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# WEAKLY DENSE IDEALS IN PRIVALOV SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. In this paper we study the structure of closed weakly dense ideals in Privalov spaces $N^p$ $(1 < p < \infty)$ of holomorphic functions on the disk $\mathbb{D} : |z| < 1$. The space $N^p$ with the topology given by Stoll's metric [21] becomes an $F$-algebra. N. Mochizuki [16] proved that a closed ideal in $N^p$ is a principal ideal generated by an inner function. Consequently, a closed subspace $E$ of $N^p$ is invariant under multiplication by $z$ if and only if it has the form $IN^p$ for some inner function $I$. We prove that if $\mathcal{M}$ is a closed ideal in $N^p$ that is dense in the weak topology of $N^p$, then $\mathcal{M}$ is generated by a singular inner function. On the other hand, if $S_\mu$ is a singular inner function whose associated singular measure $\mu$ has the modulus of continuity $O(t^{(p-1)/p})$, then we prove that the ideal $S_\mu N^p$ is weakly dense in $N^p$. Consequently, for such singular inner function $S_\mu$, the quotient space $N^p/S_\mu N^p$ is an $F$-space with trivial dual, and hence $N^p$ does not have the separation property.

## 1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\mathbb{T}$ be the boundary of $\mathbb{D}$. For $0 < q \le \infty$ we denote by $L^q(\mathbb{T})$ the *Lebesgue space* with respect to the normalized Lebesgue measure on $\mathbb{T}$. Given $1 < p < \infty$, the *Privalov class* $N^p$ consists of all holomorphic functions $f$ on $\mathbb{D}$ such that

$$\sup_{0\le r<1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \, \frac{d\theta}{2\pi} < +\infty.$$

These classes were introduced by I. I. Privalov in [17, p. 93], where $N^p$ is denoted as $A_q$ (with $q = p > 1$). The study on the spaces $N^p$ was continued by Stoll's work [21] in 1977 year. Further, topological and functional properties of these classes were investigated in [1], [5], [6], [12]-[14] and [16]; typically, the notation varied and Privalov was mentioned only in [12] and [14].

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Recall that the *Nevanlinna class* $N$ consists of all holomorphic functions $f$ on $\mathbb{D}$ such that

$$\sup_{0\le r<1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}<+\infty.$$

It is known that for each $f\in N$, the radial limit of $f$ defined as $f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$ exists for almost every $e^{i\theta}\in\mathbb{T}$ (see [7, p. 97]). The *Smirnov class* $N^+$ is the set of all functions $f\in N$ such that

$$\lim_{r\to 1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}=\int_0^{2\pi}\log^+|f(e^{i\theta})|\,\frac{d\theta}{2\pi}\,.$$

For $0<q<\infty$ we denote by $H^q$ the classical *Hardy space* on $\mathbb{D}$, and by $H^\infty$ the space of bounded holomorphic function on $\mathbb{D}$. It is known (see [16]) that

$$N^s\subset N^p\ (s>p),\qquad \bigcup_{q>0}H^q\subset\bigcap_{p>1}N^p,\quad\text{and}\quad \bigcup_{p>1}N^p\subset N^+\subset N,$$

and these inclusion relations are proper.

**Theorem A** ([21, Theorem 4.2])**.** *The Privalov space* $N^p$ $(1<p<\infty)$ (*with the notation* $(\log^+H)^p$ *in* [21]) *with the topology given by the metric* $\rho_p$ *defined as*

$$(1.1)\qquad \rho_p(f,g)=\Big(\int_0^{2\pi}\big(\log(1+|f(e^{i\theta})-g(e^{i\theta})|)\big)^p\,\frac{d\theta}{2\pi}\Big)^{1/p},\quad f,g\in N^p,$$

*is an $F$-algebra, i.e., a complete metrizable topological vector space in which multiplication is continuous.*

It is well known [2, p. 26] that every function $f\in N^+$ admits a unique factorization of the form

$$(1.2)\qquad f(z)=B(z)S_\mu(z)F(z),\quad z\in\mathbb{D},$$

where $B$ is the *Blaschke product* with respect to zeros $\{z_n\}\subset\mathbb{D}$ of $f$ (the set $\{z_n\}$ may be finite), $S_\mu$ is a *singular inner function*, $F$ is an *outer function* for $N^+$, i.e.,

$$B(z)=z^m\prod_{n=1}^{\infty}\frac{|z_n|}{z_n}\,\frac{z_n-z}{1-\bar z_n z}\,,\quad z\in\mathbb{D},$$

with $\sum_{n=1}^\infty\big(1-|z_n|\big)<\infty$, $m$ a nonnegative integer,

$$S_\mu(z)=\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right)$$

with positive singular measure $d\mu$, and

$$F(z)=\lambda\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|f\left(e^{it}\right)\right|\,dt\right),$$

where $|\lambda|=1$ and $\log|f(e^{i\theta})|\in L^1(\mathbb{T})$.

Recall that a function $I$ of the form

$$I(z) = B(z)S_\mu(z), \quad z \in \mathbb{D},$$

is called an *inner function*, and $I$ is a bounded holomorphic function whose boundary values $f(e^{i\theta})$ have modulus 1 for almost every $e^{i\theta} \in \mathbb{T}$.

The inner-outer factorization theorem for the classes $N^p$ is given by Privalov [17] as follows.

**Theorem B** ([17, pp. 98–100]; also see [6]). *A function $f \in N^+$ factorized by* (1.2) *belongs to $N^p$ if and only if* $\log^+ |F(e^{i\theta})| \in L^p(\mathbb{T})$.

A closed subspace $E$ of $N^p$ is called *invariant* if $zf \in E$ whenever $f \in E$. Here $z$ denotes the identity function on $\mathbb{D}$. By using a result of Mochizuki [16, Theorem 4], in Section 2 we exhibit the close relationship between inner functions and invariant subspaces of $N^p$. We prove (Theorem 2.3) that a closed subspace $E$ of $N^p$ is invariant if and only if it has the form $IN^p$ for some inner function $I$. Recall that a related result in more general form was proved by Matsugu [12, Theorem 1] using a classical result about invariant subspaces in $L^2(\mathbb{T})$ (cf. Remark 3).

For $0 < q < \infty$, in Section 3 we consider the space $F^q$ consisting of those functions $f$ holomorphic on $\mathbb{D}$ for which

$$\lim_{r \to 1} (1-r)^{1/q} \log^+ \left( \max_{|z| \le r} |f(z)| \right) = 0.$$

Stoll [21, Theorem 3.2] proved that the space $F^q$ with the topology given by the family of seminorms $\{\| \cdot \|_{q,c}\}_{c>0}$ defined for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^q$$

as

$$\|f\|_{q,c} = \sum_{n=0}^{\infty} |a_n| \exp\left(-cn^{1/(q+1)}\right) < \infty,$$

is a countably normed *Fréchet algebra*. By a result of Eoff [5, Theorem 4.2], $F^p$ is the *Fréchet envelope* of $N^p$, and hence $F^p$ and $N^p$ have the same topological duals. Thus, the family of seminorms $\{\| \cdot \|_{p,c}\}_{c>0}$ induces on $N^p$ the strongest locally convex (metrizable) topology that is weaker than the metric $\rho_p$ topology.

In Section 4 we characterize the weak closure $[E]_w$ of a vector subspace $E$ of $N^p$, that is, the closure with respect to *the weak topology* defined on $N^p$ in the usual way. The non locally convex phenomenon of interest here is the existence of proper, closed subspaces of $N^p$ that are dense in the weak topology. We show that for any inner function $I$, the weak closure of an invariant subspace $IN^p$ of $N^p$ is again an invariant subspace. Hence, $[IN^p]_w = I_w N^p$ for some inner function $I_w$. We shall say that $I$ is a *weak outer function* in $N^p$ if $IN^p$ is weakly dense in $N^p$; that is, if $I_w \equiv 1$. We prove (Corollary 4.7) that an inner function

$I$ is weak outer in $N^p$ if and only if the space $IN^p$ is dense in $F^p$, that is, if and only if the space $IF^p$ is dense in $F^p$.

A closed subspace $E$ of $N^p$ will be said to have the *separation property* if each $f \in N^p$ that is not in $E$ can be separated from $E$ by a continuous linear functional on $N^p$; that is, if there exists a continuous linear functional $\varphi$ on $N^p$ such that $\varphi(E) = 0$ but $\varphi(f) \neq 0$. We will say that $E$ has the *Hahn-Banach property* if each continuous linear functional on $E$ can be extended to a continuous linear functional on the whole space $N^p$. In Section 4 (Theorem 4.10) we prove that if $I$ is any Blaschke product, then $IN^p$ has the separation property. In the case if $I$ is a finite Blaschke product, we prove that $IN^p$ has the Hahn-Banach property.

