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TOEPLITZ SEQUENCES OF INTERMEDIATE COMPLEXITY

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Abstract. We present two constructions of Toeplitz sequences with an intermediate complexity function by using the generalized Oxtoby sequence. In the first one, we use the blocks from the infinite sequence, which has entropy dimension $\frac{1}{2}$. The second construction provides the Toeplitz sequences which have various entropy dimensions.

1. Introduction

Entropy is one of the basic tools for measuring the complexity of dynamical systems. Since the notion of entropy was introduced by Kolmogorov to ergodic theory, positive entropy systems are investigated extensively in many different directions, together with their applications. However, the study of zero entropy systems has been concentrated mainly around a few systems like irrational rotations, interval change maps and some constructive examples. Our goal is to find more examples that have zero entropy with different levels of complexity.

We construct infinite words and consider the dynamical systems which are generated by the orbits of them. Their topological entropies are obtained by using the complexity function of an infinite word [7]. We say that a dynamical system has intermediate complexity if its orbit has a subexponential but bigger than polynomial growth rate. It is clear that if the order of the complexity function of an infinite word is polynomial or subexponential, the topological entropy is zero. The notions of intermediate complexity and entropy dimension were introduced to help analyse the zero entropy systems. Entropy dimension classifies the different levels of intermediate complexity [8].

The Toeplitz sequences can be regarded as generalizations of periodic sequences [9]. Indeed, they are almost periodic sequences, which include Oxtoby's sequence. They have the following properties [11]: (1) The Toeplitz sequences yield minimal flows. (2) The orbit closure of a regular Toeplitz sequence is always uniquely ergodic. Most of non-regular Toeplitz sequences yield minimal

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flows which are not uniquely ergodic. Moreover, it is shown that many examples of non-regular Toeplitz flows have positive entropy [10]. It means that it has exponential complexity. There are some Toeplitz sequences that have the complexity function of polynomial order [3].

In the first example, we construct a non-regular Toeplitz sequence η_1 with density $0 < d < 1$ by using the building blocks taken from the infinite word [2], which has entropy dimension $\frac{1}{2}$. For the complexity function of η_1, we obtain $\lim_{n\to\infty} \frac{\log P_{\eta_1}(n)}{\sqrt{(1-d)n}} = 1$.

In the second example, we construct a regular Toeplitz sequence η_2 by using the binary Champernowne word [2], which has positive topological entropy. We prove that $\lim_{i\to\infty} \frac{\log P_{\eta_2}(p_i)}{\sqrt{p_i}} = 1$ and $\liminf_{n\to\infty} \frac{\log P_{\eta_2}(n)}{\sqrt{n}} = 1$ for any length n. Also, we generalize the second construction method to construct the Toeplitz sequence which has the lower entropy dimension α for any given α between 0 and 1.

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2. Preliminaries

In this section, we summarize some basic definitions of a complexity function and a Toeplitz sequence; we refer the reader to [7], [8] and [11] for more details.

We let $X = \Sigma^{\mathbb{Z}}$, where Σ is a finite alphabet. An element of X is expressed as $x = (x(n))$. We say that w is a subword of x of length n if $w = x(i)x(i+1)\cdots x(i+n-1)$ for some i. The *complexity function* of x is defined as follows:

$$P_x(n) = \sharp\{w | w = x(i)x(i+1)\cdots x(i+n-1), i \in \mathbb{Z}\}$$

i.e., the number of subwords of length n in x. Obviously, the complexity function is a nondecreasing sequence satisfying $1 \le P_x(n) \le (\sharp\Sigma)^n$ and $P_x(m+n) \le P_x(m)P_x(n)$ for all m and n in $\mathbb{N}$. With respect to complexity function, we classify the infinite words into three categories: polynomial case, intermediate case and exponential case.

The *topological entropy* of an infinite word x is defined as follows:

$$h(x) = \limsup_{n\to\infty} \frac{1}{n} \log P_x(n).$$

It is clear that the topological entropy is zero if the order of the complexity function is polynomial or intermediate. To classify the infinite words of intermediate complexity, the notion of entropy dimension in the following was introduced in [8].

Definition 1. The upper entropy dimension of x is

$$\overline{D}(x) = \inf\{\beta \ge 0 : \limsup_{n\to\infty} \frac{1}{n^\beta} \log P_x(n) = 0\}.$$

Similarly, the lower entropy dimension of x is

$$\underline{D}(x) = \inf\{\beta \geq 0 : \liminf_{n\to\infty} \frac{1}{n^\beta} \log P_x(n) = 0\}.$$

When $\overline{D}(x) = \underline{D}(x) = \alpha$, we say x has *entropy dimension* α and denote it by $D(x)$.

