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A SNOWBALL CURRENCY OPTION

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ABSTRACT. I introduce a derivative called "Snowball Currency Option" or "USDKRW Snowball Extendible At Expiry KO" which was traded once in the over-the-counter market in Korea. A snowball currency option consists of a series of maturities the payoffs at which are like those of a long position in a put option and two short position in an otherwise identical call. The strike price at each maturity depends on the exchange rate and the previous strike price so that the strike prices are random and path-dependent, which makes it difficult to find a closed form solution of the value of a snowball currency option. I analyze the payoff structure of a snowball currency option and derive an upper and a lower boundaries of the value of it in a simplified model. Furthermore, I derive a pricing formula using integral in the simplified model.

1. INTRODUCTION

Exporters usually use currency options to hedge against the risk of appreciation of the domestic currency. Firms in Korea had not expected for the KRW/USD exchange rate to rise before 2008 in which a financial crisis occurred Hence some firms, in particular exporters in Korea over-hedged their positions or speculated using currency options with the belief that KRW/USD will not rise. Consequently they suffered a large amount of loss by a steep rise of KRW/USD exchange rate in 2008. KIKO(Knock-In-Knock-Out) option is a popular example of such a currency option which was traded in the over-the-counter market in Korea. KIKO option has often been analyzed for example in Han [2] and Kim [4]. In this paper, I introduce another derivative on USD called "Snowball Currency Option" or "USDKRW Snowball Extendible At Expiry KO" which was traded once in the over-the-counter market in Korea and caused a large amount of loss to the firm who bought it. A snowball currency option consists of a series of maturities the payoffs at which are like those of a long position in a put option and two short position in an otherwise identical call. The strike price at each maturity depends on the exchange rate and the previous strike price so that the strike prices are random and path-dependent, which makes it difficult to find a closed form solution of the value of a snowball currency option. I analyze the payoff structure of the snowball currency option and derive an upper and a lower boundaries of the value of it in a simplified model. Furthermore, I derive a pricing formula using integral in the simplified model. An integral form solution provide an

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easier way to find Greeks every time in the life of the snowball currency option than that of Monte Carlo method. Hence an integral form solution makes it easier to find hedge positions than that of Monte Carlo method.

The article proceeds as follows. Section 2 explain the structure of a snowball currency option. Section 3 analyze a snowball currency option in a simplified framework, in particular find upper and lower boundaries of it. Section 4 provides a pricing formula and Greeks using integrals and Section 5 concludes.

2. THE STRUCTURE OF A SNOWBALL CURRENCY OPTION

In this section I explain the structure of a snowball currency option. The underlying asset of a snowball currency option is a foreign currency, say USD. There are basic maturities $t_1 < t_2 < ... < t_N$ satisfying $\tau := t_1 = t_2 - t_1 = ... = t_N - t_{N-1}$. The Payoff G_n at maturity t_n is the same as that of a long position in a put option and two short position in a call option with the same strike price K_n for $n = 1, ., N$. There is a constant Knock-out barrier B from the second maturity so that if the exchange rate S_{t_n} is less than B at maturity t_n, then the payoff is zero for $n = 2, ..., N$. Hence the payoffs at the basic maturities are

$$G_1 = (K_1 - S_{t_1})^+ - 2(S_{t_1} - K_1)^+$$

and

$$G_n = [(K_n - S_{t_n})^+ - 2(S_{t_n} - K_n)^+]\mathbf{1}_{\{S_{t_n} \geq B\}} \text{ for } n = 2, ..., N,$$

where S_{t_n} is the exchange rate denoting the amount of KRW corresponding 1 USD at maturity t_n. Until now the payoff structure of a snowball currency option is similar to that of KIKO. However the payoff structure of a snowball currency option has a particular aspect in that its strike prices are random and path dependent. The strike price K_1 at the first maturity is a deterministic constant but the other strike prices evolve according to

