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Rough Approximations on Preordered Sets

Yong Chan Kim[1] and Young Sun Kim[2]

[1] **Department of Mathematics, Gangneung-Wonju National University, Gangneung, 201-702, Korea**
[2] **Department of Applied Mathematics, Pai Chai University, Dae Jeon, 302-735, Korea**

Abstract

In this paper, we investigate the properties of rough approximations defined by preordered sets. We study the relations among the lower and upper rough approximations, closure and interior systems, and closure and interior operators.

Key Words: Preordered sets, Lower and upper rough approximations, Closure and interior systems, Closure and interior operators.

1. Introduction and preliminaries

An information consists of (X, Y, R) where X is a set of objects, Y is a set of attributes and R is a relation between X and Y. Rough set theory was introduced by Pawlak [6] to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. J. Järvinen et.al.[3] define rough approximations on preordered relations that are not necessarily equivalence relations. It is an important mathematical tool for data analysis and knowledge processing [1-7].

In this paper, we investigated the properties of rough approximations defined in preordered sets. We study the relations among the lower and upper approximations, closure and interior systems, closure and interior operators. In particular, we investigated the functorial relations among them.

Let X be a set. A relation $e_X \subset X \times X$ is called a preorder if it is reflexive and transitive. We can define a preorder $e_{P(X)} \subset P(X) \times P(X)$ as $(A, B) \in e_{P(X)}$ iff $A \subset B$ for $A, B \in P(X)$. If (X, e_X) is a preordered set and we define a function $(x, y) \in e_X^{-1}$ iff $(y, x) \in e_X$, then (X, e_X^{-1}) is a preordered set.

A function $I : P(X) \to P(X)$ is called an *interior operator* on X if it satisfies the following conditions:

(I1) $I(A) \subset A$,
(I2) If $A \subset B$, then $I(A) \subset I(B)$
(I3) $I(I(A)) \supset I(A)$.

The pair (X, I) is called an interior space.

Let (X, I_X) and (Y, I_Y) be interior spaces. A map $f : (X, I_X) \to (Y, I_Y)$ is called I-map if $f^{-1}(I_Y(B)) \subset I_X(f^{-1}(B))$ for each $B \in P(Y)$.

An operator $C : P(X) \to P(X)$ is called a *closure operator* on X if it satisfies the following conditions:

(C1) $C(A) \supset A$,
(C2) If $A \subset B$, then $C(A) \subset C(B)$
(C3) $C(C(A)) \subset C(A)$.

The pair (X, C) is called a closure space. Let (X, C_X) and (Y, C_Y) be closure spaces. A map $f : (X, C_X) \to (Y, C_Y)$ is called C-map if $f(C_X(A)) \subset C_Y(f(A))$ for each $A \in P(X)$.

(1) A family $\mathcal{F} = \{A \in P(X)\}$ is called an *interior system* on X if $\bigcup_{i\in\Gamma} A_i \in \mathcal{F}$ for all $A_i \in \mathcal{F}$.

(2) A family $\mathcal{G} = \{A \in P(X)\}$ is called a *closure system* on X if $\bigcap_{i\in\Gamma} A_i \in \mathcal{G}$ for all $A_i \in \mathcal{G}$.

Definition 1.1. [3-5] Let (X, e_X) be a preodered set. A set $A \in P(X)$ is called e_X*-upper set* if $(x \in A \ \&\ (x, y) \in e_X) \to y \in A$ for $x, y \in X$.

Theorem 1.2. [3-5] Let (X, e_X) be a preordered set. For $A \in P(X)$, we define operations $[e_X], \langle e_X\rangle$ as follows:

$$[e_X](A) = \{x \in X \mid (\forall z \in X)((x, z) \in e_X \to z \in A)\},$$

$$\langle e_X\rangle(A) = \{x \in X \mid (\exists z \in X)((x, z) \in e_X \ \&\ z \in A)\}.$$

Then the following properties:

(1)If $(e_X)_x = \{z \in X \mid (x, z) \in e_X\}$ and $(e_X)_x^{-1} = \{z \in X \mid (z, x) \in e_X\}$, then $(e_X)_x$ and $((e_X)_x^{-1})^c$ are e_X-upper set.

(2) A is an e_X-upper set iff $[e_X](A) = A$ iff $[e_X^{-1}](A^c) = A^c$ iff $\langle e_X^{-1}\rangle(A) = A$.

(3) If A_i is an e_X-upper set for all $i \in \Gamma$, then $\bigcup_{i\in\Gamma} A_i$ and $\bigcap_{i\in\Gamma} A_i$ are e_X-upper sets.

(4) $[e_X](A) = \bigcup_i\{A_i \mid A_i \subset A, A_i : e_X - \text{upper set}\}$.

(5) $\langle e_X\rangle(A) = \bigcap_i\{A_i \mid A \subset A_i, A_i : e_X^{-1} - \text{upper set}\}$.

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Definition 1.3. [3] In above theorem, $[e_X](A)$ and $\langle e_X\rangle(A)$ are called *rough lower approximation* and *rough upper approximation*, respectively, for $A \in P(X)$ on a preodered set.

If e_X is an equivalence relation, $[e_X](A)$ and $\langle e_X\rangle(A)$ are rough lower approximation and rough upper approximation for $A \in P(X)$ in a Pawlak sense [6].