It was proved in [4, Theorem 14] that if $S$ is a nontrivial singular inner function whose associated singular measure has the modulus of continuity $O\left(t \log \frac{1}{t}\right)$, then $I$ is a weak outer function in any space $H^q$ with $0 < q < 1$. In particular, it is not known whether an inner function can be weak outer for some values of $0 < p < 1$, but not for others. In our main result given in Section 5 (Theorem 5.5), for any fixed $1 < p < \infty$ we present a large class of positive singular measures $\mu$, depending on $p$, such that associated singular inner functions are weak outer functions in the Privalov space $N^p$. More precisely, Theorem 5.5 asserts that if $S_\mu$ is a non-trivial singular inner function with the associated measure $\mu$ whose modulus of continuity $\omega_\mu(t) = O(t^{(p-1)/p})$, then the ideal $S_\mu N^p$ is weakly dense in $N^p$. Consequently, such a singular inner function $S_\mu$ is a weak outer function in $N^p$ and the quotient space $N^p/S_\mu N^p$ is an $F$-space with trivial dual (Corollaries 5.6 and 5.7).

Recall that it was shown in [19, Proposition 4] that a closed ideal in $N^+$ is weakly dense if and only if it is generated by a singular inner function $S_\mu$ with $\mu$ a continuous singular measure.

*Remark* 1. The above notions and definitions are motivated by the famous Beurling's theorem (see [9, Ch. 7, p. 99], [8, Lecture II]) and results from [4] related to the linear space structure of the Hardy space $H^q$ with $0 < q < 1$. Beurling's invariant subspace theorem to $H^q$, tells that there is a one-to-one correspondence between inner functions and invariant subspaces; each invariant subspace of $H^q$ being of the form $IH^q$, where $I$ is an inner function. Beurling's theorem can also be viewed as a result on approximation. In this formulation it states that the polynomial multiples of an $H^q$ function form a dense subset of $H^q$ if and only if that function is outer (see [2, Section 7.3]).

In [4] Duren, Romberg, and Shields added a new dimension to Beurling's theorem by proving that when $0 < q < 1$ some inner functions (not identically 1) give rise to weakly dense invariant subspaces of $H^q$. In view of the approximation-theoretic formulation of Beurling's theorem, Duren, Romberg, and Shields called such inner functions *weakly outer*.

## 2. Invariant subspaces of $N^p$

Denote by $\mathcal{P}$ the space of all polynomials. Let $X$ be a topological vector space of holomorphic functions on $\mathbb{D}$ so that $H^\infty \subset X$ and convergence in $X$ implies uniform convergence on compact subsets of $\mathbb{D}$. Suppose that $1 \in X$ and that $f \in X$ implies $Pf \in X$ for every polynomial $P \in \mathcal{P}$. If $f \in X$, then $\mathrm{cl}(\mathcal{P}f)$ denotes the closure of $\mathcal{P}f := \{Pf : P \in \mathcal{P}\}$; so $\mathrm{cl}(\mathcal{P}f)$ is the smallest invariant (under multiplication by $z$) closed subspace containing $f$. Recall that an *invariant subspace* of $X$ is defined as a closed subspace $E$ of $X$ such that $zf \in E$ whenever $f \in E$. A function $f \in X$ is said to be *cyclic* in $X$ if $\mathrm{cl}(\mathcal{P}f) = X$. This is equivalent to the fact that there exists a sequence $\{P_n\}_n$ of polynomials such that $P_n f \to 1$ as $n \to \infty$ in the topology of $X$. Thus, a function $f \in X$ is cyclic if it generates $X$ as an invariant subspace, that is, the smallest invariant subspace of $X$ containing the function is the whole space.

By Beurling's theorem mentioned previously, a function in $H^q$ $(0 < q < \infty)$ is cyclic if and only if it is outer. Using the following result of Mochizuki, we can easily obtain the analogous characterization of cyclic functions in $N^p$.

**Theorem 2.1** ([16, Theorem 4]). *Let $\mathcal{M}$ be a closed ideal in $N^p$ which is not identically 0. Then there is a unique (modulo constants) inner function $I$ such that $\mathcal{M} = IN^p$.*

**Lemma 2.2.** *A closed subspace $E$ of $N^p$ is invariant if and only if it is an ideal.*

*Proof.* The proof is routine, by using the fact that $N^p$ is a topological algebra in which polynomials are dense (see [13, the proof of the assertion 2.3]), and hence may be omitted. □

As an immediate consequence of Theorem 2.1 and Lemma 2.2, we obtain the following $N^p$-analogue of Beurling's theorem on invariant subspaces of the Hardy space.

**Theorem 2.3.** *A closed subspace $E$ of $N^p$ is invariant if and only if it has the form $IN^p$ for some inner function $I$.*

*Remark* 2. By [19, Theorem 2] and Lemma 2.2 with $N^+$ instead of $N^p$, it follows that Theorem 2.3 is also true for the Smirnov class $N^+$.

Theorem 2.3 shows that there is a one-to-one correspondence between inner functions and invariant subspaces of $N^p$; so each invariant subspace of $N^p$ being of the form $IN^p$, where $I$ is an inner function.

The approximative version of Theorem 2.3 is given as follows.

**Theorem 2.4** ([13, the assertion 2.3 on p. 99]). *Let $f \in N^p$ with the factorization $f = BSF = IF$ given by Theorem B, and let $\mathcal{P}$ denote the space of all polynomials over $\mathbb{C}$. Then the set $BSN^p = IN^p = \{Ig : g \in N^p\}$ becomes the closure of $\mathcal{P}f := \{Pf : P \in \mathcal{P}\}$ in $N^p$ with respect to the metric topology $\rho_p$.*

Clearly, Theorem 2.4 can be formulated in terms of cyclic functions in $N^p$ as follows.

**Theorem 2.5.** *A function $f \in N^p$ is cyclic in $N^p$ if and only if $f$ is an outer function.*

*Remark* 3. In [12] Matsugu characterized invariant subspaces of the space $(\log^+)^p(\mathbb{T})$ $(1 \le p < \infty)$ consisting of all measurable functions $f$ on the circle $\mathbb{T}$ such that

$$\|f\| := \int_0^{2\pi} \left( (\log(1 + |f(e^{i\theta})|))^p \, \frac{d\theta}{2\pi} \right)^{1/p} < +\infty.$$

As noticed in [12], it can be easily shown that $(\log^+)^p(\mathbb{T})$ is an $F$-space with respect to the metric $\tilde{\rho}_p(f,g) = \|f - g\|$, $f, g \in (\log^+)^p(\mathbb{T})$. Furthermore, it follows by the Riesz Uniqueness Theorem that the Privalov space $N^p$ can be identified with the space $N_*^p$ of all boundary functions of $N^p$, i.e.,

$$N_*^p = \{f^* : \, f \in N^p\}.$$

Then $N_*^p (\equiv N^p)$ is a closed subspace of $(\log^+)^p(\mathbb{T})$ (cf. [23, Theorem 1] for $p = 1$ and [21, Theorem 4.2] for $p > 1$). By using the classical result about invariant subspaces in $L^2(\mathbb{T})$ ([8, Theorems 2 and 3]; [12, Theorem A]), Matsugu [12, Theorems 1 and 2] described the set of all invariant subspaces of $(\log^+)^p(\mathbb{T})$. These results immediately yield our Theorem 2.3 (extended for $p = 1$ with $N^1 \equiv N^+$).

## 3. Fréchet envelope and topological dual of $N^p$

In connection with the spaces $N^p$ $(1 < p < \infty)$, Stoll in [21] also studied the spaces $F^q$ $(0 < q < \infty)$ with the notation $F_{1/q}$ in [21], consisting of those functions $f$ holomorphic on $\mathbb{D}$ for which

$$\lim_{r \to 1} (1 - r)^{1/q} \log^+ M_\infty(r, f) = 0,$$

where

$$M_\infty(r, f) = \max_{|z| \le r} |f(z)|.$$

Here, as always in the sequel, we will need some Stoll's results concerning the spaces $F^q$ only with $1 < q < \infty$, and hence we will assume that $q = p > 1$ be any fixed number.

**Theorem 3.1** ([21, Theorem 2.2])**.** *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a holomorphic function on $\mathbb{D}$. Then the following statements are equivalent.*

(a) $f \in F^p$.
(b) *There exists a sequence $\{c_n\}_n$ of positive real numbers with $c_n \to 0$ such that*

$$|a_n| \le \exp\left( c_n n^{1/(p+1)} \right), \quad n = 0, 1, 2, \ldots.$$

(c) *For any* $c > 0$,

$$\|f\|_{p,c} := \sum_{n=0}^{\infty} |a_n| \exp\left(-cn^{1/(p+1)}\right) < \infty. \tag{3.1}$$

Note that in view of Theorem 3.1 ((a)⇔(c)), it is defined by (3.1) the family of seminorms $\{\|\cdot\|_{p,c}\}_{c>0}$ on $F^p$.