$$K_n = \Big(\min[K_{n-1} + A - S_{t_n}, K^{\max}] \Big)^+, \; n = 2, ., N$$

where A and $K^{\max}$(cap price) are some constants satisfying $0 < A < K^{\max}$, $K_1 \leq K^{\max}$, and $K_1 + A - K^{\max} > 0$. Since the payoff G_n at maturity t_n is the same as that of a long position in a put option and two short position in a call option with the same strike price K_n, it is decreasing in S_{t_n} for a fixed K_n and increasing in K_n if for a fixed S_{t_n}. However K_n is a decreasing function of S_{t_n} and an increasing function of K_{n-1} which is again a decreasing function of $S_{t_{n-1}}$ an increasing function of K_{n-2} and so on. Furthermore if $S_{t_{n-1}}$ is low(resp. high), then S_{t_n} tends to be low(resp. high). Because of such recursive structure of the payoffs the snowball currency option a high tail risk. The possibility of extreme loss or profit is large. Profit or loss tends to increase like a snowball. For an easily illustration of high tail risk due to the recursive payoff structure, suppose $S_{t_{n-1}}$ is high, then K_{n-1} tends to be low. Therefore G_{n-1} is low by high $S_{t_{n-1}}$ and additionally by low K_{n-1}. Since $S_{t_{n-1}}$ is high, S_{t_n} tends to be high. K_n tend to be low by high S_{t_n} and additionally by low K_{n-1} Therefore G_n is low by high S_{t_n} and additionally by low K_n. The above argument can be summarized as the following equations.

$$S_{t_{n-1}} \uparrow \Rightarrow \left.\begin{array}{l} S_{t_{n-1}} \uparrow \\ K_{n-1} \downarrow \end{array}\right\} \Rightarrow G_{n-1} \Downarrow,$$

$$\left.\begin{array}{l} S_{t_{n-1}} \uparrow \\ K_{n-1} \downarrow \end{array}\right\} \Rightarrow \left.\begin{array}{l} S_{t_n} \uparrow \\ K_{n-1} \downarrow \end{array}\right\} \Rightarrow K_n \Downarrow,$$

and

$$\left.\begin{array}{l} S_{t_n} \uparrow \\ K_n \Downarrow \end{array}\right\} \Rightarrow G_n \downarrow\downarrow\downarrow .$$

Therefore

$$S_{t_{n-1}} \uparrow \Rightarrow G_{n-1} \Downarrow, \quad G_n \downarrow\downarrow\downarrow, \text{ and so on.}$$

The above argument is vice versa but the payoff structure is not symmetric since the strike prices have an upper bound $K^{\max}$ which makes the profit limited while the loss is unlimited. Another feature of a snowball currency option is that there can be an extension event. If there is a n such that S_{t_n} is larger than the extension barrier $D > K^{\max}$ for $n = 1, ..., N$, then the contract extends with more maturities $t_{N+1}, t_{N+2}, ..., t_{2N}$ such that $t_N < t_{N+1} < t_{N+2} < ... < t_{2N}$ and $\tau = t_{N+1} - t_N = t_{N+2} - t_{N+1} = ... = t_{2N} - t_{2N-1}$. The possibility of the extension event tends to increase the possibility of a large amount of loss to the long position of the snowball currency option since the extension event occurs when the exchange rate is very high (larger than D), that is, when the long position has been suffering and have an high possibility of a large amount of loss. Generally, it is impossible to find a closed form solution like Black-Scholes formula for the price of the snowball currency option. The main difficulty arises from randomness and path-dependency of the strike prices. Monte Carlo simulation can be applied to get the price. However it can be applied with given parameters. Furthermore it is difficult or requires high computational cost to find Greeks at every time in the life of the snowball currency option by Monte Carlo simulation method. That is, it is difficult to find hedge position by Monte Carlo simulation method. In this paper I will provide a semi-closed form solution, i.e., an integral form solution which makes it easier to calculate hedge positions of the snowball currency option.