2. Rough Approximations on Preordered Sets

Theorem 2.1. Let $\mathcal{F} = \{B_i \in P(X) \mid i \in \Gamma\}$ be an interior system.

$$I_{\mathcal{F}}(A) = \bigcup\{B \mid B \subset A,\ B \in \mathcal{F}\}$$

$$C_{\mathcal{F}^*}(A) = \bigcap\{B \mid A \subset B,\ B \in \mathcal{F}^*\}$$

where $\mathcal{F}^* = \{A_i \in P(X) \mid A_i^c \in \mathcal{F}\}$.

Then; (1) $\mathcal{F}^*$ is a closure system.

(2) $I_{\mathcal{F}}$ is an interior operator on X with $\mathcal{F}_{I_{\mathcal{F}}} = \mathcal{F}$ where $\mathcal{F}_{I_{\mathcal{F}}} = \{A \in P(X) \mid I_{\mathcal{F}}(A) = A\}$.

(3) $C_{\mathcal{F}^*}$ is an closure operator on X with $\mathcal{F}^*_{C_{\mathcal{F}^*}} = \mathcal{F}^*$ where $\mathcal{F}^*_{C_{\mathcal{F}^*}} = \{A \in P(X) \mid C_{\mathcal{F}^*}(A) = A\}$.

(4) There exists a preorder $e_{\mathcal{F}}$ on X such that $\mathcal{F} \subset \mathcal{F}_{[e_{\mathcal{F}}]}$ with

$$(x, y) \in e_{\mathcal{F}} \text{ iff } (\forall B_i \in \mathcal{F})(x \in B_i \to y \in B_i).$$

In particular, $I_{\mathcal{F}}(A) \subset [e_{\mathcal{F}}](A)$ with

$$y \in [e_{\mathcal{F}}](\{x\}^c) \text{ iff } y \in I_{\mathcal{F}}(\{x\}^c) \text{ iff } (y, x) \notin e_{\mathcal{F}}.$$

(5) There exists a preorder $e_{\mathcal{F}^*}$ on X such that $\mathcal{F}^* \subset \mathcal{F}^*_{\langle e_{\mathcal{F}^*}\rangle}$ with

$$(x, y) \in e_{\mathcal{F}^*} \text{ iff } (\forall A_i \in \mathcal{F}^*)(y \in A_i \to x \in A_i).$$

In particular, $\langle e_{\mathcal{F}^*}\rangle(A) \subset C_{\mathcal{F}^*}(A)$ with

$$y \in \langle e_{\mathcal{F}^*}\rangle(\{x\}) \text{ iff } y \in C_{\mathcal{F}^*}(\{x\}) \text{ iff } (y, x) \in e_{\mathcal{F}^*}.$$

Proof. (1),(2) and (3) are easily proved.

(4) Since $\models (x \in B_j \to y \in B_j)\ \&\ (y \in B_j \to z \in B_j) \to (x \in B_j \to z \in B_j)$, then $e_{\mathcal{F}}$ is transitive. We easily show that $e_{\mathcal{F}}$ is a preorder on X. Let $B_j \in \mathcal{F}$. Since $\models (x \in B_j)\ \&\ (x \in B_j \to y \in B_j) \to y \in B_j$ and $\models (x \in B_j)\ \&\ (x, y) \in e_{\mathcal{F}} \to (x \in B_j)\ \&\ (x \in B_j \to y \in B_j)$, by M.P., $\models (x \in B_j)\ \&\ (x, y) \in e_{\mathcal{F}} \to y \in B_j$. So, B_j is $e_{\mathcal{F}}$-upper set. By Theorem 1.2 (2), $B_j \in \mathcal{F}_{[e_{\mathcal{F}}]}$. Thus, $\mathcal{F} \subset \mathcal{F}_{[e_{\mathcal{F}}]}$.

Since $I_{\mathcal{F}}(A) \in \mathcal{F}$, $\models (x \in I_{\mathcal{F}}(A))\ \&\ (x \in I_{\mathcal{F}}(A) \to y \in I_{\mathcal{F}}(A) \to y \in I_{\mathcal{F}}(A)$ iff $\models (x \in I_{\mathcal{F}}(A)) \to \Big((x \in I_{\mathcal{F}}(A) \to y \in I_{\mathcal{F}}(A)) \to y \in I_{\mathcal{F}}(A)\Big)$. Since $\models \Big((x \in I_{\mathcal{F}}(A) \to y \in I_{\mathcal{F}}(A)) \to y \in I_{\mathcal{F}}(A)\Big) \to x \in [e_{\mathcal{F}}](A))$, by M.P., $\models x \in I_{\mathcal{F}}(A) \to x \in [e_{\mathcal{F}}](A))$.

$$\begin{aligned} y \in [e_{\mathcal{F}}](\{x\}^c) \quad & \text{iff } (\forall z \in X)((y, z) \in e_{\mathcal{F}} \to z \in \{x\}^c) \\ & \text{iff } (y, x) \notin e_{\mathcal{F}}. \\ y \in I_{\mathcal{F}}(\{x\}^c) \quad & \text{iff } y \in \textstyle\bigcup_{B_i \in \mathcal{F}} B_i\ \&B_i \subset \{x\}^c \\ & \text{iff } (\exists B_i \in \mathcal{F})(x \in B_i^c\ \&\ y \in B_i) \\ & \text{iff } \Big((\forall B_i \in \mathcal{F})(y \in B_i \to x \in B_i)\Big)^c \\ & \text{iff } (y, x) \notin e_{\mathcal{F}} \end{aligned}$$