Let $x \in X$. For $p \in \mathbb{N}$ and $\sigma \in \Sigma$, we set

$$\begin{aligned}
\mathrm{Per}_p(x,\sigma) &= \{n \in \mathbb{Z} : x(n+kp) = \sigma \text{ for all } k \in \mathbb{Z}\},\\
\mathrm{Per}_p(x) &= \bigcup_{\sigma\in\Sigma} \mathrm{Per}_p(x,\sigma),\\
\mathrm{Aper}(x) &= \mathbb{Z}\backslash(\bigcup_{p\in\mathbb{N}} \mathrm{Per}_p(x)).
\end{aligned}$$

We define a *p-skeleton* of x to be the sequence obtained from x by replacing $x(n)$ by a "hole" for all $n \notin \mathrm{Per}_p(x)$. Thus, the p-skeleton of x is the part of x which is periodic with period p.

Definition 2. The sequence $\eta \in X$ is a Toeplitz sequence if $\mathrm{Aper}(\eta) = \emptyset$.

From this definition, the periodic sequences are Toeplitz sequences. Every Toeplitz sequence is almost periodic, that is, each block in η is contained in the p-skeleton for some p. In this paper, we consider the Toeplitz sequences that are not periodic.

Definition 3. A period structure for a non-periodic Toeplitz sequence η is an increasing sequence $(p_i)_{i\in\mathbb{N}}$ of natural numbers satisfying

(i) For all i, the p_i-skeleton of η is not periodic with any smaller period,
(ii) $p_i | p_{i+1}$,
(iii) $\bigcup_{i=1}^\infty \mathrm{Per}_{p_i}(\eta) = \mathbb{Z}$.

The first condition (i) means that every q satisfying $\mathrm{Per}_{p_i}(\eta,\sigma) = \mathrm{Per}_{p_i}(\eta,\sigma) -q$ for all $\sigma \in \Sigma$ implies $p_i|q$. Every non-periodic Toeplitz sequence has a period structure. Given a periodic structure (p_i) for a Toeplitz sequence η, we define a *density* in $\mathbb{Z}$ given by

$$d_i = \frac{1}{p_i} \cdot \sharp\{n \in \mathbb{Z}/p_i\mathbb{Z} : n \in \mathrm{Per}_{p_i}(\eta)\} \quad \text{for each} \quad i \in \mathbb{N}.$$

Since $\mathrm{Per}_{p_i}(\eta) \subset \mathrm{Per}_{p_{i+1}}(\eta)$ and $p_i|p_{i+1}$, it is clear that $d_i < d_{i+1}$. We set $d = \lim_{i\mapsto\infty} d_i$.

Definition 4. The Toeplitz sequence η is said to be regular if $d = 1$.

When the density d of a Toeplitz sequence is less than 1, we call it a *non-regular* Toeplitz sequence. In the first example, we build a non-regular Toeplitz sequence in $\{0,1\}^{\mathbb{Z}}$. And in the second one, we obtain a regular Toeplitz sequence in $\{0,1\}^{\mathbb{Z}}$.

3. The first example

We will introduce an infinite sequence with intermediate growth rate. We refer the reader to [2] for more details.

Let $\mathbf{c} = c_0c_1c_2\cdots$ be the *binary Champernowne word*

$$\begin{aligned}\mathbf{c} &= 11011100101110111100010011010101111001101111101111\cdots\\ &= 1.10.11.100.101.110.111.1000.1001.1010.1011.1100.1101.1110.1111\cdots.\end{aligned}$$

It has the maximal complexity, that is, $P_{\mathbf{c}}(n) = 2^n$ for all n. Consider the family of finite words (q_k) defined as follows:

$$q_k = c_{k-1}c_{k-2}00c_{k-3}0000\cdots c_0 0^{2k-2}$$

so that $|q_k| = k^2$, and the infinite word $\mathbf{u} = q_1q_2q_3\cdots$ obtained by concatenating them in order,

$$\mathbf{u} = 1.1100.010010000.1000100001000000.1100000001000000100000000\cdots.$$

The following result is proven in [2]. By the note $\alpha \sim \beta$, we mean $\frac{\alpha}{\beta} = O(1)$.

Theorem 3.1. *The complexity function of the infinite word* $\mathbf{u}$ *defined above satisfies*

$$2^{\lceil\sqrt{n}\rceil} \le P_{\mathbf{u}}(n) \le n^2 2^{\lceil\sqrt{n}\rceil}$$

for all $n \ge 1$*, and consequently* $\log P_{\mathbf{u}}(n) \sim \sqrt{n}$.

Motivated by the first example in [2], we construct a Toeplitz sequence with intermediate growth by using the generalized Oxtoby's sequence [11].

We define $Q_l = \{w : w = \square_1\square_2 00\square_3 0000\cdots\square_l 0^{2l-2},\quad \square_i = 0 \text{ or } 1\}$. Hence the number of elements in Q_l is $2^l = 2^{\sqrt{|w|}}$. It is clear that when $l' < l$, any element in $Q_{l'}$ is a prefix of some elements in Q_l. Using all elements in Q_l, we make the Toeplitz sequence η_1 through the following steps.