3. Analysis of the snowball currency option in a simplified model

In this section I investigate a snowball currency option under a simplifying assumption that $B = 0$ and $D = \infty$. That is, it is assumed that the knock out and extension event cannot occur. In particular, I find an upper and a lower bounds of the price of the snowball currency option. The exchange rate $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion as in Garman and Kohlhagen [1].

$$dS_t = S_t(\mu - r_f)dt + S_t\sigma dW_t \text{ with the current rate } S_0,$$

where $\{W_t\}_{t\geq 0}$ is a Weiner process defined on the underlying probability space $(\Omega, \mathcal{F}, P)$ and market parameters, r(domestic interest rate), r_f(foreign interest rate), μ(drift parameter), and

σ(volatility) are constants. Let $(\mathcal{F}_t)_{t=0}^{\infty}$ be the augmentation under P of the natural filtration generated by the Weiner process $\{W_t\}_{t\geq 0}$. Hence

$$dS_t = S_t(r - r_f)dt + S_t\sigma d\tilde{W}_t, \tag{3.1}$$

where $\{\tilde{W}_t\}_{t\geq 0}$ is a Weiner Process under the risk-neutral probability measure $\tilde{P}$.

K_n is a decreasing function of S_{t_n} and an increasing function of K_{n-1} as mentioned in the previous section and is shown in the following equations and Figure 1. In figure 1, I let $K^{\max} = 1200$ and $A = 950$

For $n = 2, ..., N$,

$$\begin{aligned} K_n &= \Big(\min[K_{n-1} + A - S_{t_n}, K^{\max}]\Big)^+ \\ &= \min[(K_{n-1} + A - S_{t_n})^+, K^{\max}] \\ &= \begin{cases} K^{\max} \text{ if } S_{t_n} \leq K_{n-1} + A - K^{\max}, \\ K_{n-1} + A - S_{t_n} \text{ if } K_{n-1} + A - K^{\max} \leq S_{t_n} \leq K_{n-1} + A, \\ 0 \text{ if } S_{t_n} \geq K_{n-1} + A. \end{cases} \end{aligned} \tag{3.2}$$

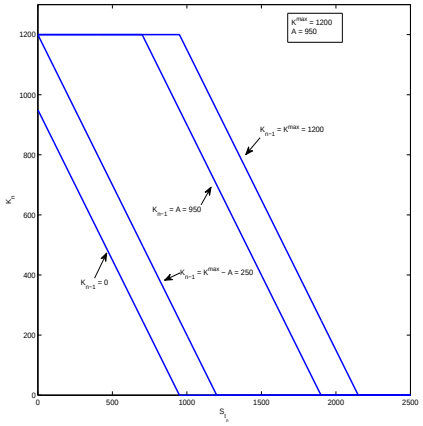

FIGURE 1. K_n as a function of S_{t_n} with various K_{n-1}'s.

The payoff G_n at time t_n can be written using Equation (3.2). For $n = 2, ..., N$,

$$\begin{aligned} G_n &= (K_n - S_{t_n})^+ - 2(S_{t_n} - K_n)^+ \\ &= \begin{cases} K_n - S_{t_n} \text{ if } S_{t_n} \le K_n, \\ -2(S_{t_n} - K_n) \text{ if } S_{t_n} \ge K_n, \end{cases} \\ &= \begin{cases} K_n - S_{t_n} \text{ if } S_{t_n} \le \frac{K_{n-1}+A}{2}, \\ -2(S_{t_n} - K_n) \text{ if } S_{t_n} \ge \frac{K_{n-1}+A}{2}, \end{cases} \\ &= \begin{cases} K^{\max} - S_{t_n} \text{ if } S_{t_n} \le K_{n-1} + A - K^{\max}, \\ K_{n-1} + A - 2S_{t_n} \text{ if } K_{n-1} + A - K^{\max} \le S_{t_n} \le \frac{K_{n-1}+A}{2}, \\ 2(K_{n-1} + A - 2S_{t_n}) \text{ if } \frac{K_{n-1}+A}{2} \le S_{t_n} \le K_{n-1} + A, \\ -2S_{t_n} \text{ if } S_{t_n} \ge K_{n-1} + A. \end{cases} \end{aligned} \tag{3.3}$$

Note here that $\frac{K_{n-1}+A}{2} > K_{n-1} + A - K^{\max}$ since $K^{\max} \ge K_{n-1}$ and $K^{\max} > A$.
It is easily checked that G_n is increasing in K_{n-1} and decreasing in S_{t_n} as is shown in Figure 2.