(5) We easily show that $e_{\mathcal{F}^*}$ is a preorder on X. Let $D_j \in \mathcal{F}^*$. Since $\models x \in D_j\ \&\ (y, x) \in e_{\mathcal{F}^*} \to x \in D_j\ \&\ (x \in D_j \to y \in D_j)$ and $\models x \in D_j\ \&\ (x \in D_j \to y \in D_j) \to y \in D_j$, by M.P., $\models x \in D_j\ \&\ (y, x) \in e_{\mathcal{F}^*} \to y \in D_j$. Hence D_j is e_X^{-1}-upper set. By Theorem 1.2(2), $\langle e_{\mathcal{F}^*}\rangle(D_j) = D_j$. Hence $D_j \in \mathcal{F}^*_{\langle e_{\mathcal{F}^*}\rangle}$. Thus, $\mathcal{F}^* \subset \mathcal{F}^*_{\langle e_{\mathcal{F}^*}\rangle}$.

Since $C_{\mathcal{F}^*}(A) \in \mathcal{F}^*$, we have $x \in \langle e_{\mathcal{F}^*}\rangle(A)$ implies $(\exists y \in Y)((y \in C_{\mathcal{F}^*}(A) \to x \in C_{\mathcal{F}^*}(A))\ \&\ y \in A)$ implies $(\exists y \in Y)((y \in C_{\mathcal{F}^*}(A) \to x \in C_{\mathcal{F}^*}(A))\ \&\ y \in C_{\mathcal{F}^*}(A)$ implies $x \in C_{\mathcal{F}^*}(A)$. Thus, $\langle e_{\mathcal{F}^*}\rangle(A) \subset C_{\mathcal{F}^*}(A)$.

$$\begin{aligned} y \in \langle e_{\mathcal{F}^*}\rangle(\{x\}) \quad & \text{iff } (\exists z \in X)((y, z) \in e_{\mathcal{F}^*}\ \&\ z \in \{x\}) \\ & \text{iff } (y, x) \in e_{\mathcal{F}^*}. \\ y \in C_{\mathcal{F}^*}(\{x\}) \quad & = \textstyle\bigcap\{B \mid \{x\} \subset B,\ B \in \mathcal{F}^*\} \\ & \text{iff } x \in B \to y \in \textstyle\bigcap_{B \in \mathcal{F}^*} B \\ & \text{iff } (\forall B \in \mathcal{F}^*)(x \in B \to y \in B) \\ & \text{iff } (y, x) \in e_{\mathcal{F}^*}. \end{aligned}$$

□

Theorem 2.2. Let (X, e_X) be a preordered set. We define $\mathcal{F}_{[e_X]}, \mathcal{G}_{\langle e_X\rangle}$ as follows:

$$\mathcal{F}_{[e_X]} = \{A \in P(X) \mid [e_X](A) = A\}$$

$$\mathcal{G}_{\langle e_X\rangle} = \{A \in P(X) \mid \langle e_X\rangle(A) = A\}$$

Then; (1) $[e_X]$ is an interior operator on X with $x \in [e_X](\{y\}^c)$ iff $(x, y) \notin e_X$.

(2) $\langle e_X\rangle$ is a closure operator on X with $x \in \langle e_X\rangle(\{y\})$ iff $(x, y) \in e_X$.

(3) $\mathcal{F}_{[e_X]}$ is an interior and closure system with $I_{\mathcal{F}_{[e_X]}} = [e_X]$ where $I_{\mathcal{F}_{[e_X]}}(A) = \bigcup\{B \mid B \subset A,\ B \in \mathcal{F}_{[e_X]}\}$.

(4) $\mathcal{G}_{\langle e_X\rangle}$ is an interior and closure system with $C_{\mathcal{G}_{\langle e_X\rangle}} = \langle e_X\rangle$ where $C_{\mathcal{G}_{\langle e_X\rangle}}(A) = \bigcap\{B \mid A \subset B,\ B \in \mathcal{G}_{\langle e_X\rangle}\}$.

(5) $e_{\mathcal{F}_{[e_X]}} = e_X$ and $e_{\mathcal{G}_{\langle e_X\rangle}} = e_X$.

Proof. (1) Since

$$\begin{aligned} & x \in [e_X]([e_X](A)) \\ \text{iff } & (\forall z \in X)((x, z) \in e_X \to z \in [e_X](A)) \\ \text{iff } & (\forall z \in X)((x, z) \in e_X \to (\forall w \in X)((z, w) \in e_X \\ & \to w \in A)) \\ \text{iff } & (\forall w \in X)((\forall z \in X)((x, z) \in e_X\ \&\ (z, w) \in e_X) \\ & \to w \in A) \end{aligned}$$

then $(\forall w \in X)((x,w) \in e_X \to w \in A)$. Hence $[e_X]([e_X](A)) \subset [e_X](A)$.

$$\begin{aligned} & x \in [e_X](\{y\}^c) \\ & \text{iff } (\forall z \in X)((x,z) \in e_X \to z \in \{y\}^c) \\ & \text{iff } (\forall z \in X)(z \in \{y\} \to (x,z) \notin e_X) \\ & \text{iff } (x,y) \notin e_X \end{aligned}$$

(3) Let $A_i \in \mathcal{F}_{[e_X]}$ for $i \in \Gamma$. Since $x \in [e_X](\bigcap_{i\in\Gamma} A_i)$ iff $(\forall z \in X)((x,z) \in e_X \to z \in \bigcap_{i\in\Gamma} A_i)$ iff $(\forall i \in \Gamma)(\forall z \in X)((x,z) \in e_X \to z \in A_i)$, we have

$$[e_X](\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma}[e_X](A_i) = \bigcap_{i\in\Gamma} A_i.$$

Hence $\bigcap_{i\in\Gamma} A_i \in \mathcal{F}_{[e_X]}$. Since $\bigcup_{i\in\Gamma} A_i = \bigcup_{i\in\Gamma}[e_X](A_i) \subset [e_X](\bigcup_{i\in\Gamma} A_i) \subset \bigcup_{i\in\Gamma} A_i$, then $\bigcup_{i\in\Gamma} A_i \in \mathcal{F}_{[e_X]}$.