Step 1: Take $p_1 = 6$, $\eta_1(n) = 0$ for $n \equiv -1 \mod p_1$ and $\eta_1(n) = 1$ for $n \equiv 0 \mod p_1$. We obtain p_1-skeleton at this step and $d_1 = \frac{2}{6}$.

Step 2: Let $J(1,k) = [kp_1 + 1, (k+1)p_1 - 1)$ for $k \in \mathbb{Z}$. $J(1,k)$ has cardinality $r_1 = p_1 - 2 = 4$. Let β_{r_1} denote the number of words in Q_2, so $\beta_{r_1} = 4$. For $k = -1, 0, \ldots, \beta_{r_1} - 2$, we fill each $J(1,k)$ with a different element of Q_2. Choose $p_2 = p_1(\beta_{r_1} + 2^2\beta_{r_1})$. For $k' \equiv k \mod \frac{p_2}{p_1}$, $k \in [-1, \beta_{r_1} - 2]$, we fill $J(1,k')$ with the block that was used to fill $J(1,k)$. We obtain p_2-skeleton at this step and we note that $d_2 = \frac{56}{120}$.

Step (i+1): Let $J(i,k)$ be the set of $n \in [kp_i, (k+1)p_i)$ for which η_1 has not been defined up till the i^{th} step. If $\sharp J(i,k) = r_i$, we take the minimum value l such that $r_i \le |w|$ where $w \in Q_l$. Then we set $\beta_{r_i} = 2^{\sqrt{|w|}}$ for $w \in Q_l$. For $k = -1, 0, \ldots, \beta_{r_i} - 2$, we fill each $J(i,k)$ with a prefix (length of r_i) of w in Q_l. We choose $p_{i+1} = p_i(\beta_{r_i} + 2^{(i+1)}\beta_{r_i})$ and fill $J(i,k')$ in the same way as $J(i,k)$ for $k' \equiv k$

mod $\frac{p_{i+1}}{p_i}$, $k \in [-1, \beta_{r_i} - 2]$. We obtain p_{i+1}-skeleton at this step and $d_{i+1} = d_i + (1 - d_i)\frac{p_i \beta_{r_i}}{p_{i+1}}$.

We note that $\mathrm{Per}_{p_i}(\eta)$ is exactly the set of coordinates on which η is defined at the end of the i^{th}-step, and $J(i,k)$'s are filled with different words at different steps of the construction.

Remark. The Toeplitz sequence η_1 is not regular. We note that $d_1 = \frac{1}{3}$ and $d_{i+1} = d_i + (1 - d_i)\frac{p_i \beta_{r_i}}{p_{i+1}}$. Hence we have

$$1 - d_{i+1} = 1 - d_i - (1 - d_i)\frac{p_i \beta_{r_i}}{p_{i+1}} = (1 - d_1)\prod_{j=1}^{i}\left(1 - \frac{p_j \beta_{r_j}}{p_{j+1}}\right).$$

Since $p_{i+1} = p_i(\beta_{r_i} + 2^{(i+1)}\beta_{r_i})$, it is clear from the above formula that $0 < d < 1$.

Theorem 3.2. *The complexity function of the infinite word η_1 satisfies*

$$\lim_{n\to\infty} \frac{\log P_{\eta_1}(n)}{\sqrt{(1-d)n}} = 1.$$

Proof. Let v be the subword of the first p_{i+1}-block of η_1. At the $(i+1)^{th}$ step, the subword v consists of $(\beta_{r_i} + 2^{i+1}\beta_{r_i})$-many p_i-blocks. All coordinates of the p_i-block from the first to $(\beta_{r_i} - 1)^{th}$ in v and the last p_i-block of v are defined at the $(i+1)^{th}$ step. There are $p_i(1 - d_i)$-many undefined coordinates for each p_i-block from $(\beta_{r_i})^{th}$ to $(2^{i+1}\beta_{r_i} + \beta_{r_i} - 1)^{th}$ in v.

For any positive integer n, we find i which satisfies that $p_i \le n < p_{i+1}$. We can approximate $P_{\eta_1}(n)$ by using the p_{i+1}-skeleton of η_1, because the p_{i+1}-skeleton is the periodic part of η_1 of period p_{i+1}, and the rest is determined by the elements of Q_l in one of the subsequent steps.

For a fixed i, we compute $P_{\eta_1}(kp_i)$ for $1 \le k \le (2^{i+1} + 1)\beta_{r_i} - 1$. There are three cases.

Case 1-1: $n_1 = p_i$.