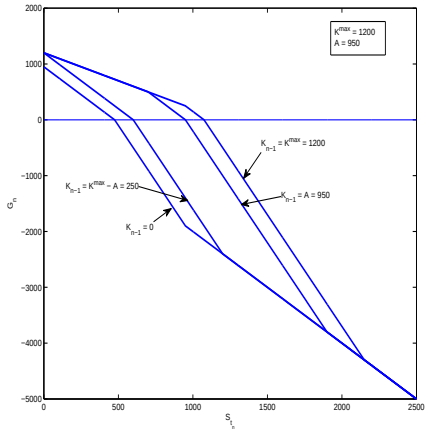

FIGURE 2. G_n as a function of S_{t_n} with various K_{n-1}'s.

The pricing formula for plain vanilla European options on a foreign currency is well known by Garman and Kohlhagen [1] and textbooks ,for example, J.C. Hull [3] or R.L. McDonald [5]. As explained in textbooks, it is identical to Merton's formula for options on dividend-paying stocks in Merton [6] with the dividend yield being replaced by the foreign interest rate. That is,

$$\begin{aligned} C(S, K) &= Se^{-r_f \tau} N(d_1) - Ke^{r\tau} N(d_2), \\ P(S, K) &= Ke^{r\tau} N(-d_2) - Se^{-r_f \tau} N(-d_1) \end{aligned}$$

with

$$d_1 := d_1(S,K) = \frac{\ln(S/K) + (r - r_f + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \text{ and } d_2 := d_2(S,K) = d_1(S,K) - \sigma\sqrt{\tau},$$

where $C(S,K)$ and $P(S,K)$ denote the prices of the European call and options respectively on the foreign currency with the current exchange rate S, the strike price K, and maturity τ, and $N(\cdot)$ is the cumulative standard normal distribution function. Since K_1 is a fixed constant, the price for the payoff G_1 is

$$\tilde{E}[e^{-r\tau}G_1] = P(S_0,K_1) - 2C(S_0,K_1) = e^{-r\tau}K_1 - e^{-r_f\tau}S_0 - C(S_0,K_1), \tag{3.4}$$

where $\tilde{E}$ is the expectation operator under the risk-neutral probability measure $\tilde{P}$ and the second equality comes from the put-call parity(See J.C. Hull [3] or R.L. McDonald [5]). The payoff G_n in Equation (3.3) can be rewritten. For $n = 2, ..., N$,

$$G_n = \begin{cases} 2[\frac{K_{n-1}+A}{2} - S_{t_n}] - [(K_{n-1} + A - K^{\max}) - S_{t_n}] \text{ if } S_{t_n} \le K_{n-1} + A - K^{\max}, \\ 2(\frac{K_{n-1}+A}{2} - S_{t_n}) \text{ if } K_{n-1} + A - K^{\max} \le S_{t_n} \le \frac{K_{n-1}+A}{2}, \\ -4(S_{t_n} - \frac{K_{n-1}+A}{2}) \text{ if } \frac{K_{n-1}+A}{2} \le S_{t_n} \le K_{n-1} + A, \\ -4(S_{t_n} - \frac{K_{n-1}+A}{2}) + 2[S_{t_n} - (K_{n-1} + A)] \text{ if } S_{t_n} \ge K_{n-1} + A. \end{cases} \tag{3.5}$$

It is easily observed by Equation (3.5) that the payoff G_n is the same as that of two long and one short position on the put options with strike price $\frac{K_{n-1}+A}{2}$ and $K_{n-1} + A - K^{\max}$ respectively, and four short and two long position on the call options with strike price $\frac{K_{n-1}+A}{2}$ and $K_{n-1} + A$ respectively. Here the payoff of the short position on the put option with strike price $K_{n-1} + A - K^{\max}$ is regarded as zero if $K_{n-1} + A - K^{\max} \le 0$. Therefore it holds that for $n = 2, ..., N$,

$$\begin{aligned} & \tilde{E}[e^{-r\tau}G_n \mid \mathcal{F}_{t_{n-1}}] \\ = \; & 2P(S_{t_{n-1}}, \frac{K_{n-1}+A}{2}) - P(S_{t_{n-1}}, K_{n-1} + A - K^{\max})\mathbf{1}_{\{K_{n-1}+A-K^{\max}>0\}} \\ & -4C(S_{t_{n-1}}, \frac{K_{n-1}+A}{2}) + 2C(S_{t_{n-1}}, K_{n-1} + A) \end{aligned} \tag{3.6}$$