Since $[e_X]([e_X](A)) = [e_X](A) \subset A$, $I_{\mathcal{F}_{[e_X]}} \supset [e_X]$.

Since $I_{\mathcal{F}_{[e_X]}}(A) \subset A$ and $I_{\mathcal{F}_{[e_X]}}(A) \in \mathcal{F}_{[e_X]}$, then $[e_X](I_{\mathcal{F}_{[e_X]}}(A)) = I_{\mathcal{F}_{[e_X]}}(A) \subset [e_X](A)$.

(4) Let $A_i \in \mathcal{G}_{\langle e_X\rangle}$ for $i \in \Gamma$. Since $x \in \langle e_X\rangle(\bigcup_{i\in\Gamma} A_i)$ iff $(\exists z \in X)((x,z) \in e_X \;\&\; z \in \bigcup_{i\in\Gamma} A_i)$ iff $(\exists i \in \Gamma)(\exists z \in X)((x,z) \in e_X \;\&\; z \in A_i)$, we have

$$\langle e_X\rangle(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma}\langle e_X\rangle(A_i) = \bigcup_{i\in\Gamma} A_i.$$

Hence $\bigcup_{i\in\Gamma} A_i \in \mathcal{G}_{\langle e_X\rangle}$. Since $\bigcap_{i\in\Gamma}(A_i) = \bigcap_{i\in\Gamma}\langle e_X\rangle(A_i) \supset \langle e_X\rangle(\bigcap_{i\in\Gamma} A_i) \supset \bigcap_{i\in\Gamma} A_i$, then $\bigcap_{i\in\Gamma} A_i \in \mathcal{G}_{\langle e_X\rangle}$.

Since $\langle e_X\rangle(\langle e_X\rangle(A)) = \langle e_X\rangle(A) \supset A$, $C_{\mathcal{G}_{\langle e_X\rangle}} \subset \langle e_X\rangle$.

Since $A \subset C_{\mathcal{G}_{\langle e_X\rangle}}$ and $C_{\mathcal{G}_{\langle e_X\rangle}}(A) \in \mathcal{G}_{\langle e_X\rangle}$, then $C_{\mathcal{G}_{\langle e_X\rangle}}(A) = C_{\mathcal{G}_{\langle e_X\rangle}}(A) \supset \langle e_X\rangle(A)$.

(5) For $B_i \in \mathcal{F}_{[e_X]}$, since $[e_X](B_i) = B_i$, by Theorem 1.2(2), B_i is e_X-upper set. Hence $\models (x,y) \in e_X \to (x \in B_i \to y \in B_i)$. Since $(x,y) \in e_{\mathcal{F}_{[e_X]}}$ iff $(\forall B_i \in \mathcal{F}_{[e_X]})(x \in B_i(x) \to y \in B_i)$, then $e_X \subset e_{\mathcal{F}_{[e_X]}}$. Since $[e_X]((e_X)_x) = (e_X)_x$ from Theorem 1.2(1), $(x,y) \in e_{\mathcal{F}_{[e_X]}}$ implies $(x \in (e_X)_x \to y \in (e_X)_x)$ implies $(x,y) \in e_X$. Hence $e_{\mathcal{F}_{[e_X]}} \subset e_X$.

For $B_i \in \mathcal{G}_{\langle e_X\rangle}$, since $\langle e_X\rangle(B_i) = B_i$, by Theorem 1.2(2), B_i is e_X^{-1}-upper set. Hence $\models (x,y) \in e_X \to (y \in B_i \to x \in B_i)$. Since $(x,y) \in e_{\mathcal{G}_{\langle e_X\rangle}}$ iff $(\forall B_i \in \mathcal{G}_{\langle e_X\rangle})(y \in B_i \to x \in B_i)$, then $e_X \subset e_{\mathcal{G}_{\langle e_X\rangle}}$. Since $\langle e_X\rangle((e_X)_y^{-1}) = (e_X)_y^{-1}$ from Theorem 1.2(1), $(x,y) \in e_{\mathcal{G}_{\langle e_X\rangle}}$ implies $(y \in (e_X)_y^{-1} \to x \in (e_X)_y^{-1})$ implies $(x,y) \in e_X$. Hence $e_{\mathcal{G}_{\langle e_X\rangle}} \subset e_X$.