In v, we consider each p_i-block which has $(1 - p_i)d_i$-many undefined coordinates. It is easy to estimate the lower bound of $P_{\eta_1}(n_1)$ because its undefined portion is filled by the prefix of an element in Q_l, which is used at $(i+2)^{th}$ step. From Theorem 3.1, it follows that the number of blocks of length $(1 - d_i)p_i$ is at most $((1 - d_i)p_i)^2 2^{\lceil \sqrt{(1-d_i)p_i} \rceil}$. So we can estimate its upper bound. If we combine the above fact with the shifted names of the filled p_i-blocks in v, then we obtain

$$2^{\lceil \sqrt{(1-d_i)p_i} \rceil - 1} \le P_{\eta_1}(n_1) \le 2n_1((1 - d_i)p_i)^2 2^{\lceil \sqrt{(1-d_i)p_i} \rceil}.$$

Since $\lim_{i\to\infty} \frac{\sqrt{(1-d_i)p_i}}{\sqrt{(1-d)p_i}} = 1$, it follows that

$$\lim_{i\to\infty} \frac{\log P_{\eta_1}(p_i)}{\sqrt{(1-d)p_i}} = 1.$$

Case 1-2: $n_2 = kp_i$ where $2 \le k \le 2^{i+1}\beta_{r_i}$.

We consider the block of length n_2 which lies from $(\beta_{r_i})^{th}$ p_i-block to $(\beta_{r_i}+k-1)^{th}$ p_i-block in v. As in the case 1-1, since there are $(1-d_i)kp_i$ undefined coordinates in this block, we obtain the lower bound and the upper bound of $P_{\eta_1}(n_2)$ as follows:

$$2^{\lceil\sqrt{(1-d_i)kp_i}\rceil-1} \le P_{\eta_1}(n_2) \le 2n_2((1-d_i)kp_i)^2 2^{\lceil\sqrt{(1-d_i)kp_i}\rceil}.$$

Since $\lim_{i\to\infty}\frac{\sqrt{(1-d_i)kp_i}}{\sqrt{(1-d)kp_i}} = 1$, we have the conclusion.

Case 1-3: $n_3 = kp_i$ where $2^{i+1}\beta_{r_i} + 1 \le k \le (1+2^{i+1})\beta_{r_i} - 1$.

In this case, a subword of length n_3 always contains all the undefined coordinates in v whose length is $(1-d_i)2^{i+1}\beta_{r_i}p_i$. We obtain the approximation by considering all possible words which are inserted into the undefined part of v. From Theorem 3.1, we know that the number of possible inserted words is $((1-d_i)k_0p_i)^2 2^{\lceil\sqrt{(1-d_i)k_0p_i}\rceil}$ where $k_0 = 2^{i+1}\beta_{r_i}$, so we have the following result:

$$2^{\lceil\sqrt{(1-d_i)k_0p_i}\rceil-1} \le P_{\eta_1}(n_3) \le n_3((1-d_i)k_0p_i)^2 2^{\lceil\sqrt{(1-d_i)k_0p_i}\rceil}.$$

Since k can be expressed as $k = k_0 + m$ with $m < \beta_{r_i}$, it follows that $\lim_{i\to\infty}\frac{\sqrt{(1-d_i)k_0p_i}}{\sqrt{(1-d)kp_i}} = 1$. Therefore, the assertion is proved.

We approximated the complexity function of η_1 for the multiples of p_i in the previous cases. Now we consider the cases for $n = kp_i + hp_{i-1}$ where $k < 2^{i+1}\beta_{r_i}$ and $1 \le h < (1+2^i)\beta_{r_{i-1}} - 1$. If $k \ge 2^{i+1}\beta_{r_i}$, it is similar with the case 1-3. For a fixed k, the complexity function of $n = kp_i + hp_{i-1}$ is computed for the three cases of h.

Case 2-1: $n_4 = kp_i + p_{i-1}$.

The number of the undefined coordinates is at most $n_e^1 = (1-d_i)kp_i + (1-d_{i-1})p_{i-1}$ in the subword of length n_4. From the same argument in case 1-1, we obtain the result:

$$2^{\lceil\sqrt{n_e^1}\rceil-1} \le P_{\eta_1}(n_4) \le 2n_4(n_e^1)^2 2^{\lceil\sqrt{n_e^1}\rceil}$$

and so we have the conclusion because $\lim_{i\to\infty}\frac{\sqrt{(1-d_i)kp_i+(1-d_{i-1})p_{i-1}}}{\sqrt{(1-d)(kp_i+p_{i-1})}} = 1$.

Case 2-2: $n_5 = kp_i + hp_{i-1}$ where $1 \le h \le 2^i\beta_{r_{i-1}}$.

The number of the undefined coordinates is at most $n_e^2 = (1-d_i)kp_i + (1-d_{i-1})hp_{i-1}$ in the subword of length n_5. By the similar argument as in case 1-2, we obtain

$$2^{\lceil\sqrt{n_e^2}\rceil-1} \le P_{\eta_1}(n_5) \le 2n_5(n_e^2)^2 2^{\lceil\sqrt{n_e^2}\rceil}.$$

From the fact that $\lim_{i\to\infty}\frac{\sqrt{(1-d_i)kp_i+(1-d_{i-1})hp_{i-1}}}{\sqrt{(1-d)(kp_i+hp_{i-1})}} = 1$, we have assertion.