Therefore, we have

$$\tilde{E}[e^{-r\tau}G_2 \mid \mathcal{F}_{t_1}] = 2P(S_{t_1}, \frac{K_1+A}{2}) - P(S_{t_1}, K_1 + A - K^{\max}) - 4C(S_{t_1}, \frac{K_1+A}{2}) + 2C(S_{t_1}, K_1 + A).$$

By the put-call parity,

$$2P(S_{t_1}, \frac{K_1+A}{2}) - 2C(S_{t_1}, \frac{K_1+A}{2}) = 2(e^{-r\tau}\frac{K_1+A}{2} - e^{-r_f\tau}S_{t_1}).$$

Hence, we have

$$\begin{aligned} \tilde{E}[e^{-r\tau}G_2 \mid \mathcal{F}_{t_1}] \;=\; & 2(e^{-r\tau}\frac{K_1+A}{2} - e^{-r_f\tau}S_{t_1}) - P(S_{t_1}, K_1 + A - K^{\max}) \\ & -2C(S_{t_1}, \frac{K_1+A}{2}) + 2C(S_{t_1}, K_1 + A) \end{aligned} \tag{3.7}$$

Since G_n is an increasing function of K_{n-1} and $K_{n-1} \leq K^{\max}$, we have, for $n = 2, ..., N$,

$$\tilde{E}[e^{-r\tau}G_n \mid \mathcal{F}_{t_{n-1}}] \leq 2P(S_{t_{n-1}}, \frac{K^{\max}+A}{2}) - P(S_{t_{n-1}}, A) - 4C(S_{t_{n-1}}, \frac{K^{\max}+A}{2}) + 2C(S_{t_{n-1}}, K^{\max}+A)$$

Applying the put-call parity to this equation, we have, for $n = 2, ..., N$,

$$\begin{aligned}\tilde{E}[e^{-r\tau}G_n \mid \mathcal{F}_{t_{n-1}}] \quad \leq \quad & 2(e^{-r\tau}\frac{K^{\max}+A}{2} - e^{-r_f\tau}S_{t_{n-1}}) - P(S_{t_{n-1}}, A) \\ & -2C(S_{t_{n-1}}, \frac{K^{\max}+A}{2}) + 2C(S_{t_{n-1}}, K^{\max}+A). \qquad (3.8)\end{aligned}$$

By Equation (3.1), we have, for $n = 2, ..., N$,

$$S_{t_{n-1}} = S_0 e^{(r-r_f-\frac{\sigma^2}{2})t_{n-1}+\sigma\tilde{W}_{t_{n-1}}},$$

and

$$\tilde{E}[S_{t_{n-1}}] = S_0 e^{(r-r_f)t_{n-1}} = S_0 e^{(r-r_f)(n-1)\tau}, \qquad (3.9)$$

where $\tilde{W}_{t_{n-1}}$ is a normal random variable with mean 0 and variance $t_{n-1} = (n-1)\tau$. Therefore $S_{t_{n-1}}$, for $n = 2, ..., N$, can be represented as

$$S_{t_{n-1}} = S_0 e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}Z}, \qquad (3.10)$$

where Z is a standard normal random variable. Note that the price V_0 at time 0 of the snowball currency option is

$$V_0 = \sum_{n=1}^{N} \tilde{E}[e^{-rt_n}G_n]. \qquad (3.11)$$

However, by the tower property and using (3.7), (3.8), (3.9), and (3.10), we have