□

Example 2.3. Let $X = \{a,b,c,d\}$ be a set. We define a preoder e_X as follows:

$$e_X = \{(a,a)(a,d),(b,b),(b,d),(c,a),$$
$$(c,b),(c,c),(c,d),(d,d)\}.$$

We obtain

$$[e_X](A) = \begin{cases} \{d\} & \text{if } \{d\} \subset A, \\ \{a,d\} & \text{if } \{a,d\} \subset A, \{b,d\} \not\subset A \\ \{b,d\} & \text{if } \{b,d\} \subset A, \{a,d\} \not\subset A, \\ \{a,b,d\} & \text{if } \{a,b,d\} \subset A, \\ & \quad \{a,d\} \not\subset A, \{b,d\} \not\subset A, \\ X & \text{if } A = X, \\ \emptyset & \quad \text{otherwise}, \end{cases}$$

$$\mathcal{F}_{[e_X]} = \{\emptyset,\{d\},\{a,d\},\{b,d\},\{a,b,d\},X\}$$

$I_{\mathcal{F}_{[e_X]}} = [e_X]$ and $e_{\mathcal{F}_{[e_X]}} = e_X$.

$$\langle e_X\rangle(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{c\} & \text{if } A = \{c\}, \\ \{a,c\} & \text{if } \{a\} \subset A, A \not\supset \{b\}, A \not\supset \{d\} \\ \{b,c\} & \text{if} \{b\} \subset A, A \not\supset \{a\}, A \not\supset \{d\} \\ \{a,b,c\} & \text{if } \{a,b\} \subset A, A \not\supset \{d\} \\ X & \text{if } \{d\} \subset A. \end{cases}$$

$$\mathcal{G}_{\langle e_X\rangle} = \{\emptyset,\{c\},\{a,c\},\{b,c\},\{a,b,c\},X\}$$

Moreover, $C_{\mathcal{G}_{\langle e_X\rangle}} = \langle e_X\rangle$ and $e_{\mathcal{G}_{\langle e_X\rangle}} = e_X$.

Theorem 2.4. Let $C : P(X) \to P(X)$ be a closure operator.

Then; (1) $\mathcal{G}_C = \{A \in P(X) \mid A = C(A)\}$ is a closure system on X with $C_{\mathcal{G}_C} = C$.

(2) Define $(x,y) \in e_C$ iff $y \in C(\{x\})$. Then e_C is a preorder e_C on X with $\langle e_C\rangle(A) = \langle e_{\mathcal{G}_C}\rangle(A) \subset C(A)$ and

$$y \in \langle e_{\mathcal{G}_C}\rangle(\{x\}) \text{ iff } y \in C(\{x\}) \text{ iff } (y,x) \in e_{\mathcal{G}_C}.$$

Proof. (1) For each $A_i \in \mathcal{G}_C$, $\cap_{i\in\Gamma}A_i \in \mathcal{G}_C$ because

$$\cap_{i\in\Gamma}A_i \subset C(\cap_{i\in\Gamma}A_i) \subset \cap_{i\in\Gamma}C(A_i) \subset \cap_{i\in\Gamma}A_i.$$

Since $C(C(A)) = C(A)$, we have $C_{\mathcal{G}_C}(A) \subset C(A)$. Since $A \subset C_{\mathcal{G}_C}(A)$, $C(A) = C_{\mathcal{G}_C}(C(A)) \subset C_{\mathcal{G}_C}(C_{\mathcal{G}_C}(A)) = C_{\mathcal{G}_C}(A)$.

(2) Since $x \in C(\{x\})$, then $(x,x) \in e_C$. For $(x,y) \in e_C$ and $(y,z) \in e_C$, since $x \in C(\{y\})$ and $y \in C(\{z\})$, then $C(\{y\}) \subset C(C(\{z\})) = C(\{z\})$. Hence $x \in C(\{z\})$;i.e. $(x,z) \in e_C$.

Since $C_{\mathcal{G}_C} = C$ from (1), $C(\{x\}) = \bigcap\{A \mid \{x\} \subset A, A \in \mathcal{G}_C$. Then $y \in C(\{x\})$ iff $\models (\forall A \in \mathcal{G}_C)(x \in A \to y \in A)$ iff $(y,x) \in e_{\mathcal{G}_C}$ iff $y \in \langle e_{\mathcal{G}_C}\rangle(\{x\})$ from Theorem 2.2(2). By a similar method in Theorem 2.1(5), $\langle e_{\mathcal{G}_C}\rangle(A) \subset C(A)$.

□

Theorem 2.5. Let $I : P(X) \to P(X)$ be an interior operator.

Then; (1) $\mathcal{F}_I = \{A \in P(X) \mid A = I(A)\}$ is an interior system on X with $I_{\mathcal{F}_I} = I$.

(2) Define $(x,y) \in e_I$ iff $x \in (I(\{x\}^c))^c$. Then e_I is a preorder e_I on X with $I(A) \subset [e_I](A) = [e_{\mathcal{F}_I}](A)$ and

$$y \notin [e_{\mathcal{F}_I}\rangle(\{x\}^c) \text{ iff } y \notin I(\{x\}^c) \text{ iff } (y,x) \notin e_{\mathcal{F}_I}.$$

Proof. (1) It is similarly proved as in Theorem 2.4(1).

(2) Since $x \in (I(\{x\}^c))^c$ iff $\{x\} \subset (I(\{x\}^c))^c$ iff $I(\{x\}^c) \subset \{x\}^c$, then $(x,x) \in e_I$. For $(x,y) \in e_I$ and $(y,z) \in e_I$, since $I(\{y\}^c) \subset \{x\}^c$ and $I(\{z\}^c) \subset \{y\}^c$, then $I(\{z\}^c) \subset I(I(\{z\}^c)) \subset I(\{y\}^c) \subset \{x\}^c$. Hence $x \in (I(\{z\}^c))^c$;i.e. $(x,z) \in e_I$.