Case 2-3: $n_6 = kp_i + hp_{i-1}$, $2^i\beta_{r_{i-1}} + 1 \le h \le (1+2^i)\beta_{r_{i-1}} - 1$.
The number of the undefined coordinates in a word of length n_6 is at most $n_e^0 = (1-d_i)kp_i + (1-d_{i-1})h_0p_{i-1}$ where $h_0 = 2^i\beta_{r_{i-1}}$. Similar to the case 1-3, we have

$$2^{\lceil\sqrt{n_e^0}\rceil-1} \le P_{\eta_1}(n_6) \le n_6(n_e^0)^2 2^{\lceil\sqrt{n_e^0}\rceil}.$$

Therefore, $\lim_{n\to\infty}\frac{\log P_{\eta_1}(n_6)}{\sqrt{(1-d)n_6}} = 1$ is obtained since $h = h_0 + m$ with $m < \beta_{r_{i-1}}$ and $\lim_{i\to\infty}\frac{\sqrt{(1-d_i)kp_i+(1-d_{i-1})h_0p_{i-1}}}{\sqrt{(1-d)(kp_i+hp_{i-1})}} = 1$.

Let n be an arbitrary length. Then n can be expressed as $n = kp_i + hp_{i-1} + l$ where $0 < l < p_{i-1}$. Since

$$P_{\eta_1}(kp_i + hp_{i-1}) \le P_{\eta_1}(n) \le P_{\eta_1}(kp_i + (h+1)p_{i-1}),$$

the following inequalities hold:

$$\begin{aligned}\lim_{i\to\infty}\frac{\log P_{\eta_1}(kp_i + hp_{i-1})}{\sqrt{(1-d)(kp_i + hp_{i-1} + l)}} &\le \lim_{n\to\infty}\frac{\log P_{\eta_1}(n)}{\sqrt{(1-d)n}}\\ &\le \lim_{i\to\infty}\frac{\log P_{\eta_1}(kp_i + (h+1)p_{i-1})}{\sqrt{(1-d)(kp_i + hp_{i-1} + l)}}.\end{aligned}$$

Since $p_i = p_{i-1}\beta_{r_{i-1}} + 2^ip_{i-1}\beta_{r_{i-1}}$ and $l/p_{i-1} < 1$, we see that the values of the first and the third limit in the above inequalities are 1. Therefore, we obtain that $\lim_{n\to\infty}\frac{\log P_{\eta_1}(n)}{\sqrt{(1-d)n}} = 1$. □

Remark. We see from Theorem 3.2 that the entropy dimension of η_1 is $\frac{1}{2}$.

4. The second example

In Section 3, we constructed the regular Toeplitz sequence η_1 by using the building blocks from the infinite word $\mathbf{u}$ which has entropy dimension $\frac{1}{2}$. In this section, we construct another Toeplitz sequence which also has the similar properties with the first one by using the building blocks from the binary Champernowne word [4]. The binary Champernowne word is of the form as

$$\mathbf{c} = 1.10.11.100.101.110.111.1000.1001.1010.1011.1100.1101.1110.1111\cdots.$$

As in the first example, we construct a Toeplitz sequence η_2 in the following steps. In each step, if we need the blocks of some length l to be inserted, then we use all subwords of length l from the binary Champernowne word.

Step 1: Let $p_1 = 4$ and set $\eta_2(n) = 0$ for $n \equiv -1 \mod p_1$ and $\eta_2(n) = 1$ for $n \equiv 0 \mod p_1$. At this step we obtain p_1-skeleton with $d_1 = \frac{2}{4}$.

Step 2: Let $J(1,k) = [kp_1 + 1, (k+1)p_1 - 1)$ for $k \in \mathbb{Z}$. The number of coordinates of $J(1,k)$ is $r_1 = p_1 - 2 = 2$. Set $\beta_{r_1} = 2^{r_1}$. For $k = -1, 0, \ldots, \beta_{r_1} - 2$, we fill each $J(1,k)$ with a different subword of length r_1, which is taken from the binary Champernowne word. Choose $p_2 = (\beta_{r_1} + m_1)p_1$ where m_1 is taken as $\lceil m_1' \rceil$, which satisfies

$\sqrt{(\beta_{r_1}+m_1')p_1} = m_1'p_1(1-d_1)$. For $k' \equiv k \mod \frac{p_2}{p_1}$, $k \in [-1, \beta_{r_2}-2]$, we fill $J(1,k')$ with the blocks that were used to fill $J(1,k)$. We get p_2-skeleton at this step and $d_2 = \frac{22}{28}$.

