla w\|$$
$$\le C\epsilon^{-1/3}\|\nabla v\|^{1/3}\|Av\|^{2/3}\|\nabla w\|^{3/2}\|Aw\|^{1/2}$$
$$\le C\epsilon^{1/2}\|\nabla v\|^{1/3}\|Av\|^{2/3}\|\nabla w\|^{2/3}\|Aw\|^{4/3}.$$

For $|b(w, v, Aw)|$, again by (2.3), (2.4), and (2.2), we have

$$|b(w, v, Aw)| \le \|\nabla v\|_6\|w\|_3\|Aw\|$$
$$\le C\epsilon^{-1/3}\|\nabla v\|^{1/3}\|Av\|^{2/3}\|w\|^{1/2}\|\nabla w\|^{1/2}\|Aw\|$$
$$\le C\epsilon^{1/2}\|\nabla v\|^{1/3}\|Av\|^{2/3}\|\nabla w\|^{2/3}\|Aw\|^{4/3}.$$

Similarly, by integration by parts and the above estimates

$$|b(w, w, Av)| \le \left|\int \nabla w \cdot \nabla w \cdot \nabla v\right| + \left|\int w \cdot \nabla^2 w \cdot \nabla v\right|$$
$$\le C\|\nabla v\|^{1/2}\|Av\|^{2/3}\|\nabla w\|^{2/3}\|Aw\|^{4/3}.$$

On $b(v, v, Av)$, by the two dimensionality of v and integration by parts,

$$b(v, v, Av) = b(v, v_3, Av_3) = \int \nabla v \cdot \nabla v_3 \cdot \nabla v_3.$$

Then, by (2.4),

$$|b(v, v, Av)| \le \|\nabla v\|\|\nabla v_3\|_4^2 \le C\|\nabla v\|\epsilon^{-1/2}\|\nabla v_3\|\|Av_3\|.$$

Finally, by integration by parts, (2.3), and (2.2),

$$|b(w,w,Aw)| = \left|\int \nabla w\cdot\nabla w\cdot\nabla w\right| \le C\|\nabla w\|^{3/2}\|Aw\|^{3/2} \le C\epsilon^{1/2}\|\nabla w\|\|Aw\|^2.$$

□

3. REGULARITY

In this section, we give our regularity result and its proof. We first reformulate (1.1-1.2) in the standard nonlinear evolutionary equation on the Hilbert space V,

$$u_t + \nu Au + B(u,u) = \mathbb{P}f. \tag{3.1}$$

We shall consider solutions of (3.1) with the initial data u_0 and $f = f(t)$ in the class

$$u_0 \in V, \quad f(t) \in L^p([0,\infty), H),\ p \ge 2. \tag{3.2}$$

Theorem 3.1. *For any $\epsilon < 1$, there exists a globally regular solution u of (3.1) if*

$$\|\nabla u_0\|^2 + \nu^{-(2p-2)/p}\|f\|_{p,2}^2 < \frac{\nu^2}{4C}\epsilon^{-1} \tag{3.3}$$

$$\|(Mu_0)_3\|^2 + \nu^{-(2p-2)/p}\|Mf\|_{p,2}^2 < \frac{\nu^2}{4C}\epsilon \tag{3.4}$$

for some $C > 0$ independent of ϵ.

Proof. We shall write differential inequalities for $\|\nabla u\|$ and $\|v_3\|$ at the same time and derive an estimate for the suitable sum of them. Since

$$\int B(u,u)_3 v_3 dx = \int u\cdot\nabla w_3 v_3 = \int w\cdot\nabla w_3 v_3,$$

taking the scalar product of (3.1) with v_3 and using (2.5), we have

$$\frac{d}{dt}\|v_3\|^2 + 2\nu\|\nabla v_3\|^2 \le 2\|Mf\|\,\|v_3\| + 2\epsilon^{5/2}\|\nabla u\|\|Au\|^2. \tag{3.5}$$

While, using (2.6)-(2.8) and the orthogonality of v and w,

$$\begin{aligned}|<B(u,u),Au>| &= |<B(v,v),Av> + <B(w,w),Aw> \\ &\quad + <B(v,w),Aw> + <B(w,v),Aw> + <B(w,w),Av>| \\ &\le C\epsilon^{-1/2}||\nabla v||||\nabla v_3||||Au|| + C\epsilon^{1/2}\|\nabla u\|\|Au\|^2.\end{aligned}$$