Moreover, Stoll defined the family of seminorms $\{|||\cdot|||_{p,c}\}_{c>0}$ on $F^p$ given as

$$|||f|||_{p,c} = \int_0^1 \exp\left(-c(1-r)^{-1/p}\right) M_p(r,f)\,dr, \quad f \in F^p, \tag{3.2}$$

where

$$M_p(r,f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \,\frac{d\theta}{2\pi}\right)^{1/p}.$$

By [21, Proposition 3.1], $\{\|\cdot\|_{p,c}\}_{c>0}$ and $\{|||\cdot|||_{p,c}\}_{c>0}$ are equivalent families of seminorms. More precisely, Stoll proved the following result.

**Theorem 3.2** ([21, Proposition 3.1])**.** *For each* $c > 0$*, there is a constant* $A$ *depending only on* $p$ *and* $c$*, such that*

$$|||f|||_{p,c} \le \|f\|_{p,c_1} \quad \textit{and} \quad \|f\|_{p,c} \le A|||f|||_{p,c_2},$$

*with* $c_1 = c^{p/(p+1)}$ *and* $c_2 = \left(\frac{c}{12}\right)^{p/(p+1)}$.

Recall that a locally convex $F$-space is called a *Fréchet space*, and a *Fréchet algebra* is a Fréchet space that is an algebra in which multiplication is continuous.

**Theorem 3.3** ([21, Theorem 3.2])**.** *The space* $F^p$ *with the topology given by the family of seminorms* $\{\|\cdot\|_{p,c}\}_{c>0}$*, or* $\{|||\cdot|||_{p,c}\}_{c>0}$*, is a countably normed Fréchet algebra. Moreover,*

$$\|fg\|_{p,c} \le \|f\|_{p,c'}\|g\|_{p,c'} \quad \textit{for all} \quad f,g \in F^p,$$

*where* $c' = c \cdot 2^{-p/(p+1)}$*. Furthermore, if* $f \in F^p$*, then* $f_r \to f$ *as* $r \to 1$ *in the topology of* $F^p$ *where* $f_r(z) = f(rz)$ *with* $z \in \mathbb{D}$ *and* $0 < r < 1$*.*

For our purposes the most important connection between spaces $N^p$ and $F^p$ is given by the following result.

**Theorem 3.4** ([21, Theorem 4.3])**.** *For any fixed* $p > 1$ *the following assertions hold.*

(a) $N^p$ *is a dense subspace of* $F^p$*.*
(b) *The topology on* $F^p$ *defined by the family of seminorms* (3.1) *or* (3.2) *is weaker than the topology on* $N^p$ *given by the metric* $\rho_p$ *defined by* (1.1).

(c) *For each $q > p$ there exists a function $f_q \in N^p$ such that*

$$\limsup_{r\to 1}(1-r)^{1/q}\log^+ M_\infty(r,f_q) > 0,$$

*i.e., $N^p$ is not contained in $F^q$ for none $q > p$.*

*Remark* 4. Recall that the spaces $F^p$ have also been studied independently by Zayed ([25] and [26]); many of the results in [24] parallel those of Stoll in [21], albeit in a more general setting. For $p = 1$, the space $F_1$ has been denoted by $F^+$ and has been studied by Yanagihara in [24] and [23]. It was shown in [24] and [23] that $F^+$ is actually the containing Fréchet space for $N^+$, i.e., $N^+$ with the initial topology embeds densely into $F^+$, under the natural inclusion, and $F^+$ and $N^+$ have the same topological duals.

Observe that the space $F^p$ topologised by the family of seminorms $\{\|\cdot\|_{p,c}\}_{c>0}$ given by (3.1) is metrizable by the metric $d_p$ defined as

$$d_p(f,g)=\sum_{n=1}^{\infty}2^{-n}\frac{\|f-g\|_{p,1/n^{p/(p+1)}}}{1+\|f-g\|_{p,1/n^{p/(p+1)}}}\,,\quad f,g\in F^p. \tag{3.3}$$

The following result describes the topological dual of the space $(F^p, d_p)$.

**Theorem 3.5** ([21, Theorem 3.3]). *If $\gamma$ is a continuous linear functional on $F^p$, then there exists a sequence $\{\gamma_n\}_n$ of complex numbers with $\gamma_n = O\big(\exp\big(-cn^{1/(p+1)}\big)\big)$ for some $c > 0$, such that*

$$\gamma(f)=\sum_{n=0}^{\infty}a_n\gamma_n, \tag{3.4}$$

*where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p$, with convergence being absolute. Conversely, if $\{\gamma_n\}$ is a sequence of complex numbers for which*

$$\gamma_n = O\left(\exp\left(-cn^{1/(p+1)}\right)\right),$$

*then* (3.4) *defines a continuous linear functional on $F^p$.*

Let us recall that if $X = (X, \tau)$ is an $F$-space whose topological dual (the set of all continuous linear functionals on $X$) $X^*$ separates the points of $X$, then its *Fréchet envelope* $\hat{X}$ is defined to be the completion of the space $(X, \tau^c)$, where $\tau^c$ is the strongest locally convex (necessarily metrizable) topology on $X$ that is weaker than $\tau$. In fact, it is known that $\tau^c$ is equal to the *Mackey topology* of the dual pair $(X, X^*)$, i.e., to the unique maximal locally convex topology on $X$ for which $X$ still has dual space $X^*$ (see [20, Theorem 1]). For each metrizable locally convex topology $\tau$ on $X$, $(X, \tau)$ is a *Mackey space*, i.e., $\tau$ coincides with the Mackey topology of the dual pair $(X, X^*)$ (see [10, Corollary 22.3, p. 210]).

Eoff ([5, the proof of Theorem 4.2]) showed that the topology of $F^p$, $p > 1$, (resp., $F^1 = F^+$) is stronger than that of the Fréchet envelope of $N^p$ (resp., $N^+$). As an immediate consequence of this result, we obtain the following statements.

**Theorem 3.6** ([5, Theorem 4.2, the case $p > 1$])**.** *For each $p > 1$, $F^p$ is the Fréchet envelope of $N^p$.*

**Theorem 3.7** ([14, Theorem 2])**.** *The spaces $N^p$ and $F^p$ have the same dual spaces in the sense that every continuous linear functional on $F^p$ (given by (3.4)) restricts to one on $N^p$, and every continuous linear functional on $N^p$ extends continuously to one on $F^p$.*

*Remark* 5. Theorem 3.7 is proved in [14] directly, by using the characterization of multipliers from $N^q$ into $H^\infty$ ([14, Theorem 1]). The dual space of the Smirnov class $N^+$ is completely described by Yanagihara in [23] and McCarthy in [15].

The following result establishes the fact that the dual of $N^p$ contains many elements.

**Theorem 3.8.** *For any fixed $\xi \in \mathbb{D}$ and all $k = 0, 1, 2 \ldots$, the functional $\delta_\xi^{(k)}$ defined as $\delta_\xi^{(k)}(f) = f^{(k)}(\xi)$, $f \in N^p$, is a continuous linear functional on $N^p$. In particular, for $k = 0$ every point evaluation $\delta_\xi$ defined as $\delta_\xi(f) = f(\xi)$, $f \in N^p$, is a multiplicative continuous linear functional on $N^p$.*

*Proof.* It is easy to verify that for a fixed $\xi \in \mathbb{D}$, the sequence $\{\gamma_n\}_n := \{\xi^n\}_n$ satisfies the condition from Theorem 3.5 for any $p > 1$, and hence it generates by (3.4) the continuous linear functional $\gamma$ such that $\gamma(f) = f(\xi) = \delta_\xi(f)$ for $f \in F^p$. Obviously, $\delta_\xi$ is multiplicative.

Similarly, for fixed $\xi \in \mathbb{D}$ and $k \in \mathbb{N}$, the sequence $\{(\xi^n)^{(k)}\}_n$ with terms

$$(\xi^n)^{(k)} = \begin{cases} n(n-1)\cdots(n-k+1)\xi^{n-k} & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

also satisfies the growth estimate from Theorem 3.5 for any $p > 1$, and thus by Theorem 3.7, every functional $\delta_\xi^{(k)} = f^{(k)}(\xi)$ is a continuous linear functional on $N^p$. □

## 4. Invariant subspaces and linear functionals on $N^p$

Recall that all the notions, statements and their proofs in this section are analogous to those for the spaces $H^q$ with $0 < q < 1$ given in [4, Section 6].

