J. Korean Math. Soc. **48** (2011), No. 2, pp. 301–309
DOI 10.4134/JKMS.2011.48.2.301

THE GROUP OF GRAPH AUTOMORPHISMS OVER A MATRIX RING

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Abstract. Let $R = \mathrm{Mat}_2(F)$ be the ring of all 2 by 2 matrices over a finite field F, X the set of all nonzero, nonunits of R and G the group of all units of R. After investigating some properties of orbits under the left (and right) regular action on X by G, we show that the graph automorphisms group of $\Gamma(R)$ (the zero-divisor graph of R) is isomorphic to the symmetric group $S_{|F|+1}$ of degree $|F| + 1$.

1. Introduction

The zero-divisor graph of a commutative ring has been studied extensively by Akbari, Anderson, Frazier, Lauve, Livinston and Mohammadian in [1, 2, 3] since its concept had been introduced by Beck in [4]. Recently, the zero-divisor graph of a noncommutative ring (resp. a semigroup) has also been studied by Redmond and Wu (resp. F. DeMeyer and L. DeMeyer) in [10, 11, 12] (resp. [5]). The zero-divisor graph has been used to study the algebraic structures of rings via their zero-divisors. In this paper, the group of the zero-divisor graph automorphisms over a matrix ring over a finite field is investigated by considering some group actions.

For a ring R with identity, let $Z(R)$ be the set of all left or right zero-divisors of R, $\Gamma(R)$ be the zero-divisor graph of R consisting of all vertices in $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero left or right zero-divisors of R, and edges $x \longrightarrow y$, which means that $xy = 0$ for $x, y \in Z(R)^*$.

For a ring R with identity, let $X(R)$ (simply, denoted by X) be the set of all nonzero, nonunits of R, $G(R)$ (simply, denoted by G) be the group of all units of R. In this paper, we will consider some group actions on X by G given by $(g, x) \longrightarrow gx$ (resp. $(g, x) \longrightarrow xg^{-1}$) from $G \times X$ to X, called the left (resp. right) regular action. If $\phi : G \times X \longrightarrow X$ is the left (resp. right) regular action,

Received September 25, 2009; Revised May 20, 2010.
2010 *Mathematics Subject Classification.* Primary 05C20; Secondary 16W22.
Key words and phrases. zero-divisor graph, left (resp. right) regular action, orbit, graph automorphisms group.
This study was supported by research funds from Dong-A University.

then for each $x \in X$, we define the *orbit* of x by $o_\ell(x) = \{\phi(g,x) = gx : g \in G\}$ (resp. $o_r(x) = \{\phi(g,x) = xg^{-1} : g \in G\}$).

In Section 2, we will show that if $R = \mathrm{Mat}_2(F)$ with F a finite field, then (1) the number of orbits under the left (resp. right) regular action on X by G is $|F|+1$; (2) if N is the set of all nonzero nilpotents in R, then $|N| = |F|^2 - 1$ and $o_\ell(x) \cap o_r(x) = o_\ell(x) \cap N = o_r(x) \cap N$ for all $x \in N$; (3) $|o_\ell(x) \cap o_r(y)| = |F| - 1$ for all $x, y \in X$.

We recall that for all $x \in X$ the set $ann_\ell(x) = \{y \in X : yx = 0\}$ (resp. $ann_r(x) = \{z \in X : xz = 0\}$) is called a left (resp. right) annihilator of x. Let $ann^*_\ell(x) = ann_\ell(x) \setminus \{0\}$ (resp. $ann^*_r(x) = ann_r(x) \setminus \{0\}$).