Taking the scalar product of (3.1) with Au and using the above estimate and the Young inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|^2 + 2\nu\|Au\|^2 &\le 2\left|\int fAu\right| + \left|\int B(u,u)Au\right| \\ &\le C\frac{\|f\|^2}{\nu} + \nu\|Au\|^2 + \nu\frac{||\nabla v_3||^2}{\epsilon^2} \\ &\quad + C\frac{\epsilon}{\nu}||\nabla v||^2||Au||^2 + C(\epsilon^{1/2}\|\nabla u\|)\|Au\|^2.\end{aligned} \tag{3.6}$$

Now, we divide (3.5) by ϵ and multiply (3.6) by ϵ and add them to have

$$\begin{aligned}\frac{d}{dt}(\epsilon\|\nabla u\|^2 + \frac{\|v_3\|^2}{\epsilon}) + \nu\epsilon\|Au\|^2 + \nu\frac{||\nabla v_3||^2}{\epsilon}\\ \le C\frac{\epsilon\|f\|^2}{\nu} + (C\frac{\epsilon}{\nu}||\nabla u||^2 + C\epsilon^{1/2}\|\nabla u\|)\|Au\|^2\\ + \frac{2}{\epsilon}\|Mf\|\,\|v_3\|. \end{aligned} \tag{3.7}$$

By the Young inequality and the Poincaré inequality for v,

$$C\epsilon^{1/2}\|\nabla u\| \le \frac{1}{2}\nu + C\frac{\epsilon}{\nu}||\nabla u||^2,$$

$$\frac{2}{\epsilon}\|Mf\|\,\|v_3\| \le \frac{C}{\epsilon}\|Mf\|\,\|\nabla v_3\| \le C\frac{\|Mf\|^2}{\nu\epsilon} + \frac{\nu}{2\epsilon}\|\nabla v_3\|^2.$$

Thus, denoting $G^2 = \epsilon\|\nabla u\|^2 + \frac{\|v_3\|^2}{\epsilon}$, (3.7) becomes

$$\frac{d}{dt}G^2 + (\frac{\nu}{2} - \frac{C}{\nu}\epsilon\|\nabla u\|^2)\epsilon\|Au\|^2 + \frac{\nu}{2}\frac{\|\nabla v_3\|^2}{\epsilon} \le \frac{C}{\nu\epsilon}\|Mf\|^2 + C\epsilon\frac{\|f\|^2}{\nu} \equiv Ch.$$

By the Poincaré inequality, we arrive at

$$\frac{d}{dt}G^2 + \frac{\nu}{4}\lambda_1 G^2 + (\frac{\nu}{4} - \frac{C}{\nu}G^2)\epsilon\|Au\|^2 \le Ch.$$

Here, λ_1 is the first eigenvalue of A. Now, we apply the Grönwall lemma to the above inequality with typical smallness argument. That is, let $G(0) < \frac{\nu^2}{4C}$ initially and suppose that $G(t)^2 > \frac{\nu^2}{4C}$ for some $t > 0$. Then there would be the first time $t = T$ such that $G(T) = \frac{\nu^2}{2C}$. However, for $0 < t \le T$,

$$\frac{d}{dt}G^2 + \frac{\nu}{4}\lambda_1 G^2 \le Ch.$$

Then, applying the Grönwall lemma to the above inequality, we would have

$$\begin{aligned} G(T)^2 &\le G(0)^2 + C\int_0^T he^{\nu\lambda_1(s-T)/4}ds\\ &\le G(0)^2 + C\|h\|_{L^{p/2}}\left(\int_0^T e^{p\nu\lambda_1(s-T)/4(p-2)}\right)^{\frac{p-2}{p}}\\ &\le G(0)^2 + C\|h\|_{L^{p/2}}\left(\frac{4(p-2)}{\nu p\lambda_1}\right)^{(p-2)/p}\\ &\le G(0)^2 + C\nu^{-(p-2)/p}\left\|\frac{C}{\nu\epsilon}\|Mf\|^2 + C\epsilon\frac{\|f\|^2}{\nu}\right\|_{L^{p/2}}\\ &\le G(0)^2 + C\nu^{-(2p-2)/p}(\epsilon^{-1}\|Mf\|_{p,2}^2 + \epsilon\|f\|_{p,2}^2). \end{aligned}$$

Note that the above estimate holds true even for $p = \infty$. Redefining C if necessary, this leads a contraction with (3.3-3.4). Therefore, $G^2 < \frac{\nu^2}{4C}$ for all $t > 0$ and u becomes globally regular. □

Clearly, the condition (3.3-3.4) is in particular satisfied by (1.5-1.6). The condition (3.3-3.4) is in a sense a condition of smallness of the initial data and external force. However, this condition allows for initial data with large H^1 norm when ϵ is small enough.

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