Since $I_{\mathcal{F}_I} = I$ from (1), $I(\{x\}^c)^c = \bigcap\{A^c \mid \{x\} \subset A^c, A \in \mathcal{F}_I\}$. Then $y \in (I(\{x\}^c))^c$ iff $\models (\forall A \in \mathcal{F}_I)(x \in A^c \to y \in A^c)$ iff $\models (\forall A \in \mathcal{F}_I)(y \in A \to x \in A)$ iff $(y,x) \in e_{\mathcal{F}_I}$. Moreover, $y \in [e_{\mathcal{F}_I}](\{x\}^c)$ iff $\models (\forall z \in X)((y,z) \in e_{\mathcal{F}_I} \to z \in \{x\}^c)$ iff $(y,x) \notin e_{\mathcal{F}_I}$. By a similar method in Theorem 2.1(4), $I(A) \subset [e_{\mathcal{F}_I}]$. □

Example 2.6. Let $X = \{a,b,c,d\}$, $B_1 = \{a,b\}$ and $B_2 = \{b,c\}$ be sets. We define functions $I, C : P(X) \to P(X)$ as follows:

$$I(A) = \begin{cases} B_1 & \text{if } B_1 \subset A, \\ B_2 & \text{if } B_2 \subset A, \\ B_1 \cup B_2 & \text{if } B_1 \cup B_2 \subset A, \\ & \quad B_1 \not\subset A, B_2 \not\subset A, \\ \emptyset & \quad \text{otherwise}, \end{cases}$$

$$C(A) = \begin{cases} B_1 & \text{if } A \subset B_1, A \not\subset B_2 \\ B_2 & \text{if } A \subset B_2, A \not\subset B_2 \\ B_1 \cap B_2 & \text{if } A \subset B_1 \cap B_2, \\ X & \text{otherwise}. \end{cases}$$

(1) I is an interior operator on X with $\mathcal{F}_I = \{A \mid I(A) = A\} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$.

(2) C is a closure operator on X with $\mathcal{G}_C = \{A \mid C(A) = A\} = \{B_1, B_2, B_1 \cap B_2, X\}$.

(3) $I_{\mathcal{F}_I} = I$ and $C_{\mathcal{G}_C} = C$.

(4) There exists a preorder $e_{\mathcal{F}_I}$ on X such that $\mathcal{F}_I \subset \mathcal{F}_{[e_{\mathcal{F}_I}]} = \{\emptyset, B_1, B_2, B_1 \cap B_2, B_1 \cup B_2, X\}$ with

$$(x,y) \in e_{\mathcal{F}_I} \text{ iff } \models (\forall B \in \mathcal{F}_I)(x \in B \to y \in B)$$

$$e_{\mathcal{F}_I} = \{(a,a)(a,b),(b,b),(c,b),(c,c),\\ (d,a),(d,b),(d,c),(d,d)\}$$

$$[e_{\mathcal{F}_I}](A) = \begin{cases} B_1 \cap B_2 & \text{if } B_1 \cap B_2 \subset A, \\ B_1 & \text{if } B_1 \subset A, B_2 \not\subset A \\ B_2 & \text{if } B_2 \subset A, B_1 \not\subset A, \\ B_1 \cup B_2 & \text{if } B_1 \cup B_2 \subset A, \\ & \quad B_1 \not\subset A, B_2 \not\subset A, \\ X & \text{if } A = X, \\ \emptyset & \quad \text{otherwise}, \end{cases}$$

Hence $I(A) \subset [e_{\mathcal{F}_I}](A)$ with $y \in [e_{\mathcal{F}_I}](\{x\}^c)$ iff $y \in I(\{x\}^c)$ iff $(y,x) \notin e_{\mathcal{F}_I} = e_I$ because

$$[e_{\mathcal{F}_I}](\{a\}^c) = I(\{a\}^c) = \{b,c\}$$
$$[e_{\mathcal{F}_I}](\{b\}^c) = I(\{b\}^c) = \emptyset$$
$$[e_{\mathcal{F}_I}](\{c\}^c) = I(\{c\}^c) = \{a,b\}$$
$$[e_{\mathcal{F}_I}](\{d\}^c) = I(\{d\}^c) = \{a,b,c\}$$

(5) There exists a preorder $e_{\mathcal{G}_C}$ on X such that $\mathcal{G}_C \subset \mathcal{G}_{\langle e_{\mathcal{G}_C} \rangle} = \{\emptyset, B_1, B_2, B_1 \cap B_2, B_1 \cup B_2, X\}$ with

$$(x,y) \in e_{\mathcal{G}_C} \text{ iff } \models (\forall B_i \in \mathcal{G}_C)(y \in B_i \to x \in B_i)$$

$$e_{\mathcal{G}_C} = \{(a,a)(a,d),(b,a),(b,b),(b,c),(b,d)\\ (c,c),(c,d),(d,d)\}$$

$$\langle e_{\mathcal{G}_C} \rangle(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ B_1 & \text{if } A \subset B_1, A \not\subset B_2 \\ B_2 & \text{if } A \subset B_2, A \not\subset B_2 \\ B_1 \cap B_2 & \text{if } \emptyset \neq A \subset B_1 \cap B_2, \\ B_1 \cup B_2 & \text{if } A \subset B_1 \cup B_2, \\ & \quad A \not\subset B_1, A \not\subset B_2 \\ X & \text{otherwise}. \end{cases}$$

Hence $\langle e_{\mathcal{G}_C} \rangle(A) \subset C(A)$ with $y \in \langle e_{\mathcal{G}_C} \rangle(\{x\})$ iff $y \in C(\{x\})$ iff $(y,x) \in e_{\mathcal{G}_C} = e_C$.