By Theorem 2.3, each invariant subspace of $N^p$ being of the form $IN^p$, where $I$ is an inner function. Thus there is a one-to-one correspondence between inner functions and invariant subspaces of $N^p$. Since $N^p$ is not locally convex, it is possible that some closed ideals are weakly dense.

Recall that we may introduce *the weak topology* on $N^p$ in the usual way. The basic weak neighborhoods of zero are defined by

$$V(\varphi_1, \ldots, \varphi_n; \varepsilon) = \{f \in N^p : |\varphi_i(f)| < \varepsilon,\, i = 1, \ldots, n\},$$

where $\varepsilon > 0$ and $n \in \mathbb{N}$ are arbitrary, and $\varphi_1, \ldots, \varphi_n$ are arbitrary continuous linear functionals on $N^p$.

For a subset $E$ of $N^p$, denote by $[E]$ the closure of $E$ in $N^p$, by $[E]_w$ the closure of $E$ in the weak topology of $N^p$, and by $[E]_{F^p}$ the closure of $E$ in $F^p$ with respect to the topology of $F^p$ given by the family of seminorms (3.1) or (3.2). Denote by $(N^p)^*$ the topological dual of $N^p$, i.e., the set of all continuous linear functional on $N^p$ with respect to the metric topology $\rho_p$. It is easy to check that a weakly closed subset of $N^p$ is also closed in $N^p$ with respect to the metric topology $\rho_p$; that is, the weak topology is weaker than the initial $N^p$ topology. The weak topology is always locally convex. Clearly, by Theorem 3.8, $N^p$ has the *point separation property.* This means that for every function $f \in N^p$ which is not identically equal to zero, there is a non-trivial continuous linear functional $\varphi$ on $N^p$ such that $\varphi(f) \neq 0$, and hence the weak topology is also Hausdorff.

For vector subspaces of $N^p$ there is another description of the weak closure that we shall use (see [4, p. 54]). The weak topology of $N^p$ is locally convex, and hence by [10, p. 154, Corollary 17.3], a linear functional on $N^p$ is weakly continuous if and only if it is continuous with respect to the metric topology $d_p$ given by (3.3). Thus $[E]_w$ consists of all those functions $f \in N^p$ which cannot be separated from $E$ by a linear functional in $(N^p)^*$. Namely, by [10, p. 154, 17.1], in any locally convex topological vector space a convex set $E$ is closed if and only if it is weakly closed, or equivalently, if and only if each point not in it can be separated from it by a linear functional. In the case when $E$ is a vector subspace, this becomes: $f \in [E]_w$ if and only if $\phi(f) = 0$ for every functional $\phi \in E^\perp$, where $E^\perp$ is the set of all $\phi \in (N^p)^*$ which vanish on $E$. Thus we have the following result.

**Lemma 4.1.** *A closed subspace of $N^p$ has the separation property if and only if it is weakly closed.*

**Lemma 4.2.** *The weak closure of any invariant subspace $IN^p$ of $N^p$ is again an invariant subspace.*

*Proof.* We follow the proof of Lemma 7 in [4, pp. 54–55]. Suppose that $f \in [IN^p]_w$, where $I$ is an inner function. We must prove that a function $zf(z)$ is in the weak closure of $IN^p$. According to the argument preceding Lemma 4.1, this is equivalent to the fact that $\phi(zf) = 0$ for each $\phi \in (IN^p)^\perp$. Now define the linear functional $\phi_1$ on $N^p$ as

$$\phi_1(f) = \phi(zf), \quad f \in N^p.$$

Since $\phi_1(Ig) = \phi(Izg) = 0$ for each $g \in N^p$, we see that $\phi_1 \in (IN^p)^\perp$. From this and the assumption $f \in [IN^p]_w$ it follows that $\phi_1(f) = 0$, and hence $\phi(zf) = 0$. This concludes the proof. □

**Theorem 4.3.** *For any inner function $I$ there is a unique* (*modulo constants*) *inner function $I_w$ such that*

$$[IN^p]_w = I_w N^p.$$

*Furthermore, $J = I/I_w$ is again an inner function.*

*Proof.* By Lemma 4.2, $[IN^p]_w$ is an invariant subspace of $IN^p$ for any inner function $I$. Thus by Theorem 2.3, there is an inner function $I_w$ for which

$$[IN^p]_w = I_w N^p. \tag{4.1}$$

The uniqueness (modulo constants) of a function $I_w$ follows immediately from the uniqueness of the factorization of $N^p$ functions (Theorem B). Finally, since the weak topology is weaker than the metric topology $\rho_p$ on $N^p$, we conclude that $IN^p = [IN^p] \subseteq [IN^p]_w$. Therefore, it follows by (4.1) that $IN^p \subseteq I_w N^p$, and hence $J = I/I_w$ is an inner function. This completes the proof. □

**Lemma 4.4.** *If $E$ is a vector subspace of $N^p$, then*

$$[E]_w = [E]_{F^p} \cap N^p. \tag{4.2}$$

*Furthermore, $[E] \subseteq [E]_w$ and $[E]_w$ is closed in $N^p$.*

*Proof.* As noticed above, $[E]_w$ consists of all those functions $f \in N^p$ that cannot be separated from $E$ by a linear functional on the dual space $(N^p)^*$ of $N^p$. On the other hand, by Theorem 3.7, the spaces $N^p$ and $F^p$ have the same dual spaces, and hence $[E]_{F^p} \cap N^p$ consists of all those $f \in N^p$ that cannot be separated from it by a linear functional. Therefore, $[E]_w = [E]_{F^p} \cap N^p$. Further, $[E] \subseteq [E]_w$ is immediate from the fact that the weak topology is weaker than the metric topology $\rho_p$ on $N^p$. Finally, it remains to show that $[E]_w$ is closed in $N^p$. Suppose $\{f_n\}$ is a sequence in $[E]_w$ such that $f_n \to f$ in $N^p$ as $n \to \infty$ for some $f \in N^p$. We must show that $f \in [E]_w$. As $\varphi(f_n) \to \varphi(f)$ as $n \to \infty$ for each $\varphi \in (N^p)^*$, this means that $f_n \to f$ weakly in $N^p$ as $n \to \infty$. Hence, $f \in [[E]_w]_w = [E]_w$, as desired. □

**Corollary 4.5.** *If $\mathcal{M}$ is an ideal in $N^p$, then both $[\mathcal{M}]$ and $[\mathcal{M}]_w$ are also ideals in $N^p$.*

*Proof.* Clearly, since the multiplication is continuous in $N^p$, if $\mathcal{M}$ is an ideal in $N^p$, so is $[\mathcal{M}]$. Moreover, since the multiplication is also continuous in $F^p$ and by Theorem 3.4(a), $N^p$ is a dense subspace of $F^p$, it follows that $[\mathcal{M}]_{F^p}$ is an ideal in $F^p$. This fact together with (4.2) yields that $[\mathcal{M}]_w$ is an ideal in $N^p$. □

Although the weak topology of $N^p$ is not metrizable, the following corollary shows that, for vector subspaces of $N^p$ the weak closure can be formed by adjoining limits of sequences.

**Corollary 4.6.** *If $E$ is a vector subspace of $N^p$, then each function in the weak closure of $E$ is the weak limit of a sequence of elements of $E$.*

*Proof.* Assume $f \in [E]_w$. By Lemma 4.4, $f \in [E]_{F^p}$ and so there is a sequence $\{f_n\}_n$ in $E$ such that $f_n \to f$ in the topology of $F^p$. Hence $f_n \to f$ weakly in $F^p$, which is the same thing as $f_n \to f$ weakly in $N^p$. □

**Corollary 4.7.** *Let $I$ be an inner function. The following three statements are equivalent.*

(i) $[IN^p]_w = N^p$.
(ii) *$IN^p$ is dense in $F^p$.*
(iii) *$IF^p$ is dense in $F^p$.*

*Proof.* (i)⇔(ii) follows immediately from (4.2) of Lemma 4.4 and the fact that $N^p$ is a dense subspace of $F^p$.

(ii)⇒(iii) is obvious, in view of the fact that $IN^p \subset IF^p$.

