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A REMARK ON H-CONTACT UNIT TANGENT SPHERE BUNDLES

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Abstract. We shall give some curvature conditions for the unit tangent sphere bundle of an $n(\geq 4)$-dimensional Riemannian manifold to be H-contact. Furthermore, we provide an example illustrating Main Theorem.

1. Introduction

Studying the relationships between the geometric structures of Riemannian manifolds and their respective unit tangent sphere bundles is one of interesting topics in Riemannian geometry. A unit vector field V on M determines a map between (M, g) and (T_1M, g'). If the Riemannian manifold (M, g) is compact and orientable, the energy of V is defined as the energy of the corresponding map:

$$E(V) = \frac{1}{2}\int_M |dV|^2 dv_g = \frac{m}{2}\text{vol}(M, g) + \frac{1}{2}\int_M |\nabla V|^2 dv_g,$$

where $m = \dim M$ [14]. V is said to be a *harmonic vector field* if it is a critical point for the energy functional E in the set of all unit vector fields of M [14].

Perrone defined an H-contact manifold as a contact metric manifold whose characteristic vector field ξ is harmonic, and proved that a contact metric manifold is an H-contact manifold if and only if the characteristic vector field ξ is an eigenvector of the Ricci operator [13]. Boeckx and Vanhecke [3] proved that the unit tangent sphere bundle of a 2-dimensional or 3-dimensional Riemannian manifold is H-contact if and only if the base manifold has constant sectional curvature. Calvaruso and Perrone [5] obtained the same result in the case of the $n(\geq 4)$-dimensional conformally flat manifold. The authors proved that the unit tangent sphere bundle T_1M of an $n(\geq 3)$-dimensional Einstein manifold is H-contact if and only if the base manifold is 2-stein ([8], Main Theorem). The main purpose of the present paper is to prove the following:

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Main Theorem. *Let $M = (M, g)$ be an $n(\geq 2)$-dimensional Riemannian manifold whose unit tangent sphere bundle T_1M equipped with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ is H-contact. If $\dim M \neq 4$, then the scalar curvature τ of M, the square norm $|\rho|^2$ of the Ricci tensor and the square norm $|R|^2$ of the curvature tensor are all constant. If $\dim M = 4$, then τ and $|\rho|^2$ are constant, however, $|R|^2$ is not necessary constant.*

This Main Theorem together with Theorem 2 in Section 5 can be comparable with the results ([7], Theorem 1 and Theorem 2). After the proof of Main Theorem, we shall provide an example concerning Main Theorem.

The authors would like to express their thanks to the referee for the insightful suggestion concerning Question 1.

2. Standard contact metric structure on a unit tangent sphere bundle

All manifolds in this paper are assumed to be of class C^∞. We refer to [2] for the basic concepts and terminologies on contact metric manifolds.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ the associated Levi Civita connection. Its Riemann curvature tensor R is defined by $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for all vector fields X, Y and Z on M. The tangent bundle of (M, g) is denoted by TM and consists of pairs (p, u), where p is a point in M and u a tangent vector to M at p. The mapping $\pi : TM \to M$, $\pi(p, u) = p$ is the natural projection from TM onto M. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X) \circ \pi$, where ω is a 1-form on M. For a Levi Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where i is the immersion. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p, u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of a vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$.

We now define the standard contact metric structure of the unit tangent sphere bundle T_1M of a Riemannian manifold (M, g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a (1,1)-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since $g'(\bar{X}, \phi'\bar{Y}) = 2d\eta'(\bar{X}, \bar{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. If we rescale by

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}g',$$

we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. From now on, we consider $T_1M = (T_1M, \eta, \bar{g}, \phi, \xi)$ with the standard contact metric structure.