A *graph automorphism* f of a graph $\Gamma(R)$ (where R denotes a ring) is defined to be a bijection $f : \Gamma(R) \longrightarrow \Gamma(R)$ which preserves adjacency. Note that the set $\mathrm{Aut}(\Gamma(R))$ of all graph automorphisms of $\Gamma(R)$ forms a group under the usual composition of functions. In [3], Anderson and Livingston have shown that $\mathrm{Aut}(\Gamma(\mathbb{Z}_n)$ is a (finite) direct product of symmetric groups for $n \geq 4$ a nonprime integer. For the case of noncommutative rings, it was shown by [8] that when $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ (p is a prime), $\mathrm{Aut}(\Gamma(R)) \simeq S_{p+1}$, the symmetric group of degree $p+1$. In Section 3, for the continuation of these investigation, we prove that $\mathrm{Aut}(\Gamma(R)) \simeq S_{|F|+1}$ when $R = \mathrm{Mat}_2(F)$ with F a finite field.

2. Orbits under the regular action in $\mathrm{Mat}_2(F)$

Recall that G is *transitive* on X (or G acts transitively on X) under the left (resp. right) regular action on X by G if there is an $x \in X$ with $o_\ell(x) = X$ (resp. $o_r(x) = X$) and the left (resp. right) regular action of G on X is said to be *half-transitive* if G is transitive on X or if $o_\ell(x)$ (resp. $o_r(x)$) is a finite set with $|o_\ell(x)| > 1$ (resp. $|o_r(x)| > 1$) and $|o_\ell(x)| = |o_\ell(y)|$ (resp. $|o_r(x)| = |o_r(y)|$) for all x and $y \in X$. In [7, Theorem 2.4 and Lemma 2.7], it was shown that if $R = \mathrm{Mat}_2(F)$ with F a finite field, then G is half-transitive on X by the left (resp. right) regular action and $|o_\ell(x)| = |F|^2 - 1$ (resp. $|o_r(x)| = |F|^2 - 1$) for all $x \in X$.

Lemma 2.1. *Let $R = \mathrm{Mat}_2(F)$ with F a finite field. Then the number of orbits under the left* (*resp. right*) *regular action on X by G is $|F| + 1$.*

Proof. Let μ be the number of orbits under the left (resp. right) regular action on X by G. Note that $|G| = (|F|^2 - 1)(|F|^2 - |F|)$. Thus $|X| = |R| - |G| - 1 = |F|^4 - (|F|^2 - 1)(|F|^2 - |F|) - 1 = (|F| + 1)(|F|^2 - 1)$. Since the cardinality of any orbit under the left (resp. right) regular action on X by G is $|F|^2 - 1$ by [7, Lemma 2.7], $\mu = |X|/(|F|^2 - 1) = |F| + 1$. □

The following theorem was shown in [6].

Theorem 2.2. *The probability that n by n matrix over $GF(p^\alpha)$ be nilpotent is $p^{\alpha n}$*

Proof. Refer [6, Theorem 1]. □

By Theorem 2.2, we note that the number of all 2 by 2 nonzero nilpotent matrices over a finite field F is equal to $|F|^2 - 1$.

Theorem 2.3. *Let $R = \mathrm{Mat}_2(R)$ with F a finite field and N be the set of all nonzero nilpotents in R. Then under the left (resp. right) regular action on X by G, we have the following.*
(i) $|o_\ell(x) \cap N| = |F| - 1$;
(ii) $o_\ell(x) \cap N = o_r(x) \cap N = o_\ell(x) \cap o_r(x)$ *for each* $x \in N$.

Proof. (i) Consider the set $S_x = \{(\alpha I)x | \alpha \in F \setminus \{0\}\}$ for each $x \in N$ where I is the identity matrix in R. Since $(\alpha I)x = x(\alpha I)$ for all $(\alpha I)x \in S$, $S_x \subseteq o_\ell(x) \cap N, o_r(x) \cap N$. Note that for all $\alpha, \beta \in F \setminus \{0\}(\alpha \neq \beta)$, $(\alpha I)x \neq (\beta I)x$, and so $|S_x| = |F| - 1$. Next, we will show that $o_\ell(x) \cap N \subseteq S_x$. Let $y \in o_\ell(x) \cap N$ be arbitrary. Let

$$x = \begin{bmatrix} -\alpha b & b \\ -\alpha^2 b & \alpha b \end{bmatrix} \in N \quad \text{for some} \quad b(\neq 0), \alpha \in F.$$