$$\langle e_{\mathcal{G}_C} \rangle(\{a\}) = C(\{a\}) = \{a,b\}$$
$$\langle e_{\mathcal{G}_C} \rangle(\{b\}) = C(\{b\}) = \{b\}$$
$$\langle e_{\mathcal{G}_C} \rangle(\{c\}) = C(\{c\}) = \{b,c\}$$
$$\langle e_{\mathcal{G}_C} \rangle(\{d\}) = C(\{d\}) = \{a,b,c,d\}$$

Theorem 2.7. Let (X, e_X) and (Y, e_Y) be preordered sets. Then the following statements are equivalent:

(1) $f : (X, e_X) \to (Y, e_Y)$ is an order preserving map,

(2) $f^{-1}(B)$ is an e_X-upper set for e_Y-upper set B,

(3) $f^{-1}(D)$ is an e_X^{-1}-upper set for e_Y^{-1}-upper set D,

(4) $f(\langle e_X \rangle(A)) \subset \langle e_Y \rangle(f(A))$, for each $A \in P(X)$,

(5) $f([e_X](A)) \subset [e_Y](f(A))$, for each $A \in P(X)$,

(6) $f^{-1}([e_Y](B)) \subset [e_X](f^{-1}(B))$, for each $B \in P(Y)$.

(7) $f^{-1}(\langle e_Y \rangle(B)) \subset \langle e_X \rangle(f^{-1}(B))$, for each $B \in P(Y)$.

Proof. (1)⇒ (2). Let B be an e_Y-upper set. Since f is an order preserving map, $\models x \in f^{-1}(B) \;\&\; (x,z) \in e_X \to f(x) \in B \;\&\; (f(x), f(z)) \in e_Y$ and $\models (f(x), f(z)) \in e_Y \;\&\; f(z) \in B \to z \in f^{-1}(B)$. By M.P., $\models x \in f^{-1}(B) \;\&\; (x,z) \in e_X \to z \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an e_X-upper set.

(2)⇒ (4). Since B_i is an e_Y^{-1}-upper set iff B_i^c is an e_Y-upper set, then $f^{-1}(B_i^c) = (f^{-1}(B_i))^c$ is an e_X-upper set iff $f^{-1}(B_i)$ is an e_X^{-1}-upper set. Thus,

$$\begin{aligned} &\langle e_Y \rangle(f(A)) \\ &= \textstyle\bigcap_i \{B_i \mid f(A) \subset B_i, B_i : e_Y^{-1} - \text{upper set}\} \\ &= \textstyle\bigcap_i \{B_i \mid A \subset f^{-1}(B_i), B_i : e_Y^{-1} - \text{upper set}\} \\ &\supset \textstyle\bigcap_i \{f(f^{-1}(B_i)) \mid A \subset f^{-1}(B_i), \\ &\quad f^{-1}(B_i) : e_X^{-1} - \text{upper set}\} \\ &\supset f(\textstyle\bigcap_i \{f^{-1}(B_i) \mid A \subset f^{-1}(B_i), \\ &\quad f^{-1}(B_i) : e_X^{-1} - \text{upper set}\} \\ &\supset f(\langle e_X \rangle(A)) \end{aligned}$$

(4)⇒ (6). Put $A = f^{-1}(B)$. $f(\langle e_X\rangle(f^{-1}(B)) \subset \langle e_Y\rangle(f(f^{-1}(B))) \subset \langle e_Y\rangle(B)$ implies $\langle e_X\rangle(f^{-1}(B)) \subset f^{-1}(\langle e_Y\rangle(B))$. $([e_X](f^{-1}(B^c)))^c \supset (f^{-1}([e_Y](B^c)))^c$ implies $[e_X](f^{-1}(B^c)) \supset f^{-1}([e_Y](B^c))$.

(6)⇒ (1). Put $B = \{f(z)\}^c$. Since

$$f^{-1}([e_Y](\{f(z)\}^c) \subset [e_X](f^{-1}((\{f(z)\}^c)) \subset [e_X]((\{z\}^c)$$

Then $(f(x), f(z)) \notin e_Y$ iff $x \in f^{-1}([e_Y](\{f(z)\}^c))$ implies $x \in [e_X](f^{-1}(\{f(z)\}^c))$ implies $x \in [e_X](\{z\}^c)$ iff $(x, z) \notin e_X$.

Similarly, (1)⇒ (3)⇒ (5)⇒ (7) ⇒ (1).

□

Theorem 2.8. Let $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ be closure systems. Then the following statements are equivalent:

(1) $f^{-1}(B) \in \mathcal{G}_X$ for each $B \in \mathcal{G}_Y$,

(2) $f : (X, C_{\mathcal{G}_X}) \to (X, C_{\mathcal{G}_Y})$ is a C-map.

(3) $C_{\mathcal{G}_X}(f^{-1}(B)) \subset f^{-1}(C_{\mathcal{G}_Y}(B))$ for each $B \in P(Y)$.

(4) $f : (X, I_{\mathcal{G}^*_X}) \to (X, I_{\mathcal{G}^*_Y})$ is an I-map.

Moreover, if $f^{-1}(B) \in \mathcal{G}_X$ for each $B \in \mathcal{G}_Y$, then $f : (X, e_{\mathcal{G}_X}) \to (Y, e_{\mathcal{G}_Y})$ is an order preserving map.