Step (i+1): Let $J(i,k)$ be the set of $n \in [kp_i, (k+1)p_i)$ for which η_2 has not been defined after i^{th} step, and let r_i denote the number of coordinates of $J(i,k)$. Let $\beta_{r_i} = 2^{r_i}$. For $k = -1, 0, \ldots, \beta_{r_i}-2$, we fill each $J(i,k)$ with a different subword that is taken from the binary Champernowne word of length r_i. We then choose $p_{i+1} = (\beta_{r_i}+m_i)p_i$ where m_i is taken as $\lceil m_i' \rceil$ which satisfies $\sqrt{(\beta_{r_i}+m_i')p_i} = m_i'p_i(1-d_i)$. We fill $J(i,k')$ in the same way as $J(i,k)$ for $k' \equiv k \mod \frac{p_{i+1}}{p_i}$, $k \in [-1, \beta_{r_i}-2]$. At this step, we obtain p_{i+1}-skeleton with $d_{i+1} = d_i + (1-d_i)\frac{p_i\beta_{r_i}}{p_{i+1}}$.

Lemma 4.1. *The infinite sequence η_2 is well-defined.*

Proof. For each step, we define $p_{i+1} = (\beta_{r_i}+m_i)p_i$, where m_i is taken as $\lceil m_i' \rceil$ which satisfies $\sqrt{(\beta_{r_i}+m_i')p_i} = m_i'p_i(1-d_i)$. So it is enough to check the existence of m_i' in the $(i+1)^{th}$ step. If we solve the above equation, we obtain the following

$$m_i' = \frac{1+\sqrt{1+4p_i(1-d_i)^2\beta_{r_i}}}{2p_i(1-d_i)^2}.$$

By the induction, we have $p_i(1-d_i) = m_{i-1}p_{i-1}(1-d_{i-1}) \sim \sqrt{p_i}$. We see that $\beta_{r_i} = 2^{p_i(1-d_i)} \sim 2^{\sqrt{p_i}}$ and m_i' can be expressed as the following:

$$\begin{aligned} m_i' &= \frac{1+\sqrt{1+4p_i(1-d_i)^2\beta_{r_i}}}{2p_i(1-d_i)^2} \\ &= \frac{p_i+\sqrt{p_i^2+4p_i^3(1-d_i)^2\beta_{r_i}}}{2p_i^2(1-d_i)^2} \\ &\sim \frac{p_i+\sqrt{p_i^2+4p_i^2\beta_{r_i}}}{2p_i} = \frac{1}{2}+\sqrt{\frac{1}{4}+\beta_{r_i}} \\ &\sim \sqrt{\beta_{r_i}} \sim 2^{\sqrt{p_i}/2}. \end{aligned}$$

□

Remark. By Proposition 4.1 in [11], it is easy to show that η_2 is the regular Toeplitz sequence. We have the following

$$\sum_{i=1}^{\infty}\frac{p_i\beta_{r_i}}{p_{i+1}} = \sum_{i=1}^{\infty}\frac{p_i\beta_{r_i}}{p_i(\beta_{r_i}+m_i)} = \infty.$$

Theorem 4.2. *The complexity function of the infinite word η_2 satisfies*

$$\lim_{i\to\infty}\frac{\log P_{\eta_2}(p_i)}{\sqrt{p_i}} = 1$$

and

$$\liminf_{n\to\infty}\frac{\log P_{\eta_2}(n)}{\sqrt{n}} = 1.$$

Proof. The proof is similar to that in Theorem 3.2. Let v be the subword of the first p_{i+1}-block of η_2. At the $(i+1)^{th}$ step, the subword v consists of $(\beta_{r_i} + m_i)$-many p_i-blocks. All coordinates of p_i-block from the first to the $(\beta_{r_i} - 1)^{th}$ in v and the last p_i-block of v are filled at the $(i+1)^{th}$ step. There are $p_i(1 - d_i)$-many undefined coordinates for each p_i-block from $(\beta_{r_i})^{th}$ to $(\beta_{r_i} + m_i - 1)^{th}$ in v.

For any positive integer n, we choose p_i which satisfies that $p_i \le n < p_{i+1}$. We count the subwords of η_2 of length n by considering all building blocks which are inserted into the undefined coordinates. We note that $P_{\eta_2}(n)$ is easily approximated because we know the exact number of building blocks from the binary Champernown word.

First, we compute the complexity function of the length kp_i where $1 \le k \le \beta_{r_i} + m_i - 1$.

Case 1-1: $n_1 = p_i$.

We consider the p_i-block which contains $(1 - d_i)p_i$-many undefined coordinates in v. Its undefined part is filled by a subword of length $(1 - d_i)p_i$ from the binary Champernowne word. There are at least $2^{p_i(1-d_i)}$'s p_i-blocks. Combining it with the shifted names of the filled p_i-blocks in v, we obtain the following result:

$$2^{(1-d_i)p_i} \le P_{\eta_2}(p_i) \le 2p_i 2^{(1-d_i)p_i}.$$

Because $m_{i-1} = m'_{i-1} + \gamma$, where $0 < \gamma < 1$ and $m'_{i-1}(1 - d_{i-1})p_{i-1} = \sqrt{p_i}$, we have the following equation

$$\begin{aligned}\lim_{i\to\infty} \frac{(1 - d_i)p_i}{\sqrt{p_i}} &= \lim_{i\to\infty} \frac{m_{i-1}(1 - d_{i-1})p_{i-1}}{\sqrt{p_i}} \\ &= \lim_{i\to\infty} \frac{(m'_{i-1} + \gamma)(1 - d_{i-1})p_{i-1}}{\sqrt{p_i}} = 1.\end{aligned}$$

Therefore, we know that have $\lim_{i\to\infty} \frac{\log P_{\eta_2}(p_i)}{\sqrt{p_i}} = 1$.