Let $\{e_1, \ldots, e_n = u\}$ be an orthonormal basis of T_pM. Then the Ricci tensor $\bar{\rho}$ of T_1M is given by

$$\begin{aligned} \bar{\rho}(X^t, Y^t) &= (n-2)(g(X,Y) - g(X,u)g(Y,u)) \\ &\quad + \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i), \\ \bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u\rho)(X,Y) - (\nabla_X\rho)(u,Y)), \\ \bar{\rho}(X^h, Y^h) &= \rho(X,Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y), \end{aligned} \tag{2.1}$$

where ρ denotes the Ricci curvature tensor of M. We refer to [4, 7, 11, 12] for the formula (2.1).

3. H-contact unit tangent sphere bundles

Let $M = (M, g)$ be an $n(\geq 3)$-dimensional Riemannian manifold and $\{e_i\}_{i=1}^n$ be a local orthonormal frame field at an arbitrary point $p \in M$. Now, we assume that T_1M is H-contact with respect to the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then the base manifold M satisfies the following conditions [5].

$$\nabla_i\rho_{jk} - \nabla_j\rho_{ik} = 0, \tag{3.1}$$

$$2\rho_{ab} = \sum_{i,j=1}^{n} R_{aibj}R_{aiaj} \quad (a \neq b). \tag{3.2}$$

From (3.1), we see easily that the scalar curvature τ of M is constant.
Now, we shall deduce several easy consequences of formula (3.2) for the later use. We set

$$\begin{cases} u = \cos\theta e_a + \sin\theta e_b, \\ x = -\sin\theta e_a + \cos\theta e_b \qquad \text{for all } a \neq b. \end{cases} \tag{3.3}$$

Substituting (3.3) into the left hand side of (3.2), we get (using some standard trigonometric identities)

$$2\rho(\cos\theta e_a+\sin\theta e_b,-\sin\theta e_a+\cos\theta e_b)=2\rho_{ab}\cos(2\theta)+(\rho_{bb}-\rho_{aa})\sin(2\theta). \tag{3.4}$$

Similarly, substituting (3.3) into the right hand side of (3.2), and taking account of (3.3), we get

$$\begin{aligned}
&\sum_{i,j=1}^{n} R(\cos\theta e_a+\sin\theta e_b, e_i, -\sin\theta e_a+\cos\theta e_b, e_j)\\
&\qquad \times R(\cos\theta e_a+\sin\theta e_b, e_i, \cos\theta e_a+\sin\theta e_b, e_j)\\
&= 2\rho_{ab}\cos(2\theta)+\frac{1}{4}\Big\{\sum_{i,j=1}^{n}(R_{bibj})^2-\sum_{i,j=1}^{n}(R_{aiaj})^2\Big\}\sin(2\theta)\\
&\quad +\frac{1}{4}\Big\{\sum_{i,j=1}^{n}(R_{aibj})^2+\sum_{i,j=1}^{n}R_{aibj}R_{biaj}+\sum_{i,j=1}^{n}R_{aiaj}R_{bibj}\\
&\qquad -\frac{1}{2}\sum_{i,j=1}^{n}(R_{aiaj})^2-\frac{1}{2}\sum_{i,j=1}^{n}(R_{bibj})^2\Big\}\sin(4\theta).
\end{aligned} \tag{3.5}$$

Then, comparing the finite Fourier series in (3.4) and (3.5), we obtain two equations:

$$4(\rho_{aa}-\rho_{bb})=\sum_{i,j=1}^{n}(R_{aiaj})^2-\sum_{i,j=1}^{n}(R_{bibj})^2, \tag{3.6}$$

$$\begin{aligned}
&2\Big\{\sum_{i,j=1}^{n}(R_{aibj})^2+\sum_{i,j=1}^{n}R_{aibj}R_{biaj}+\sum_{i,j=1}^{n}R_{aiaj}R_{bibj}\Big\}\\
&=\sum_{i,j=1}^{n}(R_{aiaj})^2+\sum_{i,j=1}^{n}(R_{bibj})^2.
\end{aligned} \tag{3.7}$$

We shall recall the following fact ([8], Lemma 4.1) which plays an important role in the proof of Main Theorem.