Since $y \in o_\ell(x)$, $y = gx$ for some $g \in G$. Let $g = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in G$. Then

$$y = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} -\alpha b & b \\ -\alpha^2 b & \alpha b \end{bmatrix} = \begin{bmatrix} -(p + q\alpha)\alpha b & (p + q\alpha)b \\ -(r + s\alpha)\alpha b & (r + s\alpha)b \end{bmatrix} \in o_\ell(x).$$

Since $y \in N$, we have $(p + q\alpha)\alpha = (r + s\alpha)(\neq 0)$ by the proof of Lemma 2.2, and so

$$y = \begin{bmatrix} p + q\alpha & 0 \\ 0 & p + q\alpha \end{bmatrix} \begin{bmatrix} -\alpha b & b \\ -\alpha^2 b & \alpha b \end{bmatrix} = ((p + q\alpha)I)x \in S_x.$$

Therefore, $o_\ell(x) \cap N \subseteq S_x$, and consequently we have $S_x = o_\ell(x) \cap N$. By the similar argument, we have also $S_x = o_r(x) \cap N$. Hence $o_\ell(x) \cap N = o_r(x) \cap N$ and $|o_\ell(x) \cap N| = |o_r(x) \cap N| = |S_x| = |F| - 1$ for each $x \in N$.

(ii) By the proof of (i), we have that $o_\ell(x) \cap N = o_r(x) \cap N$ for each $x \in N$. Note that $S_x = o_\ell(x) \cap N (= o_r(x) \cap N) \subseteq o_\ell(x) \cap o_r(x)$ for each $x \in N$ where S_x is the set considered in the proof of (i). Let $y \in o_\ell(x) \cap o_r(x)$ be arbitrary and let

$$x = \begin{bmatrix} -\alpha\beta & \beta \\ -\alpha^2\beta & \alpha\beta \end{bmatrix} \in N \quad (\forall \alpha \in F, \forall \beta \in F \setminus \{0\})$$

be arbitrary. Then there exist $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], h = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in G$ such that $y = gx = xh$. Thus

$$(1) \qquad gx = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -\alpha\beta & \beta \\ -\alpha^2\beta & \alpha\beta \end{bmatrix} = \begin{bmatrix} -\alpha\beta(a + b\alpha) & \beta(a + b\alpha) \\ -\alpha\beta(c + d\alpha) & \beta(c + d\alpha) \end{bmatrix},$$

$$(2) \qquad xh = \begin{bmatrix} -\alpha\beta & \beta \\ -\alpha^2\beta & \alpha\beta \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} -\beta(\alpha p - r) & -\beta(\alpha q - s) \\ -\alpha\beta(\alpha p - r) & -\alpha\beta(\alpha q - s) \end{bmatrix}.$$

Let $\gamma = -\beta(\alpha q - s)$ (= (1,2) - entry of $y = xh$). From (1) and (2), we have that

$$y = \begin{bmatrix} -\alpha\gamma & \gamma \\ -\alpha^2\gamma & \alpha\gamma \end{bmatrix} \in N,$$

and so $o_\ell(x) \cap o_r(x) \subseteq o_\ell(x) \cap N$ for each $x \in N$. Hence $o_\ell(x) \cap o_r(x) = o_\ell(x) \cap N$ for each $x \in N$. Similarly, we have $o_\ell(x) \cap o_r(x) = o_r(x) \cap N$ for each $x \in N$. □

Remark 1. Let $R = \mathrm{Mat}_2(R)$ with F a finite field and N be the set of all nonzero nilpotents in R. Choose $x_1 \in N$ so that $S_{x_1} = \{(\alpha I)x_1 | \alpha \in F \setminus \{0\}\} \subset N$. By Theorem 2.3, $o_\ell(x_1) \cap N = S_{x_1}$. Since $|N| = |F|^2 - 1$ by Theorem 2.2 and $|S_{x_1}| = |F| - 1$ by Theorem 2.3, we can choose $x_2 \in N \setminus S_{x_1}$. Then $S_{x_1} = o_\ell(x_1) \cap N$ and $S_{x_2} = o_\ell(x_2) \cap N$ are disjoint. Continuing in this way, we can choose $x_1, x_2, \ldots, x_{|F|+1} \in N$ so that $x_{i+1} \in N(R) \setminus (S_{x_1} \cup S_{x_2} \cup \cdots \cup S_{x_i})$ for all $i = 1, \ldots, |F|$. Then we have