$$\begin{aligned}\tilde{E}[e^{-rt_2}G_2] \quad = \quad & e^{-r\tau}\tilde{E}[\tilde{E}[e^{-r\tau}G_2 \mid \mathcal{F}_{t_1}]] \\ = \quad & e^{-r\tau}\Big[2(e^{-r\tau}\frac{K_1+A}{2} - e^{-r_f\tau}S_0 e^{(r-r_f)\tau}) \\ & - \int_{-\infty}^{\infty} P(S_0 e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A - K^{\max})\phi(z)dz \\ & -2\int_{-\infty}^{\infty} C(S_0 e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, \frac{K_1+A}{2})\phi(z)dz \\ & +2\int_{-\infty}^{\infty} C(S_0 e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A)\phi(z)dz\Big], \qquad (3.12)\end{aligned}$$

and for $n = 3, ..., N$,

$$\begin{aligned}
\tilde{E}[e^{-rt_n}G_n] &= e^{-r(n-1)\tau}\tilde{E}[\tilde{E}[e^{-r\tau}G_n \mid \mathcal{F}_{t_{n-1}}]] \\
&\le e^{-r(n-1)\tau}\Big[2(e^{-r\tau}\frac{K^{\max}+A}{2} - e^{-r_f\tau}S_0e^{(r-r_f)(n-1)\tau}) \\
&\quad - \int_{-\infty}^{\infty} P(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, A)\phi(z)dz \\
&\quad -2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, \frac{K^{\max}+A}{2})\phi(z)dz \\
&\quad +2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, K^{\max}+A)\phi(z)dz\Big] \quad (3.13)
\end{aligned}$$

where $\phi(z) := \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ is the standard normal density. Therefore we get an upper bound V_{ub} of the price V_0 of the snowball currency option by using (3.11), (3.4), (3.12), and (3.13). That is,

$$\begin{aligned}
V_{ub} &= e^{-r\tau}K_1 - e^{-r_f\tau}S_0 - C(S_0, K_1) \\
&\quad +e^{-r\tau}\Big[2(e^{-r\tau}\frac{K_1+A}{2} - e^{-r_f\tau}S_0e^{(r-r_f)\tau}) \\
&\quad -\int_{-\infty}^{\infty} P(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1+A-K^{\max})\phi(z)dz \\
&\quad -2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, \frac{K_1+A}{2})\phi(z)dz \\
&\quad +2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1+A)\phi(z)dz\Big] \\
&\quad +\sum_{n=3}^{N} e^{-r(n-1)\tau}\Big[2(e^{-r\tau}\frac{K^{\max}+A}{2} - e^{-r_f\tau}S_0e^{(r-r_f)(n-1)\tau}) \\
&\quad -\int_{-\infty}^{\infty} P(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, A)\phi(z)dz \\
&\quad -2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, \frac{K^{\max}+A}{2})\phi(z)dz \\
&\quad +2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, K^{\max}+A)\phi(z)dz\Big]. \quad (3.14)
\end{aligned}$$

Since G_n is an increasing function of K_{n-1} and $K_{n-1} \ge 0$, we have, for $n = 2, ..., N$,

$$\tilde{E}[e^{-r\tau}G_n \mid \mathcal{F}_{t_{n-1}}] \ge 2P(S_{t_{n-1}}, \frac{A}{2}) - 4C(S_{t_{n-1}}, \frac{A}{2}) + 2C(S_{t_{n-1}}, A)$$

Applying the put-call parity to this equation, we have, for $n = 2, ..., N$,

$$\tilde{E}[e^{-r\tau}G_n \mid \mathcal{F}_{t_{n-1}}] \geq 2(e^{-r\tau}\frac{A}{2} - e^{-r_f\tau}S_{t_{n-1}}) - 2C(S_{t_{n-1}}, \frac{A}{2}) + 2C(S_{t_{n-1}}, A).$$