Lemma 1. *Let $S^n (n \geq 2)$ be an n-dimensional unit sphere centered at the origin 0 in an $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ and f be a real-valued function on S^n satisfying the condition $f(u) = f(v)$ for any $u, v \in S^n$ such that $u \perp v$. Then, f is constant on S^n.*

4. Proof of Main Theorem

Let $M = (M, g)$ be an $n(\geq 3)$-dimensional Riemannian manifold satisfying the hypothesis of Main Theorem and $\{e_i\}_{i=1}^n$ be any orthonormal basis of T_pM

at an arbitrary point $p \in M$. Then from the equality (3.6), we get

$$4\rho_{aa} - \sum_{i,j=1}^{n} (R_{aiaj})^2 = 4\rho_{bb} - \sum_{i,j=1}^{n} (R_{bibj})^2 \tag{4.1}$$

for all $a \neq b$. We may regard (T_pM, g_p) as an n-dimensional Euclidean space $\mathbb{E}^n$ and the unit tangent sphere $U_p = \{u \in T_pM \| u| = 1\}$ as an $(n-1)$-dimensional unit sphere $S^{n-1}(\subset \mathbb{E}^n)$, respectively. We now consider the smooth function F on $\mathbb{E}^n = (T_pM, g_p)$ defined by

$$F(u) = 4\rho(u,u)g(u,u) - \sum_{i,j=1}^{n} (R_{uiuj})^2 \tag{4.2}$$

for all $u \in T_pM$. Further, we denote by f the restriction of the function F to $S^{n-1} = U_p$. Then, applying Lemma 1 to the function f on S^{n-1}, we see that there exists a function C on M satisfying the following equality for any $u \in S^{n-1}$, at each point $p \in M$:

$$4\rho(u,u) - \sum_{i,j=1}^{n} (R_{uiuj})^2 = C(p). \tag{4.3}$$

From (4.3), we have also

$$4\rho(u,u)g(u,u) - \sum_{i,j=1}^{n} (R_{uiuj})^2 = C(p)g(u,u)g(u,u) \tag{4.4}$$

for any $u \in T_pM = \mathbb{E}^n$, at each point $p \in M$. We set $u = \sum_{i=1}^{n} u_i e_i$. Then, from (4.4), we have

$$\begin{aligned} &\sum_{a,b,c,d=1}^{n} \Big\{ \sum_{(a,b,c,d)\in\mathfrak{S}_4} \big(4\rho_{ab}g_{cd} - \sum_{i,j=1}^{n} R_{aibj}R_{cidj}\big) \Big\} u_a u_b u_c u_d \\ &= C(p) \sum_{a,b,c,d=1}^{n} \Big\{ \sum_{(a,b,c,d)\in\mathfrak{S}_4} g_{ab}g_{cd} \Big\} u_a u_b u_c u_d \end{aligned} \tag{4.5}$$

for any $(u_i) \in \mathbb{E}^n$, where $\mathfrak{S}_4$ denotes the set of all permutations of the letters a, b, c, d. Since we may regard both sides of the equality (4.5) as homogeneous symmetric polynomials of degree 4, from (4.5), we have

$$\sum_{(a,b,c,d)\in\mathfrak{S}_4} \Big(4\rho_{ab}g_{cd} - \sum_{i,j=1}^{n} R_{aibj}R_{cidj}\Big) = C(p) \sum_{(a,b,c,d)\in\mathfrak{S}_4} g_{ab}g_{cd} \tag{4.6}$$

at each point $p \in M$. From (4.6), by direct calculations, we get

$$
\begin{aligned}
&4\big(\rho_{ab}g_{cd}+\rho_{ac}g_{bd}+\rho_{ad}g_{bc}+\rho_{bc}g_{ad}+\rho_{bd}g_{ac}+\rho_{cd}g_{ab}\big)\\
&-\sum_{i,j=1}^{n}\big(R_{aibj}R_{cidj}+R_{aibj}R_{dicj}+R_{aicj}R_{bidj}+R_{aicj}R_{dibj}\\
&\qquad+R_{aidj}R_{bicj}+R_{aidj}R_{cibj}\big)\\
=\;&2C(p)\big(g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}\big)
\end{aligned} \tag{4.7}
$$

for any $1 \le a,b,c,d \le n$. Transvecting g_{cd} with (4.7), we have

$$
4(n+4)\rho_{ab}+4\tau g_{ab}+2\sum_{i,j=1}^{n}\rho_{ij}R_{aibj}-3\sum_{i,j,k=1}^{n}R_{kiaj}R_{kibj}=2(n+2)C(p)g_{ab}. \tag{4.8}
$$