(iii)⇒(ii). If $f \in F^p$ is arbitrary, then by (iii) there is a sequence $\{f_n\}_n$ in $F^p$ such that $If_n \to f$ in $F^p$. Since $N^p$ is a dense subspace of $F^p$, there exists a sequence $\{g_n\}_n$ in $N^p$ such that $f_n - g_n \to 0$ in $F^p$. From this fact and the continuity of multiplication in $F^p$, it follows that $If_n - Ig_n \to 0$ in $F^p$, and hence $Ig_n \to f$ in $F^p$. Thus $f$ belongs to $[IN^p]_{F^p}$, and so (ii) is satisfied. □

If $T$ is a continuous linear operator on $N^p$, then the *adjoint operator* $T^*$ on $(N^p)^*$ is defined by the usual formula

$$(T^*\varphi)(f) = \varphi(Tf), \quad \varphi \in (N^p)^*, f \in N^p.$$

**Lemma 4.8.** *Every continuous linear operator $T$ on $N^p$ is weakly continuous.*

*Proof.* Assume $f_\alpha \to f$ weakly, that is, $\psi(f_\alpha) \to \psi(f)$ for all $\psi \in (N^p)^*$. Then we have

$$\psi(Tf_\alpha) = (T^*\psi)(f_\alpha) \to (T^*\psi)(f) = \psi(Tf),$$

whence it follows that $Tf_\alpha \to Tf$ weakly, as desired. □

**Lemma 4.9.** *Let $I$ and $J$ be arbitrary inner functions. Then*

$$[I_w JN^p] \subseteq [IJN^p]_w\,,$$

*where $I_w$ is the inner function such that $[IN^p]_w = I_w N^p$.*

*Proof.* For a fixed function $g \in N^p$, define the linear operator $T_g$ on $N^p$ as

$$T_g(f) = Jgf, \quad f \in N^p.$$

Since the multiplication in $N^p$ is continuous, it follows that $T_g$ is a continuous linear operator, and hence by Lemma 4.8, $T_g$ is weakly continuous. Therefore, if we choose a sequence $\{f_n\}_n$ in $N^p$ such that $If_n \to I_w$ weakly, then $T_g(If_n) \to T_g(I_w)$ weakly, i.e., $IJ(gf_n) \to JI_w g$ weakly. Hence, $JI_w g \in [IJN^p]_w$, which implies $I_w JN^p \subseteq [IJN^p]_w$. Therefore, in view of the fact that by Lemma 4.4, $[[IJN^p]_w] = [IJN^p]_w$, we obtain

$$[I_w JN^p] \subseteq [[IJN^p]_w] = [IJN^p]_w\,.$$

This yields a desired inclusion. □

Recall that a closed subspace $E$ of $N^p$ will be said to have the *separation property* if each function $f \in N^p$ that is not in $E$ can be separated from $E$ by a continuous linear functional on $N^p$; that is, if there exists a continuous linear functional $\varphi$ on $N^p$ such that $\varphi(E) = 0$ but $\varphi(f) \neq 0$. We will say that $E$ has the *Hahn-Banach property* if each continuous linear functional on $E$ can be extended to a continuous linear functional on the whole space $N^p$ (we do not require that the norm of the functional be preserved in the extension). We are now ready to state the following result.

**Theorem 4.10.** *Let $I$ be an inner function. Then the following assertions hold.*

(i) *If $I$ is a finite Blaschke product, then $IN^p$ has both the separation property and the Hahn-Banach property.*
(ii) *If $I$ is any Blaschke product, then $IN^p$ has the separation property.*
(iii) *If the space $IN^p$ does not have the separation property and if $J$ is any inner function, then also $IJN^p$ does not have the separation property.*

*Proof.* (i) If $I$ is a finite Blaschke product, then the space $IN^p$ has a finite co-dimension. Hence, the assertion (i) follows immediately from the fact that in any topological vector space a closed subspace of finite co-dimension has both the separation and the Hahn-Banach properties.

(ii) If $I$ is a Blaschke product and if $h \notin IN^p$, then either $h$ does not vanish at one of the zeros of $I$ or $h$ has a zero of lower multiplicity. In the first case, let $\xi \in \mathbb{D}$ be a zero of $I$ such that $h(\xi) \neq 0$. For such a $\xi$ consider the point evaluation $\delta_\xi$ on $N^p$ defined as $\delta_\xi(f) = f(\xi)$, $f \in N^p$. Then by Theorem 3.8, $\delta_\xi$ is a continuous linear functional on $N^p$. Obviously, $\delta_\xi(g) = 0$ for all $g \in IN^p$ but $\delta_\xi(h) = h(\xi) \neq 0$, and hence $\delta_\xi$ is a desired separating linear functional. In the second case, suppose that $\xi \in \mathbb{D}$ is a zero of $I$ of the multiplicity $k \geq 1$, and hence of the multiplicity less than $k$ for $h$, that is, $h^{(k)}(\xi) \neq 0$. Now consider a linear functional on $N^p$ by $\delta_\xi^{(k)}(f) = f^{(k)}(\xi)$, $f \in N^p$. By Theorem 3.8, $\delta_\xi^{(k)}$ is also a continuous linear functional on $N^p$. Clearly, $\delta_\xi^{(k)}(g) = 0$ for all $g \in IN^p$ while $\delta_\xi^{(k)}(h) = h^{(k)}(\xi) \neq 0$, and thus $\delta_\xi^{(k)}$ is a desired separating linear functional.

(iii) Suppose that $IJN^p$ has the separation property for some inner function $J$, and so by Lemma 4.1,

$$[IJN^p]_w = IJN^p. \tag{4.3}$$

By the assumption, the space $IN^p$ does not have the separation property, and hence by Lemma 4.1, it is not weakly closed. This means that $I \neq I_w$ (modulo constants), where $I_w$ is an inner function associated to $I$ (see Theorem 4.3) such that

$$[IN^p]_w = I_w N^p.$$

Since $IN^p$ is a weakly dense subspace of $I_w N^p$, by (4.3) and Lemma 4.9, we have

$$IJN^p = [IJN^p]_w \supseteq [I_w JN^p]_w \supseteq I_w JN^p \supseteq IJN^p.$$

From the above inclusions we see that must be $I_w JN^p = IJN^p$. Thus, $I_w J$ is in $IJN^p$, and hence there is an inner function $I_1$ such that $I_w J = IJI_1$. It follows that $I_w = II_1$, and hence $I_w/I = I_1$ is an inner function. On the other hand, by Theorem 4.3, $I/I_w$ is also an inner function. Therefore, we infer that

$I = I_w$ (modulo constants). This contradiction yields that $IJN^p$ does not have the separation property. This completes the proof of the theorem. □

## 5. Weakly dense ideals in $N^p$

Beurling's invariant subspaces theorem holds for each Hardy space $H^q$, $0 < q < \infty$, the Smirnov class $N^+$ (see Remark 2), and for all Privalov spaces $N^p$ with $1 < p < \infty$ (Theorem 2.3). This means that if $E$ is a closed subspace of one of these spaces, denoting by $X$, and if $E$ is invariant under multiplication by $z$, then $E = IX$ for some $X$-inner function $I$.

Since the spaces $N^p$ are not locally convex, it is possible that some closed ideals are weakly dense (dense in the weak topology of $N^p$). In this section we give a construction of such ideals for any $p > 1$.

Recall that an inner function $I$ is called a weak outer function in $N^p$ if $IN^p$ is closed in the weak topology of $N^p$; that is, if $[IN^p]_w = N^p$. This means that $I_w = 1$, where $I_w$ is an inner function as in Theorem 4.3 such that

$$[IN^p]_w = I_w N^p.$$

In other words, an inner function $I$ is a weak outer function in $N^p$ if and only if the principal ideal $IN^p$ is weakly dense. The main result of this paper (Theorem 5.5) gives a large class of positive singular measures $\mu$, depending on $p$, such that associated singular inner functions are weak outer functions in the space $N^p$.

By Theorem 3.2, $\{\|\cdot\|_{p,c}\}_{c>0}$ and $\{|||\cdot|||_{p,c}\}_{c>0}$ defined by (3.1) and (3.2), respectively, are equivalent families of seminorms. For simplicity, in this section we write $\|\cdot\|_c$ instead of $\|\cdot\|_{p,c}$.

For every $c > 0$, we define the function $\|\cdot\|_c^{\sim}$ on $F^p$ by

$$\|f\|_c^{\sim} = \left(\frac{1}{\pi}\iint_{\mathbb{D}} |f(re^{i\theta})|^2 \exp\left(-\frac{c}{(1-r)^{1/p}}\right) r dr d\theta\right)^{1/2}, \quad f \in F^p. \tag{5.1}$$

It follows easily by Minkowski's inequality that $\|\cdot\|_c^{\sim}$ satisfies the triangle inequality for any $c > 0$, and hence $\|\cdot\|_c^{\sim}$ is a norm on $F^p$.