Similarly to (3.14), we get a lower bound V_{lb} of the price V_0 of the snowball currency option:

$$\begin{aligned}
V_{lb} &= e^{-r\tau}K_1 - e^{-r_f\tau}S_0 - C(S_0, K_1) \\
&\quad + e^{-r\tau}\Big[2(e^{-r\tau}\frac{K_1 + A}{2} - e^{-r_f\tau}S_0e^{(r-r_f)\tau}) \\
&\quad - \int_{-\infty}^{\infty} P(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A - K^{\max})\phi(z)dz \\
&\quad - 2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, \frac{K_1 + A}{2})\phi(z)dz \\
&\quad + 2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A)\phi(z)dz\Big] \\
&\quad + \sum_{n=3}^{N} e^{-r(n-1)\tau}\Big[2(e^{-r\tau}\frac{A}{2} - e^{-r_f\tau}S_0e^{(r-r_f)(n-1)\tau}) \\
&\quad - 2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, \frac{A}{2})\phi(z)dz \\
&\quad + 2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})(n-1)\tau+\sigma\sqrt{(n-1)\tau}z}, A)\phi(z)dz\Big].
\end{aligned}$$

4. A PRICING FORMULA USING INTEGRAL IN A SIMPLIFIED MODEL

As in the previous section, I assume that $B = 0$ and $D = \infty$. Using arguments similar to the previous section, we can find an integral form of the price V_0 of a snowball currency option. I focus on the case where $N = 2$ for a simple illustration.[1] In this case, using (3.4) and (3.12), we get

$$\begin{aligned}
V_0 &= e^{-r\tau}K_1 - e^{-r_f\tau}S_0 - C(S_0, K_1) \\
&\quad + e^{-r\tau}\Big[2(e^{-r\tau}\frac{K_1 + A}{2} - e^{-r_f\tau}S_0e^{(r-r_f)\tau}) \\
&\quad - \int_{-\infty}^{\infty} P(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A - K^{\max})\phi(z)dz \\
&\quad - 2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, \frac{K_1 + A}{2})\phi(z)dz \\
&\quad + 2\int_{-\infty}^{\infty} C(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A)\phi(z)dz\Big].
\end{aligned}$$

[1] An integral form solution in the case where $N \geq 3$ can be found using similar arguments in the previous section. The form is more complex since the backward induction needs more steps.

The Greek letters can be calculated using this pricing formula. For example, I will find the formula of delta(Δ). As is shown in many textbooks(for example,J.C. Hull [3]), we have

$$\frac{\partial C(S,K)}{\partial S} = e^{-r_f\tau}N(d_1), \quad \frac{\partial P(S,K)}{\partial S} = e^{-r_f\tau}[N(d_1)-1],$$

where

$$d_1 := d_1(S,K) = \frac{\ln(S/K) + (r - r_f + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \text{ and } d_2 := d_2(S,K) = d_1(S,K) - \sigma\sqrt{\tau}.$$

Therefore, delta(Δ) of the snowball currency option becomes

$$\begin{aligned}
\Delta &= \frac{\partial V_0}{\partial S_0} \\
&= -e^{-r_f\tau} - e^{-r_f\tau}N(d_1(S_0,K_1)) \\
&\quad + e^{-r\tau}\Big[- 2e^{-r_f\tau}e^{(r-r_f)\tau} \\
&\quad - \int_{-\infty}^{\infty} e^{-r_f\tau}\Big[N\Big(d_1(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A - K^{\max})\Big) - 1\Big]e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}\phi(z)dz \\
&\quad - 2\int_{-\infty}^{\infty} e^{-r_f\tau}N\Big(d_1(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, \frac{K_1+A}{2})\Big)e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}\phi(z)dz \\
&\quad + 2\int_{-\infty}^{\infty} e^{-r_f\tau}N\Big(d_1(S_0e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}, K_1 + A)\Big)e^{(r-r_f-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}\phi(z)dz\Big].
\end{aligned}$$

Note that the pricing formula can be found for the case where $t_1 \neq \tau = t_2 - t_1$ in a similar way. Therefore, delta(Δ) of the snowball currency option can be found dynamically, i.e., at every time t such that $0 \leq t \leq t_2$, which is very difficult in Monte Carlo method. That is, an integral form solution makes it easier to do delta hedging than that of Monte Carlo method.

5. Conclusion

I have introduced a derivative, a snowball currency option, which was once traded in the over-the-counter market in Korea. I have analyzed the payoff structure. I have found an upper and a lower boundary of the snowball currency option under a simplifying assumption. I have also provided an integral form of solution in the simplified model, which makes it easier to find Greeks dynamically than that of Monte Carlo method.

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