From (4.8), we have immediately

$$
\begin{aligned}
2n(n+2)C(p)&=4(n+4)\tau+4n\tau-2|\rho|^2-3|R|^2\\
&=8(n+2)\tau-2|\rho|^2-3|R|^2.
\end{aligned} \tag{4.9}
$$

From (4.9), we may see that C gives rise to a smooth function on M. Thus, from (4.9), since τ is constant, we have

$$
2n(n+2)uC=-2u|\rho|^2-3u|R|^2 \tag{4.10}
$$

for any tangent vector u at any point $p \in M$. Now, since τ is constant, from (4.8), taking account of the second Bianchi identity and (3.1), we have

$$
2\sum_{a,i,j=1}^{n}(\nabla_a\rho_{ij})R_{aibj}-3\sum_{a,i,j,k=1}^{n}R_{kiaj}(\nabla_aR_{kibj})=2(n+2)\nabla_bC,
$$

and hence,

$$
-\frac{3}{4}\nabla_b|R|^2=2(n+2)\nabla_bC. \tag{4.11}
$$

Thus, from (4.11), we have

$$
-3u|R|^2=8(n+2)uC. \tag{4.12}
$$

Thus, from (4.10) and (4.12), we have

$$
8u|\rho|^2+3(4-n)u|R|^2=0 \tag{4.13}
$$

for any tangent vector u at each point $p \in M$. Therefore, if $n=4$, then $|\rho|^2$ is constant.

On the other hand, operating ∇_a on both sides of (4.7) and taking sum with respect to a, we have

$$\begin{aligned} &4\big(\nabla_d\rho_{bc}+\nabla_c\rho_{bd}+\nabla_b\rho_{cd}\big)\\ &\quad-\sum_{i,j=1}^{n}\big(R_{aibj}\nabla_aR_{cidj}+R_{aibj}\nabla_aR_{dicj}+R_{aicj}\nabla_aR_{bidj}\\ &\qquad+R_{aicj}\nabla_aR_{dibj}+R_{aidj}\nabla_aR_{bicj}+R_{aidj}\nabla_aR_{cibj}\big)\\ &=2\big((\nabla_bC)g_{cd}+(\nabla_cC)g_{bd}+(\nabla_dC)g_{bc}\big). \end{aligned} \tag{4.14}$$

Here, we get

$$\begin{aligned} \sum_{a,i,j=1}^{n}R_{aibj}\nabla_aR_{cidj}&=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aibj}(\nabla_aR_{cidj}-\nabla_iR_{cadj})\\ &=-\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aibj}\nabla_cR_{iadj}\\ &=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aibj}\nabla_cR_{aidj}. \end{aligned} \tag{4.15}$$

Similarly, we have the following.

$$\begin{aligned} \sum_{a,i,j=1}^{n}R_{aibj}\nabla_aR_{dicj}&=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aibj}\nabla_dR_{aicj},\\ \sum_{a,i,j=1}^{n}R_{aicj}\nabla_aR_{bidj}&=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aicj}\nabla_bR_{aidj},\\ \sum_{a,i,j=1}^{n}R_{aicj}\nabla_aR_{dibj}&=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aicj}\nabla_dR_{aibj},\\ \sum_{a,i,j=1}^{n}R_{aidj}\nabla_aR_{bicj}&=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aidj}\nabla_bR_{aicj},\\ \sum_{a,i,j=1}^{n}R_{aidj}\nabla_aR_{cibj}&=\frac{1}{2}\sum_{a,i,j=1}^{n}R_{aidj}\nabla_cR_{aibj}. \end{aligned} \tag{4.16}$$