$$\begin{aligned} N &= S_{x_1} \cup S_{x_2} \cup \cdots \cup S_{x_{|F|+1}} \\ &= [o_\ell(x_1) \cap N] \cup [o_\ell(x_2) \cup N] \cup \cdots \cup [o_\ell(x_{|F|+1}) \cap N], \end{aligned}$$

which is a disjoint union of N. Observe that $o_\ell(x_1), o_\ell(x_2), \ldots, o_\ell(x_{|F|+1})$ are disjoint (equivalently, they are all distinct). Indeed, assume that there exist $o_\ell(x_i)$ and $o_\ell(x_j)$ for some $i, j (i < j, i \neq j)$ such that $o_\ell(x_i) = o_\ell(x_j)$. Then $x_j \in o_\ell(x_i) \cap N = S_{x_i}$, and so $S_{x_j} \subseteq S_{x_i}$, which is a contradiction. Since the number of orbits under the left regular action on X by G is $|F| + 1$ by Lemma 2.1, $X = o_\ell(x_1) \cup o_\ell(x_2) \cup \cdots \cup o_\ell(x_{|F|+1})$. By the similar argument, we have $X = o_r(x_1) \cup o_r(x_2) \cup \cdots \cup o_r(x_{|F|+1})$.

Lemma 2.4. *Let $R = \mathrm{Mat}_2(R)$ with F a finite field and N be the set of all nonzero nilpotents in R. Then for all $x, y \in N$, $y = gxg^{-1}$ for some $g \in G$.*

Proof. Consider a group action on X by G given by $(g, x) \longrightarrow gxg^{-1}$ from $G \times X$ to X, called conjugation.

Take $a = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in N$. Let $o_c(a) = \{gag^{-1} | g \in G\}$ be the orbit of a under conjugation, and $\mathrm{stab}_c(a) = \{g \in G | ga = ag\}$ be the stabilizer of a under conjugation. Then we have

$$\mathrm{stab}_c(a) = \left\{ \begin{bmatrix} s & t \\ 0 & s \end{bmatrix} \in G | s (\neq 0), t \in F \right\}$$

by easy computation, and so $|\mathrm{stab}_c(a)| = (|F| - 1)|F|$. Hence

$$|o_c(a)| = \frac{|G|}{|\mathrm{stab}_c(a)|} = \frac{(|F|^2 - |F|)(|F|^2 - 1)}{(|F| - 1)|F|} = |F|^2 - 1 = |N|.$$

Since $o_c(a) \subseteq N$, $o_c(a) = N$. Therefore we have the result. □

Theorem 2.5. *Let $R = \mathrm{Mat}_2(R)$ with F a finite field and N be the set of all nonzero nilpotents in R. Then under the left (resp. right) regular action on X by G, $|o_\ell(x) \cap o_r(y)| = |F| - 1$ for each $x, y \in X$.*

Proof. First, we will show that $o_\ell(x) \cap o_r(y) \neq \emptyset$ for each $x, y \in X$. By Remark 1, we can choose $x_1, \dots, x_{|F|+1}$ (resp. $y_1, \dots, y_{|F|+1}$) in N so that $X = o_\ell(x_1) \cup \cdots \cup o_\ell(x_{|F|+1})$ (resp. $X = o_r(y_1) \cup \cdots \cup o_r(y_{|F|+1})$. Thus $o_\ell(x) = o_\ell(x_i)$ and $o_r(y) = o_r(y_j)$ for some $x_i, y_j \in N$. Observe that $o_\ell(x_i) \cap o_r(y_j) \neq \emptyset$. Indeed, since x_i and y_j are nonzero nilpotents in R, $y_j = g x_i g^{-1}$ for some $g \in G$ by Lemma 2.4. Hence $o_\ell(x_i) \cap o_r(y_j) = o_\ell(x_i) \cap o_r(g x_i g^{-1}) = o_\ell(g x_i) \cap o_r(g x_i)$ contains an element $g x_i \in X$, and so $o_\ell(x) \cap o_r(y) = o_\ell(x_i) \cap o_r(y_j) \neq \emptyset$.