Case 1-2: $n_2 = kp_i$ where $2 \le k \le m_i$.

We consider a block of length kp_i which lies between $(\beta_{r_i})^{th}$ p_i-block and $(\beta_{r_i} + m_i - 1)^{th}$ p_i-block. Then it has $(1 - d_i)kp_i$-many undefined coordinates. We obtain the estimation by similar argument as in case 1-1. If we combine the above fact with the shifted names of the filled p_i-blocks in v, then we obtain the following result:

$$2^{kp_i(1-d_i)} \le P_{\eta_2}(n_2) \le 2n_2 2^{kp_i(1-d_i)}.$$

Thus, we see that

$$\frac{\log P_{\eta_2}(n_2)}{\sqrt{n_2}} \ge \frac{kp_i(1 - d_i)}{\sqrt{kp_i}} \ge \frac{k\sqrt{p_i}}{\sqrt{kp_i}} \ge 1,$$

because $p_i(1 - d_i) \ge \sqrt{p_i}$.

Case 1-3: $n_3 = kp_i$ where $m_i + 1 \leq k \leq m_i + \beta_{r_i} - 1$.
A subword of length kp_i in v always contains $(1 - d_i)m_i p_i$-many undefined coordinates. We estimate $P_{\eta_2}(n_3)$ by considering all possible words which are inserted into the undefined coordinates of p_i-blocks in v. If we set $k_0 = m_i$, then we obtain the following result:

$$2^{k_0 p_i(1-d_i)} \leq P_{\eta_2}(n_3) \leq kp_i 2^{k_0 p_i(1-d_i)}.$$

By a similar argument with the case 1-2, we get the following estimation:

$$\frac{\log P_{\eta_2}(n_3)}{\sqrt{n_3}} \geq \frac{k_0 p_i(1-d_i)}{\sqrt{kp_i}} \geq \frac{\sqrt{p_{i+1}}}{\sqrt{p_{i+1}}} = 1,$$

because $k_0 p_i(1 - d_i) = m_i p_i(1 - d_i) \geq \sqrt{p_{i+1}}$ and $p_{i+1} > kp_i$.

In the previous cases, we approximated the complexity function of η_2 for the multiples of p_i. Now we consider the cases for $n = kp_i + hp_{i-1}$, where $k < m_i$ and $1 \leq h \leq m_{i-1} + \beta_{r_i - 1} - 1$. If $k \geq m_i$, it is similar to the case 1-3. For a fixed integer k, the complexity function of $n = kp_i + hp_{i-1}$ is computed for the various values of h in the following cases. We also consider the subword v, which consists of only the p_{i+1}-skeleton.

Case 2-1: $n_4 = kp_i + p_{i-1}$.
In the subword of length $n_4 = kp_i + p_{i-1}$ in v, the number of the undefined coordinates is at most $n_e^1 = (1 - d_i)kp_i + (1 - d_{i-1})p_{i-1}$. By the same argument as in the case 1-1, we see that

$$2^{n_e^1} \leq P_{\eta_2}(n_4) \leq 2n_4 2^{n_e^1}.$$

By the above inequality, we get

$$\begin{aligned} \frac{\log P_{\eta_2}(n_4)}{\sqrt{n_4}} &\geq \frac{(1-d_i)kp_i + (1-d_{i-1})p_{i-1}}{\sqrt{kp_i + p_{i-1}}} \\ &\geq \frac{k\sqrt{p_i} + \sqrt{p_{i-1}}}{\sqrt{kp_i + p_{i-1}}} \\ &\geq \frac{\sqrt{kp_i + p_{i-1}}}{\sqrt{kp_i + p_{i-1}}} = 1, \end{aligned}$$

since $(1 - d_i)p_i \geq \sqrt{p_i}$ and $(1 - d_{i-1})p_{i-1} \geq \sqrt{p_{i-1}}$.