Thus, from (4.15) and (4.16), transvecting $(\nabla_bC)g_{cd}+(\nabla_cC)g_{bd}+(\nabla_dC)g_{bc}$ with the left hand side of (4.14), we have

$$\begin{aligned} &4\sum\big((\nabla_bC)g_{cd}+(\nabla_cC)g_{bd}+(\nabla_dC)g_{bc}\big)\times\big(\nabla_d\rho_{bc}+\nabla_c\rho_{bd}+\nabla_b\rho_{cd}\big)\\ &-\frac{1}{2}\sum\big((\nabla_bC)g_{cd}+(\nabla_cC)g_{bd}+(\nabla_dC)g_{bc}\big)\\ &\qquad\times\big(\nabla_b(R_{aicj}R_{aidj})+\nabla_c(R_{aibj}R_{aidj})+\nabla_d(R_{aibj}R_{aicj})\big). \end{aligned} \tag{4.17}$$

Similarly, transvecting $(\nabla_b C)g_{cd} + (\nabla_c C)g_{bd} + (\nabla_d C)g_{bc}$ with the right hand side of (4.14), we have

$$
\begin{aligned}
&2\sum \big((\nabla_b C)g_{cd} + (\nabla_c C)g_{bd} + (\nabla_d C)g_{bc}\big)\\
&\qquad \times \big((\nabla_b C)g_{cd} + (\nabla_c C)g_{bd} + (\nabla_d C)g_{bc}\big)\\
&= 2\sum \{4(\nabla_b C)^2 + 4(\nabla_c C)^2 + 4(\nabla_d C)^2 + (\nabla_b C)^2 + (\nabla_d C)^2\\
&\qquad + (\nabla_c C)^2 + (\nabla_d C)^2 + (\nabla_b C)^2 + (\nabla_c C)^2\}\\
&= 36|\nabla C|^2,
\end{aligned} \tag{4.18}
$$

and hence, from (4.17) and (4.18), we get

$$
\begin{aligned}
&36|\nabla C|^2\\
= &-\frac{1}{2}\Big\{2\sum(\nabla_b C)(\nabla_c(R_{aibj}R_{aicj})) + \sum(\nabla_b C)(\nabla_b|R|^2)\\
&\quad + 2\sum(\nabla_c C)(\nabla_b(R_{aicj}R_{aibj})) + \sum(\nabla_c C)(\nabla_c|R|^2)\\
&\quad + 2\sum(\nabla_d C)(\nabla_b(R_{acbj}R_{aidj})) + \sum(\nabla_d C)(\nabla_d|R|^2)\Big\}\\
= &-\frac{9}{4}\sum(\nabla_i C)\nabla_i|R|^2.
\end{aligned}
$$

Thus, we have

$$16|\nabla C|^2 = -\sum(\nabla_i C)\nabla_i|R|^2. \tag{4.19}$$

From (4.11) and (4.19), we have

$$128(n+2)|\nabla C|^2 = 3\big|\nabla|R|^2\big|^2. \tag{4.20}$$

On one hand, from (4.11), we have

$$64(n+2)^2|\nabla C|^2 = 9\big|\nabla|R|^2\big|^2. \tag{4.21}$$

Thus, from (4.21) and (4.20), we have

$$6\big|\nabla|R|^2\big|^2 = (n+2)\big|\nabla|R|^2\big|^2,$$

and hence,

$$(n-4)\big|\nabla|R|^2\big|^2 = 0. \tag{4.22}$$

Thus, if $n \neq 4$, then $\nabla|R|^2 = 0$, and hence, $|R|^2$ is a constant. Therefore, from (4.13), it follows that $|\rho|^2$ is a constant. This completes the proof of Main Theorem.

Here, we remark that an η-Einstein structure is a special case of an H-contact. In [7], we proved that if T_1M is η-Einstein, then τ, $|R|^2$ and $|\rho|^2$ are all constants.