Next, we will show that $|o_\ell(x) \cap o_r(y)| = |F| - 1$ for each $x, y \in X$. Since $o_\ell(x) \cap o_r(y) \neq \emptyset$ for each $x, y \in X$, we choose $z \in o_\ell(x) \cap o_r(y)$, and then $o_\ell(z) \cap o_r(z) = o_\ell(x) \cap o_r(y)$. Consider a set $S_z = \{(\alpha I)z | \alpha \in F \setminus \{0\}\}$ where I is the identity matrix in R. Since $(\alpha I)z = z(\alpha I)$ for all $(\alpha I)z \in S$, $S_z \subseteq o_\ell(x) \cap o_r(x)$. Note that for all $\alpha, \beta \in F \setminus \{0\} (\alpha \neq \beta)$, $(\alpha I)z \neq (\beta I)z$, and so $|S_z| = |F| - 1$. Thus $|o_\ell(x) \cap o_r(y)| = |o_\ell(z) \cap o_r(z)| \geq |S_z| = |F| - 1$. Since $X = o_\ell(x_1) \cup \cdots \cup o_\ell(x_{|F|+1})$, we have $o_r(y) = X \cap o_r(y) = [o_\ell(x_1) \cap o_r(y)] \cup \cdots \cup [o_\ell(x_{|F|+1}) \cap o_r(y)]$. Clearly, $o_\ell(x_1) \cap o_r(y), \dots, o_\ell(x_{|F|+1}) \cap o_r(y)$ are disjoint, and thus $|o_r(y)| = |F|^2 - 1 = |o_\ell(x_1) \cap o_r(y)| + \cdots + |o_\ell(x_{|F|+1}) \cap o_r(y)| \geq (|F| - 1)(|F| + 1) = |F|^2 - 1$, which implies that $|o_\ell(x_1) \cap o_r(y)| = \cdots = |o_\ell(x_{|F|+1}) \cap o_r(y)| = |F| + 1$. Since $o_\ell(x) = o_\ell(x_i)$ for some $x_i \in N$, we have that $|o_\ell(x) \cap o_r(y)| = |o_\ell(x_i) \cap o_r(y)| = |F| - 1$ for each $x, y \in X$. □

The following example illustrates Theorem 2.3 and Theorem 2.5 for a certain finite field.

Example 1. Consider $F = \mathbb{Z}_2[x]/\langle 1+x+x^2 \rangle$, a field of order 4 where $\mathbb{Z}_2$ is the Galois field of order 2. To simplify notation, we denote $f(x) + \langle 1+x+x^2 \rangle \in F$ by $f(x)$ for all $f(x) \in \mathbb{Z}_2[x]$. Thus $F = \{0, 1, x, 1+x\}$. Let $R = \mathrm{Mat}_2(F)$ and let N be the set of all nonzero nilpotents of R. Then $|X| = (|F|+1)(|F|^2-1) = 75$ and $|N| = |F|^2 - 1 = 15$. Note that under the left (resp. right) regular action on X by G, there are $z_1, z_2, z_3, z_4, z_5 \in N$ such that $X = o_\ell(z_1) \cup o_\ell(z_2) \cup o_\ell(z_3) \cup o_\ell(z_4) \cup o_\ell(z_5)$ (resp. $X = o_r(z_1) \cup o_r(z_2) \cup o_r(z_3) \cup o_r(z_4) \cup o_r(z_5)$), where $z_1 = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$, $z_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $z_4 = \begin{bmatrix} 1 & x \\ 1+x & 1 \end{bmatrix}$ and $z_5 = \begin{bmatrix} 1 & 1+x \\ x & 1 \end{bmatrix}$.