Case 2-2: $n_5 = kp_i + hp_{i-1}$ where $2 \leq h \leq m_{i-1}$.
Similar to the above case, the number of the undefined coordinates is at most $n_e^2 = (1 - d_i)kp_i + (1 - d_{i-1})hp_{i-1}$. By using the argument of case 1-2, we obtain the result:

$$2^{n_e^2} \leq P_{\eta_2}(n_5) \leq 2n_5 2^{n_e^2}.$$

We have

$$\frac{\log P_{\eta_2}(n_5)}{\sqrt{n_5}} \geq \frac{(1-d_i)kp_i + (1-d_{i-1})hp_{i-1}}{\sqrt{kp_i + hp_{i-1}}} \geq \frac{k\sqrt{p_i} + h\sqrt{p_{i-1}}}{\sqrt{kp_i + hp_{i-1}}} \geq \frac{\sqrt{kp_i + hp_{i-1}}}{\sqrt{kp_i + hp_{i-1}}} = 1,$$

because $(1-d_i)p_i \geq \sqrt{p_i}$ and $(1-d_{i-1})p_{i-1} \geq \sqrt{p_{i-1}}$.

Case 2-3: $n_6 = kp_i + hp_{i-1}$ where $m_{i-1} + 1 \leq h \leq m_{i-1} + \beta_{r_{i-1}} - 1$.
In this case, the word of length $n_6 = kp_i + hp_{i-1}$ contains n_e^0-many undefined coordinates where $n_e^0 = (1-d_i)kp_i + (1-d_{i-1})h_0 p_{i-1}$ and $h_0 = m_{i-1}$. By using the argument in the case 1-3, we have the inequality:

$$2^{n_e^0} \leq P_{\eta_2}(n_6) \leq n_6 2^{n_e^0}.$$

Hence, we get

$$\frac{\log P_{\eta_2}(n_6)}{\sqrt{n_6}} \geq \frac{(1-d_i)kp_i + (1-d_{i-1})m_{i-1}p_{i-1}}{\sqrt{kp_i + hp_{i-1}}} \geq \frac{k\sqrt{p_i} + \sqrt{p_i}}{\sqrt{kp_i + p_i}} \geq \frac{\sqrt{kp_i + p_i}}{\sqrt{kp_i + p_i}} = 1,$$

because $(1-d_i)p_i \geq \sqrt{p_i}$ and $(1-d_{i-1})m_{i-1}p_{i-1} \geq \sqrt{p_i}$.

We have estimated the lower bound of the complexity function of the specified length such as $kp_i + hp_{i-1}$. Let n be any length with $p_i \leq n < p_{i+1}$. We express n as $n = kp_i + hp_{i-1} + l$ with $0 \leq l < p_{i-1}$. From the results in the above cases, we have $\lim_{i\to\infty} \frac{\log P_{\eta_2}(p_i)}{\sqrt{p_i}} = 1$ and

$$\lim_{n\to\infty} \frac{\log P_{\eta_2}(n)}{\sqrt{n}} \geq \lim_{i\to\infty} \frac{\log P_{\eta_2}(kp_i + hp_{i-1})}{\sqrt{kp_i + hp_{i-1} + l}} \geq 1.$$

Hence, for an arbitrary length n, we obtain the result

$$\liminf_{n\to\infty} \frac{\log P_{\eta_2}(n)}{\sqrt{n}} = 1$$

by using $\lim_{i\to\infty} \frac{p_{i-1}}{p_i} = 0$. □

Remark. From Theorem 4.2, we know that the lower entropy dimension of η_2 is $\frac{1}{2}$. The upper entropy dimension of η_2 is obtained from the following argument: if we use the case 1-2 in the proof of Theorem 4.2, we obtain the following equation

$$\lim_{i\to\infty} \frac{m_i\sqrt{p_i}}{(m_i p_i)^\gamma} = \infty \quad \text{for any} \quad \frac{1}{2} < \gamma < 1,$$

because $m_i \sim 2^{\sqrt{p_i}/2}$ by Lemma 4.1. Hence the upper entropy dimension of η_2 is 1.

By modifying the second construction, we can construct the Toeplitz sequences of various intermediate complexities. To construct a Toeplitz sequence, we choose $p_1 \geq 3$ and $0 < d_1 < 1$. If the integer m_i satisfies $((\beta_{r_i} + m_i)p_i)^\alpha = m_i p_i(1 - d_i)$ for a given $0 < \alpha < 1$, then we obtain a Toeplitz sequence η with the following properties.

Theorem 4.3. *The complexity function of the infinite word η satisfies*

$$\lim_{i\to\infty} \frac{\log P_\eta(p_i)}{{p_i}^\alpha} = 1$$

and

$$\liminf_{n\to\infty} \frac{\log P_\eta(n)}{n^\alpha} = 1.$$

Proof. The proof is similar to Theorem 4.2. □

Remark. The lower entropy dimension of the Toeplitz sequence η is α. Its upper entropy dimension is 1.

Example. For any given $\alpha = \frac{1}{n}$ where $n > 1$, we set $p_1 = 2^{1/\alpha}$ and $d_1 = \frac{2^n-2}{2^n}$. Then we can construct a Toeplitz sequence with lower entropy dimension α by using $((\beta_{r_i} + m_i)p_i)^\alpha = m_i p_i(1 - d_i)$.

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