Lastly, we provide an example illustrating the latter part of Main Theorem.

Let M be a 4-dimensional real half-space given by $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ x_1 > 0, (x_2, x_3, x_4) \in \mathbb{R}^3\}$ and define a Riemannian metric g on M by

$$g = (g_{ij}) = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_1 + \frac{x_3^2}{4x_1} & -\frac{x_2 x_3}{4x_1} & \frac{x_3}{2x_1} \\ 0 & -\frac{x_2 x_3}{4x_1} & x_1 + \frac{x_2^2}{4x_1} & -\frac{x_2}{2x_1} \\ 0 & \frac{x_3}{2x_1} & -\frac{x_2}{2x_1} & \frac{1}{x_1} \end{pmatrix}, \tag{4.23}$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ [9, 10].

Then we see that (M, g) is Ricci flat and 2-stein, and hence, the unit tangent sphere bundle T_1M equipped with the standard contact metric structure is H-contact. However, we may also check that the square norm of the curvature tensor is not constant [8]. Reflecting on Main Theorem, the following question will naturally arise.

Question 1. Is an $n(\neq 4)$-dimensional Riemannian manifold, whose unit tangent sphere bundle equipped with the standard contact metric structure having H-contact, locally symmetric?

5. An application

In this section, we shall provide an application concerning Main Theorem. First, we define symmetric (0,2)-tensor field α on M by

$$\alpha(x, y) = \sum_{i,j,k} R_{xijk} R_{yijk} \tag{5.1}$$

for any $x, y \in T_pM$ at each point $p \in M$. An n-dimensional Einstein manifold $M = (M, g)$ is called a *super-Einstein* manifold [4] if M additionally satisfies the condition

$$\alpha = \frac{|R|^2}{n} g. \tag{5.2}$$

We here remark that the constancy of $|R|^2$ follows from the condition (5.2) for an $n(\neq 4)$-dimensional super-Einstein manifold ([4], Lemma 3.3). For a 4-dimensional super-Einstein manifold, the constancy of $|R|^2$ is usually required ([4], p. 531). We may easily check that a 2-stein manifold satisfies the condition (5.2). It is also well-known that every harmonic space is super-Einstein [1].

For the remainder of this section, we assume that $M = (M, g)$ is an n-dimensional Riemannian manifold whose unit tangent sphere bundle T_1M (equipped with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$) is H-contact, unless otherwise specified. Then, since the characteristic vector field ξ is an eigenvector field of the Ricci operator $\bar{Q}$ of T_1M, there exists a smooth function $\bar{\lambda}$ (call it the corresponding eigenvalues of $\bar{Q}$) satisfying

$$\bar{Q}\xi = \bar{\lambda}\xi. \tag{5.3}$$

Thus, from (2.1), (4.3), (4.9) and (5.3), we have

$$\begin{aligned}\bar{\lambda} &= 4\rho(u,u) - 2\alpha(u,u) \\ &= -4\rho(u,u) + \frac{8}{n}\tau - \frac{2|\rho|^2}{n(n+2)} - \frac{3|R|^2}{n(n+2)}\end{aligned} \tag{5.4}$$

on T_1M.

Further, from (2.1), we also see that the scalar curvature $\bar{\tau}$ of T_1M is given by

$$\bar{\tau} = 4(n-1)(n-2) + 4\tau - \alpha(u,u) \tag{5.5}$$

for any $u \in T_1M$. Now, we shall prove the following:

Theorem 2. *Let $M = (M, g)$ be an $n(\neq 4)$-dimensional Riemannian manifold whose unit tangent sphere bundle T_1M is H-contact. Then, the followings are equivalent*:

1. (1) *the corresponding eigenvalue $\bar{\lambda}$ of the Ricci operator $\bar{Q}$ of T_1M is constant.*
2. (2) *the scalar curvature $\bar{\tau}$ of T_1M is constant.*
3. (3) *M is 2-stein.*

Proof. It suffices to prove the equivalence of (1) and (3) and the equivalence of (2) and (3). Since $\dim M \neq 4$, from (5.4), taking account of Main Theorem and the result ([8]), we have the equivalence of (1) and (3).