We compute the followings by a computer programming (using Mathematica Ver. 6):

$$o_\ell(z_1) \cap N = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+x \\ 0 & 0 \end{bmatrix} \right\},$$
$$o_\ell(z_2) \cap N = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1+x & 0 \end{bmatrix} \right\},$$
$$o_\ell(z_3) \cap N = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} x & x \\ x & x \end{bmatrix}, \begin{bmatrix} 1+x & 1+x \\ 1+x & 1+x \end{bmatrix} \right\},$$
$$o_\ell(z_4) \cap N = \left\{ \begin{bmatrix} 1 & x \\ 1+x & 1 \end{bmatrix}, \begin{bmatrix} x & 1+x \\ 1 & x \end{bmatrix}, \begin{bmatrix} 1+x & 1 \\ x & 1+x \end{bmatrix} \right\},$$
$$o_\ell(z_5) \cap N = \left\{ \begin{bmatrix} 1 & 1+x \\ x & 1 \end{bmatrix}, \begin{bmatrix} x & 1 \\ 1+x & x \end{bmatrix}, \begin{bmatrix} 1+x & x \\ 1 & 1+x \end{bmatrix} \right\},$$

with $o_\ell(z_i) \cap N = o_r(z_i) \cap N$ for all $i = 1, \ldots, 5$.

Also we compute the followings by a computer programming (using Mathematica Ver. 6):

$$o_\ell(z_1) \cap o_r(z_1) = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+x \\ 0 & 0 \end{bmatrix} \right\},$$
$$o_\ell(z_1) \cap o_r(z_2) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1+x \end{bmatrix} \right\},$$
$$\cdots$$
$$\cdots$$
$$\cdots$$
$$o_\ell(z_5) \cap o_r(z_4) = \left\{ \begin{bmatrix} 1 & 1+x \\ 1+x & x \end{bmatrix}, \begin{bmatrix} x & 1 \\ 1 & 1+x \end{bmatrix}, \begin{bmatrix} 1+x & x \\ x & 1 \end{bmatrix} \right\},$$
$$o_\ell(z_5) \cap o_r(z_5) = \left\{ \begin{bmatrix} 1 & 1+x \\ x & 1 \end{bmatrix}, \begin{bmatrix} x & 1 \\ 1+x & x \end{bmatrix}, \begin{bmatrix} 1+x & x \\ 1 & 1+x \end{bmatrix} \right\}.$$

3. Automorphism of graph over $\mathbf{Mat_2}(F)$

Lemma 3.1. *Let R be a ring with identity and $f : \Gamma(R) \longrightarrow \Gamma(R)$ be a graph automorphism of $\Gamma(R)$. Then for all $x \in X$, $f(ann_\ell^*(x)) = ann_\ell^*(f(x))$ (and $f(ann_r^*(x)) = ann_r^*(f(x))$).*

Proof. Let $y \in f(ann_\ell^*(x))$ be arbitrary. Then $y = f(z)$ for some $z \in ann_\ell^*(x)$. Since $zx = 0$ and f preserves adjacency, $0 = f(z)f(x) = yf(x)$ and so $y \in ann_\ell^*(f(x))$. Hence $f(ann_\ell^*(x)) \subseteq ann_\ell^*(f(x))$. Let $z \in ann_\ell^*(f(x))$ be arbitrary. Then $zf(x) = 0$. Since f is one-to-one, there exists $z_1 \in X$ such that $f(z_1) = z$. Then $0 = zf(x) = f(z_1)f(x)$. Since f preserves adjacency, $z_1x = 0$. Since $z_1 \in ann_\ell^*(x)$ and $z = f(z_1) \in f(ann_\ell^*(x))$, $ann_\ell^*(f(x)) \subseteq f(ann_\ell^*(x))$. By the similar argument, we have $f(ann_r^*(x)) = ann_r^*(f(x))$. □