Proof. We easily proved that the statements (1), (2),(3) and (4) are equivalent. Let $f^{-1}(B) \in \mathcal{G}_X$ for each $B \in \mathcal{G}_Y$. Then

$$\begin{array}{l} (f(x), f(y)) \in e_{\mathcal{G}_Y} \\ \text{iff } (\forall B \in \mathcal{G}_Y)(f(y) \in B \to f(x) \in B) \\ \text{iff } (\forall B \in \mathcal{G}_Y)(y \in f^{-1}(B) \to x \in f^{-1}(B)) \end{array}$$

Hence

$$\begin{array}{ll} (x, y) \in \mathcal{G}_X & \text{iff } (\forall D \in \mathcal{G}_Y)(y \in D \to x \in D) \\ & \text{implies } (f(x), f(y)) \in e_{\mathcal{G}_Y}. \end{array}$$

□

Theorem 2.9. Let (X, C_X) and (Y, C_Y) be closure spaces. Then the following statements are equivalent:

(1) $f : X \to Y$ is a C-map.

(2) $C_X(f^{-1}(B)) \subset f^{-1}(C_Y(B))$ for each $B \in L^Y$.

(3) $f^{-1}(B) \in \mathcal{G}_{C_X}$ for each $B \in \mathcal{G}_{C_Y}$,

Moreover, if $f : X \to Y$ is a C-map, then $f : (X, e_{C_X}) \to (Y, e_{C_Y})$ is an order preserving map. If $C_X(\cup_{i\in\Gamma} A_i) = \cup_{i\in\Gamma} C_X(A_i)$, the converse holds.

Proof. We easily proved that the statements (1), (2) and (3) are equivalent. Since $f(C_X(\{x\})) \subset C_Y(\{f(x)\})$, then $z \in C_X(\{x\})$ implies $f(z) \in C_Y(\{f(x)\})$. Hence $(z, x) \in e_{C_X}$ implies $(f(z), f(x)) \in e_{C_Y}$.

Since $(z, x) \in e_{C_X}$ iff $z \in C_X(\{x\})$ implies $(f(z), f(x))$ iff $f(z) \in C_Y(\{f(x)\})$, then $f(C_X(\{x\}) \subset C_Y(\{f(x)\})$. For each $A = \cup_{x\in A}\{x\}$,

$$f(C(A)) = f(C_X(\cup_{x\in A}\{x\}) = \cup_{x\in A} f(C_X(\{x\})$$

$$\subset \cup_{x\in A} C_Y(\{f(x)\}) \subset C_Y(\cup_{x\in A}\{f(x)\}) = C_Y(f(A))$$

□

Theorem 2.10. Let (X, I_X) and (Y, I_Y) be interior spaces. Then the following statements are equivalent:

(1) $f : X \to Y$ is an I-map.

(2) $f^{-1}(B) \in \mathcal{F}_{I_X}$ for each $B \in \mathcal{F}_{I_Y}$,

Moreover, if $f : X \to Y$ is an I-map, then $f : (X, e_{I_X}) \to (Y, e_{I_Y})$ is an order preserving map. If $I_X(\cap_{i\in\Gamma} A_i) = \cap_{i\in\Gamma} I_X(A_i)$, the converse holds.

Proof. We easily proved that the statements (1) and (2) are equivalent. Let $f : X \to Y$ be an I-map. Then

$$f^{-1}(I_Y(\{f(x)\}^c)) \subset I_X(f^{-1}(\{f(x)\}^c)) \subset I_X(\{x\}^c).$$

Hence $z \in I_X(\{x\}^c)^c$ implies $f(z) \in I_Y(\{f(x)\}^c)^c$. Thus, $(z, x) \in e_{I_X}$ implies $(f(z), f(x)) \in e_{I_Y}$.

Since $(x, z) \in e_{I_X}$ iff $x \in I_X(\{z\}^c)^c$ implies $(f(x), f(z))$ iff $f(x) \in I_Y(\{\{f(z)\}^c)^c$, then $f(I_X(\{x\}^c)^c) \subset I_Y(\{f(x)\}^c)^c$. Thus $I_X(\{x\}^c)^c \subset f^{-1}(I_Y(\{f(x)\}^c)^c)$ implies $f^{-1}(I_Y(\{f(x)\}^c)) \subset I_X(\{x\}^c)$. Put $f(x) = y$. Then $f^{-1}(I_Y(\{y\}^c)) \subset I_X(\{f^{-1}(y)\}^c)$. For each $B = \cap_{y\in B^c}\{y\}^c$,

$$\begin{array}{ll} f^{-1}(I_Y(B)) & = f^{-1}(I_Y(\cap_{y\in B^c}\{y\}^c)) \\ & \subset \cap_{y\in B^c} f^{-1}(I_Y(\{y\}^c)) \\ & \subset \cap_{y\in B^c} I_X(f^{-1}(\{y\}^c)) \\ & = I_X(\cap_{y\in B^c} f^{-1}(\{y\}^c)) \\ & = I_X(f^{-1}(\cap_{y\in B^c}\{y\}^c)) = I_X(f^{-1}(B)). \end{array}$$

□

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Yong Chan Kim
Professor of Gangneung-Wonju University
Research Area: Fuzzy topology, Fuzzy logic.

E-mail: yck@gwnu.ac.kr

Young Sun Kim
Professor of Pai Chai University
Research Area: Fuzzy topology, Fuzzy logic.

E-mail: yskim@pcu.ac.kr