Similarly, from (5.1), (5.5) and Main Theorem, we see easily that $\bar{\tau}$ is constant on T_1M if and only if the equality (5.2) holds on M. Thus, from (4.9) and (4.14) $\sim$ (4.16), taking account of Main Theorem, we have

$$\underset{b,c,d}{\mathfrak{S}} \nabla_b \rho_{cd} = 0. \tag{5.6}$$

Thus, from (3.1) and (5.6), it follows immediately that $\nabla\rho = 0$. By the result of Calvaruso and Perrone ([5], Theorem 4.2), we see that M is Einstein (and hence, 2-stein). This completes the proof of the equivalence of (2) and (3). □

Now, let $M = (M, g)$ be a locally symmetric space whose unit tangent sphere bundle T_1M is H-contact. Then, we also see that M is Einstein (and hence, 2-stein). Thus, in this case, we see that the corresponding eigenvalue $\bar{\lambda}$ of the Ricci operator $\bar{Q}$ of T_1M and the scalar curvature of the unit tangent sphere bundle T_1M of M are both constant by virtue of Theorem 2. So, in order to determine the base manifold whose unit tangent sphere bundle T_1M is H-contact, it seems reasonable to start with the case where the scalar curvature of T_1M is constant. Therefore, as a special case of Question 1, the following question will be raised.

Question 2. Is $n(\neq 4)$-dimensional 2-stein manifold locally symmetric?

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References

[1] A. L. Besse, *Manifolds All of Whose Geodesics are Closed*, Springer-Verlag, Berlin-New York, 1978.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, Inc., Boston, MA, 2002.

[3] E. Boeckx and L. Vanhecke, *Harmonic and minimal vector fields on tangent and unit tangent bundles*, Differential Geom. Appl. **13** (2000), no. 1, 77–93.

[4] ———, *Unit tangent sphere bundles with constant scalar curvature*, Czechoslovak Math. J. **51(126)** (2001), no. 3, 523–544.

[5] G. Calvaruso and D. Perrone, *H-contact unit tangent sphere bundles*, Rocky Mountain J. Math. **37** (2007), no. 5, 1435–1458.

[6] P. Carpenter, A. Gray, and T. J. Willmore, *The curvature of Einstein symmetric spaces*, Quart. J. Math. Oxford Ser. (2) **33** (1982), no. 129, 45–64.

[7] Y. D. Chai, S. H. Chun, J. H. Park, and K. Sekigawa, *Remarks on η-Einstein unit tangent bundles*, Monatsh. Math. **155** (2008), no. 1, 31–42.

[8] S. H. Chun, J. H. Park, and K. Sekigawa, *H-contact unit tangent sphere bundles of Einstein manifolds*, Quart. J. Math., (to appear), DOI:10.1093/qmath/hap025.

[9] P. Nurowski and M. Pruzanowski, *A four-dimensional example of a Ricci flat metric admitting almost-Kähler non-Kähler structure*, Classical Quantum Gravity **16** (1999), no. 3, L9–L13.

[10] T. Oguro, K. Sekigawa, and A. Yamada, *Four-dimensional almost Kahler Einstein and weakly $*$-Einstein manifolds*, Yokohama Math. J. **47** (1999), no. 1, 75–92.

[11] J. H. Park and K. Sekigawa, *When are the tangent sphere bundles of a Riemannian manifold η-Einstein?*, Ann. Global Anal. Geom. **36** (2009), no. 3, 275–284.

[12] ———, *Notes on tangent sphere bundles of constant radii*, J. Korean Math. Soc. **46** (2009), no. 6, 1255–1265.

[13] D. Perrone, *Contact metric manifolds whose characteristic vector field is a harmonic vector field*, Differential Geom. Appl. **20** (2004), no. 3, 367–378.

[14] C. M. Wood, *On the energy of a unit vector field*, Geom. Dedicata **64** (1997), no. 3, 319–330.

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