Lemma 3.2. *Let R be a ring with identity. If $ann_\ell^*(x) \neq \emptyset$ (resp. $ann_r^*(x) \neq \emptyset$) for some $x \in X$, then $ann_\ell^*(x)$ (resp. $ann_r^*(x)$) is a union of orbits under the left (resp. right) regular action on X by G.*

Proof. Let $y \in ann_\ell^*(x)$ be arbitrary. Then we have $o_\ell(y) \subseteq ann_\ell^*(x)$, and so $\bigcup_{y \in ann_\ell^*(x)} o_\ell(y) \subseteq ann_\ell^*(x)$. Since $ann_\ell^*(x) \neq \emptyset$, it is clear that $ann_\ell^*(x) \subseteq \bigcup_{y \in ann_\ell^*(x)} o_\ell(y)$. Hence $ann_\ell^*(x) = \bigcup_{y \in ann_\ell^*} o_\ell^*(y)$, i.e., $ann_\ell^*(x)$ is a union of orbits under the left regular action on X by G. By the similar argument, $ann_r^*(x)$ is a union of orbits under the right regular action on X by G. □

Corollary 3.3. *Let R be a finite ring with identity. Then for all $x \in X$, $ann_\ell^*(x)$ (resp. $ann_r^*(x)$) is a union of finite number of orbits under the left (resp. right) regular action on X by G.*

Proof. By [8, Proposition 1.2], all $x \in X$ are zero-divisors, and so $ann_\ell^*(x) \neq \emptyset$ (resp. $ann_r^*(x) \neq \emptyset$) for all $x \in X$. Hence for all $x \in X$ $ann_\ell^*(x)$ (resp. $ann_r^*(x)$)

is a union of finite number of orbits under the left (resp. right) regular action on X by G by Lemma 3.2. □

The following lemma is well-known in [9].

Lemma 3.4. *Let p be a prime number and α, β be positive integers. Then $p^{\alpha} - 1$ is a divisor of $p^{\beta} - 1$ if and only if α is a divisor of β.*

Proof. Refer [9, Lemma 3, p 32]. □

By using the preceding lemma, we describe $ann^*_\ell(x)$ (and $ann^*_r(x)$) for all $x \in X$ effectively as follows:

Theorem 3.5. *Let $R = \mathrm{Mat}_2(F)$ with F a finite field. Then $ann^*_\ell(x) = o_\ell(y)$ for all $y \in ann^*_\ell(x)$ (and $ann^*_r(x) = o_r(z)$ for all $z \in ann^*_r(x)$).*

Proof. By [7, Lemma 2.7], we have $|o_\ell(x)| = |F|^2 - 1$ for all $x \in X$. Hence we observe that

(1) since $ann^*_\ell(x)$ is a union of a finite number of orbits under the left regular action of G on X by Corollary 3.3 and the left regular action of G on X is half-transitive by [7, Theorem 2.4], $|o_\ell(y)|$ is a divisor of $|ann^*_\ell(x)|$ for all $y \in ann^*_\ell(x)$;

(2) $|ann_\ell(x)|$ is a divisor of $|F|$ since $ann_\ell(x)$ is an additive subgroup of F for all $x \in X$.

Let $|F| = p^{\alpha}$ for some prime p and some positive integer α. Then $|o_\ell(x)| = p^{2\alpha} - 1$ and $|R| = p^{4\alpha}$. Since $ann_\ell(x) \neq R$, we have $|ann_\ell(x)| = p^k$ for some positive integer k ($2\alpha \leq k < 4\alpha$) by (2). By (1) and Lemma 3.4, $|ann^*_\ell(x)| = p^{2\alpha} - 1$, and so $|ann^*_\ell(x)| = |o_\ell(y)|$. Since $o_\ell(y) \subseteq ann^*_\ell(x)$, $ann^*_\ell(x) = o_\ell(y)$ for all $y \in ann^*_\ell(x)$. Similarly, we can show that $ann^*_r(x)) = o_r(z)$ for all $z \in\in ann^*_r(x)$. □