**Lemma 5.1.** *Let $p > 1$ and $c > 0$ be any fixed. Then there exist positive constants $A$ and $B$, depending only on $p$ and $c$, such that*

$$\|f\|_c^{\sim} \le A\|f\|_{c_1}, \quad f \in F^p, \tag{5.2}$$

*and*

$$\|f\|_c \le B\|f\|_{c_2}^{\sim}, \quad f \in F^p, \tag{5.3}$$

*where $c_1 = \frac{c^{p/(p+1)}}{2}$ and $c_2 = \left(\frac{c^{p+1}}{3\cdot 6^{p+1}}\right)^{1/p}$. Hence, the families $\{\|\cdot\|_c\}_{c>0}$ and $\{\|\cdot\|_c^{\sim}\}_{c>0}$ induce the same topology on $F^p$.*

*Proof.* Suppose that $f \in F^p$, $f \not\equiv 0$, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$. Given any constant $c > 0$ put $c_2 = \left(\frac{c^{p+1}}{3\cdot 6^{p+1}}\right)^{1/p}$. Then by [21, the first inequality on p. 146], there is $m \in \mathbb{N}$ so that for all $n \geq m$ holds

$$\int_0^1 r^n \exp\left(-\frac{c_2}{(1-r)^{1/p}}\right) dr \geq \exp\left(-6(c_2)^{p/(p+1)} n^{1/(p+1)}\right). \tag{5.4}$$

Assuming that $L \leq 1$ is a positive constant such that

$$\int_0^1 r^{2n+1} \exp\left(-\frac{c_2}{(1-r)^{1/p}}\right) dr \geq L \exp\left(-6(c_2)^{p/(p+1)} (3n)^{1/(p+1)}\right)$$

for all $n = 0, 1, \ldots, [m/2] - 1$, then by (5.1) and (5.4) for each $c_2 > 0$ we have

$$\begin{aligned}
\left(\|f\|_{c_2}^{\sim}\right)^2 &= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 f(re^{i\theta})\overline{f(re^{i\theta})} \exp\left(-\frac{c_2}{(1-r)^{1/p}}\right) r dr d\theta \\
&= \frac{2\pi}{\pi}\sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} \exp\left(-\frac{c_2}{(1-r)^{1/p}}\right) dr \\
&= 2\left(\sum_{n=0}^{\left[\frac{m}{2}\right]-1} + \sum_{n=\left[\frac{m}{2}\right]}^{\infty}\right) \\
&\geq 2\left(L \sum_{n=0}^{\left[\frac{m}{2}\right]-1} |a_n|^2 \exp\left(-6(c_2)^{p/(p+1)}(3n)^{1/(p+1)}\right)\right. \\
&\qquad \left. + \sum_{n=\left[\frac{m}{2}\right]}^{\infty} |a_n|^2 \exp\left(-6(c_2)^{p/(p+1)}(2n+1)^{1/(p+1)}\right)\right) \\
&\geq 2L\left(\sum_{n=0}^{\left[\frac{m}{2}\right]-1} |a_n|^2 \exp\left(-6(c_2)^{p/(p+1)}(3n)^{1/(p+1)}\right)\right. \\
&\qquad \left. + \sum_{n=\left[\frac{m}{2}\right]}^{\infty} |a_n|^2 \exp\left(-6(c_2)^{p/(p+1)}(3n)^{1/(p+1)}\right)\right) \\
&= 2L\sum_{n=0}^{\infty} |a_n|^2 \exp\left(-6(c_2)^{p/(p+1)}(3n)^{1/(p+1)}\right).
\end{aligned} \tag{5.5}$$

By Cauchy-Schwarz inequality, for positive numbers $x_n, y_n$, with $n = 1, 2, \ldots, k$, holds

$$\left(\sum_{n=1}^{k} x_n y_n\right)^2 = \left(\sum_{n=1}^{k} (x_n\sqrt{y_n})\sqrt{y_n}\right)^2 \leq \left(\sum_{n=1}^{k} x_n^2 y_n\right)\left(\sum_{n=1}^{k} y_n\right).$$

The above inequality with $x_n = |a_n|$ and $y_n = \exp\left(-cn^{1/(p+1)}\right)$ as $k \to \infty$ yields

$$
\begin{aligned}
(\|f\|_c)^2 &= \left(\sum_{n=0}^{\infty} |a_n| \exp\left(-cn^{1/(p+1)}\right)\right)^2 \\
&\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \exp\left(-cn^{1/(p+1)}\right)\right)\left(\sum_{n=0}^{\infty} \exp\left(-cn^{1/(p+1)}\right)\right) \\
&= K \sum_{n=0}^{\infty} |a_n|^2 \exp\left(-cn^{1/(p+1)}\right) \\
&= K \sum_{n=0}^{\infty} |a_n|^2 \exp\left(-6(c_2)^{p/(p+1)}(3n)^{1/(p+1)}\right),
\end{aligned}
\tag{5.6}
$$

where $0 < K < +\infty$. The inequalities (5.5) and (5.6) immediately yield

$$\left(\|f\|_{c_2}^{\sim}\right)^2 \geq \frac{2L}{K} \left(\|f\|_c\right)^2.$$

On the other hand, we have

$$
\begin{aligned}
(\|f\|_c^{\sim})^2 &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(re^{i\theta})\overline{f(re^{i\theta})} \exp\left(-\frac{c}{(1-r)^{1/p}}\right) r dr d\theta \\
&= 2 \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} \exp\left(-\frac{c}{(1-r)^{1/p}}\right) dr.
\end{aligned}
$$

Since by [21, the first inequality on p. 145 with $\beta = 1/p$], for $n = 0, 1, 2, \dots,$

$$r^n \exp\left(-\frac{c}{(1-r)^{1/p}}\right) \leq \exp\left(-c^{p/(p+1)} n^{1/(p+1)}\right), \quad 0 < r < 1,$$

it follows that

$$
\begin{aligned}
(\|f\|_c^{\sim})^2 &\leq 2 \left(\sum_{n=0}^{\infty} |a_n| \exp\left(-\frac{c^{p/(p+1)}}{2} n^{1/(p+1)}\right)\right)^2 \\
&= 2 \left(\|f\|_{c_1}\right)^2,
\end{aligned}
$$

with $c_1 = \left(c^{p/(p+1)}\right)/2$. By setting $A = \sqrt{2}$, the above inequality yields (5.2). This concludes the proof. □

Recall that the *modulus of continuity* $\omega_\mu$ of a finite Borel measure $\mu$ on the unit circle $\mathbb{T}$ is defined by

$$\omega_\mu(t) = \sup_{|L| \leq t} \mu(L) \quad (t > 0),$$

where the supremum is taken on every subarcs $L$ of $\mathbb{T}$ whose normalized Lebesgue measure (lenght) $|L| \leq t$.

Observe that from the condition $\omega_\mu(t) = O(t)$ it follows that the measure $\mu$ is *absolutely continuous* with respect to the normalized Lebesgue measure $|\cdot|$

on $\mathbb{T}$. Furthermore, it is known that there are positive singular measures with prescribed modulus of continuity (of higher order than $O(t)$); for example, with the modulus of continuity $\omega_\mu(t) = O(t \log \frac{1}{t})$. It is pointed out in [4] that such a measure can be constructed as the Lebesgue function over a Cantor set with variable ratio of dissection. Another example is based on Riesz products (see [3]) that we use here in order to obtain the following result.

**Lemma 5.2.** *For any positive number* $\alpha$ *such that* $0 < \alpha < 1$ *there exists a positive singular Borel measure* $\mu$ *on* $\mathbb{T}$ *with the modulus of continuity*

$$\omega_\mu(t) = O\left(t^\alpha\right) \quad (t > 0). \tag{5.7}$$

*Proof.* We will construct a continuous nondecreasing function $F$ on $[0, 2\pi]$ which generates measure $\mu$ such that

$$\mu([x, y]) = F(x) - F(y) \tag{5.8}$$

for any segment $[x, y]$ with $0 \leq y \leq x \leq 2\pi$, and $\omega_\mu(t)$ satisfies (5.7). Recall that the *modulus of continuity* $\omega(t; F)$ of a continuous complex-valued function $F$ defined on $[0, 2\pi]$ is given by

$$\omega(t; F) = \sup_{|x-y| \leq t} |F(x) - F(y)|.$$

Moreover, $F$ belongs to the *Lipschitz class* $\Lambda_\alpha$ with $0 < \alpha \leq 1$ if $\omega(t; F) = O(t^\alpha)$ as $t \to 0$.

Given $0 < \alpha < 1$, consider a constant sequence $\{a_j\}_j$ with $a_j = 1$ for all $j = 1, 2, \dots$, and a sequence $\{n_j\}_j$ with $n_j = q^j$ for all $j = 1, 2, \dots$, where $q > 3$ is a positive integer for which $q^{1-\alpha} \geq 2$. Now define a sequence $\{p_k(t)\}_k$ of trigonometric polynomials as a *Riesz product*

$$\begin{aligned} p_k(t) &= \prod_{j=1}^{k} (1 + a_j \cos n_j t) \\ &= 1 + \sum_{i=1}^{m_k} c_i \cos it, \quad k = 1, 2, \dots, \end{aligned}$$

with $m_k = n_1 + \cdots + n_k = \frac{q(1-q^k)}{1-q}$ and suitable coefficients $c_i$. Since $n_{j+1}/n_j = q > 3$ for all $j = 1, 2, \dots$, as noticed in [3, pp. 1264–1265], it follows by [18] and [27, p. 208] that

$$F(x) := \lim_{k \to \infty} \int_0^x p_k(t)\, dt$$

exists for all $x, 0 \leq x \leq 2\pi$, and $F$ is a continuous nondecreasing function on $[0, 2\pi]$. We say that the function $F$ is generated by a Riesz product. Further, we have

$$\prod_{j=1}^{k} (1 + |a_j|) = 2^k \leq q^{(1-\alpha)k} = (n_k)^{1-\alpha},$$

and hence sequences $\{a_j\}_j$ and $\{n_j\}_j$ satisfy the asymptotic condition (6) from [3, Theorem 1]. Then by the same theorem from [3], the function $F$ belongs to the Lipschitz class $\Lambda_\alpha$, and so for a measure $\mu$ defined by (5.8) we obtain

$$\begin{aligned}\omega_\mu(t) &= \omega(t;F)\\ &= \sup_{|x-y|\le t} |F(x)-F(y)|\\ &= O(t^\alpha).\end{aligned}$$

Finally, since $\sum_{j=1}^{\infty} a_j^2 = \infty$, by [3, Theorem A], we conclude that $F$ is a singular function. Consequently, $\mu$ is a singular Borel measure on $\mathbb{T}$ whose modulus of continuity satisfies (5.7). This completes the proof. □

We will need the following result in the proof of the main result.

**Lemma 5.3.** *Let*

$$S_\mu(z) = \exp\left(-\int_0^{2\pi} H(z,e^{it})\,d\mu(t)\right)$$

*be a singular inner function, where* $H(z,e^{it}) = (e^{it}+z)(e^{it}-z)^{-1}$, *and let* $\mu$ *be a positive singular Borel measure with the modulus of continuity* $\omega_\mu(t) = O\left(t^{(p-1)/p}\right)$. *Then*

$$|S_\mu(re^{i\theta})| \ge \exp\left(-\frac{C}{(1-r)^{1/p}}\right), \quad 0 \le r < 1, \tag{5.9}$$

*for some constant* $C > 0$.

*Proof.* Obviously,

$$-\log|S_\mu(re^{i\theta})| = \int_0^{2\pi} P(r,\theta-t)\,d\mu(t), \tag{5.10}$$

with the Poisson kernel

$$P(r,\theta-t) = \Re H(z,e^{it}) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,, \quad z = re^{i\theta}. \tag{5.11}$$

Since $\sin x \ge 2x/\pi$ for each $0 \le x \le \pi/2$, we have

$$\begin{aligned} 1-2r\cos(\theta-t)+r^2 &= (1-r)^2 + 4r\sin^2\frac{\theta-t}{2}\\ &\ge (1-r)^2 + \frac{4r}{\pi^2}(\theta-t)^2. \end{aligned} \tag{5.12}$$

As for $r \ge 1/17$, $\left(372r/\pi^2\right)(\theta-t)^2 \ge 2(\theta-t)^2$, and since $|\theta-t| \le 2\pi$, the inequality $91(1-r)^2 > 2(\theta-t)^2$ holds for $r < 1/17$, in both cases it is obvious that

$$2\left((1-r)^2 + (\theta-t)^2\right) \le 93\left((1-r)^2 + \frac{4r}{\pi^2}(\theta-t)^2\right). \tag{5.13}$$

Now from (5.11)-(5.13) we immediately obtain

$$
\begin{aligned}
P(r,\theta-t) &\leq \frac{2(1-r)}{1-2r\cos(\theta-t)+r^2} \\
&\leq \frac{93(1-r)}{(1-r)^2+(\theta-t)^2}\,, \quad z=re^{i\theta}.
\end{aligned}
\tag{5.14}
$$

If we put $\delta = 2\pi/n$, then by (5.10), (5.11) and (5.14) and the assumption $\omega_\mu(\delta) = O(\delta^{(p-1)/p})$ of the lemma, we obtain

$$
\begin{aligned}
-\log|S_\mu(re^{i\theta})| &\leq c\delta^{\frac{p-1}{p}} \sum_{k=0}^{n-1} \max_{k\delta\leq t\leq (k+1)\delta} \left\{ \frac{1-r}{(1-r)^2+(\theta-t)^2} \right\} \\
&\leq c\delta^{\frac{p-1}{p}} \left( \frac{1}{1-r} + \sum_{k=1}^{n-1} \frac{1-r}{(1-r)^2+k^2\delta^2} \right)
\end{aligned}
\tag{5.15}
$$

for a positive constant $c$. Then assuming $n\in\mathbb{N}$ such that $1/(1-r)\leq n < 1+1/(1-r)$ for such $n$, we have $2\pi(1-r)/(2-r) < \delta \leq 2\pi(1-r)$, and from the inequality (5.15) we obtain

$$
\begin{aligned}
-\log|S_\mu(re^{i\theta})| &\leq c\delta^{\frac{p-1}{p}} \left( \frac{1}{1-r} + \sum_{k=1}^{n-1} \frac{1}{1-r}\cdot\frac{1}{1+\pi^2k^2} \right) \\
&\leq c(2\pi)^{\frac{p-1}{p}}(1-r)^{\frac{p-1}{p}} \left( \frac{1}{1-r} + \frac{1}{(1-r)\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \right) \\
&= c(2\pi)^{\frac{p-1}{p}}(1-r)^{\frac{p-1}{p}} \cdot \frac{7}{6(1-r)} \\
&= C(1-r)^{-\frac{1}{p}}
\end{aligned}
$$

for a constant $C = \left(7c(2\pi)^{(p-1)/p}\right)/6$. Hence we have

$$
|S_\mu(re^{i\theta})| \geq \exp\left( -\frac{C}{(1-r)^{1/p}} \right).
$$

This is the desired estimate (5.9). □

**Lemma 5.4.** *For any $c>0$ denote by $[E]_c$ the closure of a subspace $E$ of $N^p$ in the normed space $(F^p, \|\cdot\|_c^{\sim})$. Then for arbitrary inner functions $I$ and $J$*

$$
[IJN^p]_c = [I[JN^p]_c]_c\,.
\tag{5.16}
$$

*In particular, if $[JN^p]_c = N^p$, then*

$$
[IJN^p]_c = [IN^p]_c\,.
$$

*Proof.* Since $JN^p \subseteq [JN^p]_c$, we have $[IJN^p]_c \subseteq [I[JN^p]_c]_c$. Obviously, for any function $h\in H^\infty$ with the norm $\|h\|_\infty = \max_{z\in\mathbb{D}}|h(z)|$ and each $f\in N^p$ holds

$$
\|hf\|_c^{\sim} \leq \|h\|_\infty \|f\|_c^{\sim}.
$$

Therefore, for each $f$ and $g$ in $N^p$ we obtain

$$\|IJg - If\|_c^{\sim} \leq \|I\|_\infty \|Jg - f\|_c^{\sim},$$

and hence $[I[JN^p]_c]_c \subseteq [IJN^p]_c$. This concludes the proof. $\square$

According to Theorem 2.5, the principal ideal in $N^p$ is dense in $N^p$ if and only if it is generated by an outer function. Although we have been unable to give a complete characterization of weakly dense ideals in $N^p$, the following theorem describes a large class of such ideals in $N^p$.

**Theorem 5.5.** *Let $\mathcal{M}$ be a closed ideal in $N^p$ that is weakly dense in $N^p$. Then $\mathcal{M}$ is a principal ideal generated by a singular inner function. Conversely, if $S_\mu$ is a non-trivial singular inner function with the associated measure $\mu$ whose modulus of continuity $\omega_\mu$ satisfies*

$$\omega_\mu(t) = O\left(t^{\frac{p-1}{p}}\right), \quad t > 0,$$

*then the ideal $S_\mu N^p$ is weakly dense in $N^p$.*

*Proof.* Let $\mathcal{M}$ be a closed ideal with respect to the metric topology $\rho_p$ of $N^p$ that is weakly dense in $N^p$. Then by Theorem 2.1, there is a unique (modulo constants) inner function $I$ such that $\mathcal{M} = IN^p$. Suppose that a function $I$ vanishes in $\mathbb{D}$, i.e., $I(z) = B(z)S_\mu(z)$, where $B$ is a nontrivial Blaschke factor of $I$, and $S_\mu$ is a singular inner factor of $I$. Let $\xi$ be an arbitrary zero of $B$. Since by Theorem 3.8, the evaluation functional $\delta_\xi$ defined as $\delta_\xi(f) = f(\xi)$, $f \in N^p$, is continuous, the assumption that $\mathcal{M}$ is weakly dense implies that there is a sequence $\{f_n\}_n \subset N^p$ such that $\delta_\xi(BS_\mu f_n) \to \delta_\xi(1)$ as $n \to \infty$. This means that $B(\xi)S_\mu(\xi)f_n(\xi) \to 1$ as $n \to \infty$, what is impossible in view of the fact that $B(\xi) = 0$. This contradiction shows that $I$ is a singular inner function.

Conversely, suppose that $S_\mu$ is a singular inner function with the associated measure $\mu$ whose modulus of continuity $\omega_\mu(t) = O(t^{(p-1)/p})$. By Theorem 3.3, $F^p$ is a Fréchet algebra in which $f_r \to f$ as $r \to 1$ for each $f \in N^p$ with $f_r(z) = f(rz)$, $z \in \mathbb{D}$. Since each function $f_r$ can be uniformly approximated on the closed disk $\bar{\mathbb{D}} : |z| \leq 1$ by partial sums of its Taylor expansion, it follows that the space $\mathcal{P}$ of polynomials is dense in $F^p$. Hence, the density of $S_\mu N^p$ in $F^p$ is equivalent to the fact that the set $\mathcal{P}S_\mu = \{PS_\mu : P \in \mathcal{P}\}$ is dense in $F^p$. Since by Lemma 5.1, families of seminorms $\{\|\cdot\|_c^{\sim}\}_{c>0}$ and $\{\|\cdot\|_c\}_{c>0}$ given by (5.1) and (3.1), respectively, define the same topology on $F^p$, it follows from (i)$\Leftrightarrow$(ii) of Corollary 4.7 that the ideal $S_\mu N^p$ is weakly dense in $N^p$ if and only if it is dense in the normed spaces $(F^p, \|\cdot\|_c^{\sim})_{c>0}$ for each $c > 0$.

By (5.9) of Lemma 5.3, there is a constant $C > 0$ such that for the minimum modulus $m(r) = \min_{|z|=r} |S_\mu(z)|$ of $S_\mu$ holds

$$m(r) \geq \exp\left(-\frac{C}{(1-r)^{1/p}}\right) \quad \text{for} \quad 0 \leq r < 1. \tag{5.17}$$

Denote by $p_n$ $n$'th *Cesàrov's sum* ($n$'th arithmetic mean of the partial sums of the Taylor expansion) of the function $1/S_\mu(z)$. Then $p_n(z) \to 1/S_\mu(z)$

uniformly on compact subsets of $\mathbb{D}$, and hence $p_n(z)S_\mu(z) \to 1$ for each $z \in \mathbb{D}$. By [11, Kap.1, Satz 1, p. 22] there holds

$$\max_{|z|=r} |p_n(z)| \le \max_{|z|=r} \frac{1}{|S_\mu(z)|}\,,$$

whence by (5.17) immediately follows

$$\max_{|z|=r} |p_n(z)| \le \frac{1}{m(r)} \le \exp\left(\frac{C}{(1-|z|)^{1/p}}\right) \text{ for } 0 \le |z| = r < 1. \tag{5.18}$$

Since $|S_\mu(z)| < 1$ for each $z \in \mathbb{D}$, combining (5.18) and the inequality $|1+v|^2 \le 2\left(1+|v|^2\right)$, for any positive constant $c$ and $0 \le |z| < 1$ we obtain

$$\begin{aligned} & |1-p_n(z)S_\mu(z)|^2 \exp\left(-\frac{c}{(1-|z|)^{1/p}}\right) \\ \le\ & 2|p_n(z)|^2 \exp\left(-\frac{c}{(1-|z|)^{1/p}}\right) + 2\exp\left(-\frac{c}{(1-|z|)^{1/p}}\right) \\ \le\ & 2\exp\left(-\frac{c-C}{2(1-|z|)^{1/p}}\right) + 2\exp\left(-\frac{c}{(1-|z|)^{1/p}}\right). \end{aligned}$$

It follows from the above inequality that for a constant $c$ such that $c \ge C$, a sequence $\{f_n\}_n$ of functions defined as

$$f_n(z) = |1-p_n(z)S_\mu(z)|^2 \exp\left(-\frac{c}{(1-|z|)^{1/p}}\right), \quad z \in \mathbb{D},\ n \in \mathbb{N},$$

is bounded by modulus by a function which is integrable on $\mathbb{D}$ with respect to the area normalized Lebesgue measure $rdrd\theta/\pi$. Since $p_n(z)S_\mu(z) \to 1$ as $n \to \infty$ for each $z \in \mathbb{D}$, it follows by the Dominated Convergence Theorem that for any fixed $c \ge C$ a sequence $\left\{ \left(\|p_nS_\mu - 1\|_c^\sim\right)^2 \right\}_n$ defined as

$$\begin{aligned} & \left(\|p_nS_\mu - 1\|_c^\sim\right)_n^2 \\ =\ & \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |1-p_n(re^{i\theta})S_\mu(re^{i\theta})|^2 \exp\left(-\frac{c}{(1-|re^{i\theta}|)^{1/p}}\right) rdrd\theta, \quad n \in \mathbb{N}, \end{aligned}$$

converges to zero as $n \to \infty$. Hence, the space $S_\mu\mathcal{P}$ is dense in $(F^p, \|\cdot\|_c^\sim)$ for each $c \ge C$.

Now suppose that $0 < c < C$. Let $n$ be a positive integer such that $1/n < c/C$. Then applying the above argument to the singular inner function $(S_\mu)^{1/n}$, we conclude that the space $(S_\mu)^{1/n}\mathcal{P}$ is dense in $(F^p, \|\cdot\|_c^\sim)$, i.e., according to the notation from Lemma 5.4, we have $[(S_\mu)^{1/n}N^p]_c = F^p$. Therefore, by (5.16) of Lemma 5.4, we have

$$\begin{aligned} [(S_\mu)^{2/n}N^p]_c &= \Big[(S_\mu)^{1/n}[(S_\mu)^{1/n}N^p]_c\Big]_c \\ &= [(S_\mu)^{1/n}F^p]_c \\ &= F^p. \end{aligned} \tag{5.19}$$

Hence, repeating the application of (5.19) $n-1$ times, we obtain $[S_\mu N^p]_c = F^p$. Therefore, $S_\mu N^p$ is dense in the space $(F^p, \{\|\cdot\|_c^{\sim}\})$ for each $c > 0$. This completes the proof. □

Recall that an inner function $I$ is said to be a *weak outer function* in $N^p$ if the set $IN^p := \{If : f \in N^p\}$ is dense in the weak topology of $N^p$. Then the second assertion of Theorem 5.5 can be formulated as follows.

**Corollary 5.6.** *Every non-trivial singular inner function $S_\mu$ with the associated measure $\mu$ whose modulus of continuity $\omega_\mu(t) = O(t^{(p-1)/p})$ is a weak outer function in $N^p$.*

For a closed subspace $E$ of $N^p$, and for any $f \in N^p$, let $\bar{f}$ denote the coset of $f + N^p$ in the quotient space $N^p/E$. Define

$$\|\bar{f}\| = \inf\left\{\|g\| : g \in \bar{f}\right\}, \quad f \in N^p,$$

where $\|g\| = \rho_p(g, 0)$. Then the function $\bar{\rho}_p$ defined as

$$\bar{\rho}_p(\bar{f}, \bar{g}) = \|\bar{f} - \bar{g}\|, \quad \bar{f}, \bar{g} \in N^p/E,$$

is an additively invariant metric that induces the quotient topology on $N^p/E$. It is well known that $N^p/E$ is an $F$-space with respect to the metric $\bar{\rho}_p$ (cf. [22, Theorem 12.3.5, p. 264]).

**Corollary 5.7.** *Let $S_\mu$ be a singular inner function described in Theorem* 5.5. *Then the following statements about the closed ideal $S_\mu N^p$ in $N^p$ hold.*

(i) *If $\phi$ is a continuous linear functional on $N^p$ which annihilates $S_\mu N^p$, then $\phi$ is the zero functional.*
(ii) *The quotient space $(N^p/S_\mu N^p, \bar{\rho}_p)$ is an $F$-space with the trivial dual.*
(iii) *The space $S_\mu N^p$ does not have the separation property, and hence does not have the Hahn-Banach property.*

*Proof.* Properties (i) and (ii) immediately follow from the fact that by Theorem 5.5, the ideal $S_\mu N^p$ is weakly dense in $N^p$ (cf. [4, Theorem 16, p. 59], where the analogous assertions are given for any topological vector space with enough continuous linear functionals to separate points). Since $[S_\mu N^p]_w = N^p \neq S_\mu N^p$, we see that $S_\mu N^p$ is not weakly closed, and by Lemma 4.1, $S_\mu N^p$ does not have the separation property. □

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