Theorem 3.6. *Let $R = \mathrm{Mat}_2(F)$ with F a finite field. Then $\mathrm{Aut}(\Gamma(R)) \neq \{1\}$.*

Proof. If $|F| = 2$, then F is isomorphic to $\mathbb{Z}_2$, and so $\mathrm{Aut}(\Gamma(R)) \neq \{1\}$ by [8, Theorem 3.5]. Suppose that $|F| \geq 3$ and let $N(R)$ be the set of all nonzero nilpotents in R. By Theorem 2.3, $|o_\ell(x) \cap N(R)| = |F| - 1 \geq 2$ for each $x \in X$. Take $x_1, x_2 \in o_\ell(x) \cap N(R)$ so that $x_1 \neq x_2$. Since x_1 and x_2 are nilpotents, we have $ann^*_\ell(x_1) = o_\ell(x_1) = o_\ell(x_2) = ann^*_\ell(x_2)$ by Theorem 3.5. Observe that $ann^*_r(x_1) = ann^*_r(x_2)$. Indeed, if $a \in ann^*_r(x_1)$, then $0 = x_1 a = g x_2 a = 0$ for some $g \in G$ since $x_2 \in o_\ell(x_1)$, which implies that $a \in ann^*_r(x_2)$, and so $ann^*_r(x_1) \subseteq ann^*_r(x_2)$. Similarly, we have $ann^*_r(x_2) \subseteq ann^*_r(x_1)$. By a similar argument, we have $ann^*_r(x_1) = o_r(x_1) = o_r(x_2) = ann^*_r(x_2)$ by Theorem 3.5. Since $o_\ell(x_1) = o_\ell(x_2)$, $x_2 = g x_1$ for some $g \in G$. Let $f = (x_1, x_2)$ be a transposition in S_X, the symmetric group on X. Since $x_1 \neq x_2$, $f \neq 1$. We will show that $f \in \mathrm{Aut}(\Gamma(R))$. Let $yz = 0$ for some $y, z \in X$. Then we consider the following cases.

Case 1. $y = z = x_1$.

Then $f(y)f(z) = x_2x_2 = 0$ since $x_2 \in N(R)$.

Case 2. $y = z = x_2$.
Then $f(y)f(z) = x_1x_1 = 0$ since $x_1 \in N(R)$.

Case 3. $y = x_1, z = x_2$.
Then $f(y)f(z) = x_2x_1 = gx_1x_1 = 0$ since $x_1 \in N(R)$.

Case 4. $y = x_2, z = x_1$.
Then $f(y)f(z) = x_1x_2 = g^{-1}x_2x_2 = 0$ since $x_2 \in N(R)$.

Case 5. $y, z \neq x_1, x_2$.
Then $f(y)f(z) = yz = 0$.

Consequently, if $yz = 0$ for some $y, z \in X$, then $f(y)f(z) = 0$, which implies that $f \in \mathrm{Aut}(\Gamma(R))$, and so $\mathrm{Aut}(\Gamma(R)) \neq \{1\}$. □

Corollary 3.7. *Let $R = \mathrm{Mat}_2(F)$ with F a finite field and $N(R)$ be the set of all nonzero nilpotents in R. Consider $X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{|F|+1})$ as mentioned in Remark 1. For all $j = 1, \ldots, |F| + 1$, let $s_j = (1, j)$ be a transposition in $S_{|F|+1}$, the symmetric group of degree $|F|+1$. If $f_{s_j} = (x_1, x_j)$ is a transposition in S_X, the symmetric group on X, then f_{s_j} is a graph automorphism in $\Gamma(R)$.*

Proof. By Lemma 3.1 and Theorem 3.5, $f_{s_j}(o_\ell(x_1)) = o_\ell(f_{s_j}(x_1)) = o_\ell(x_j)$. Then f_{s_j} is a graph automorphism in $\Gamma(R)$ by the similar argument as given in the proof in Theorem 3.6. □

Theorem 3.8. *Let $R = \mathrm{Mat}_2(F)$ with F a finite field. Then $\mathrm{Aut}(\Gamma(R)) \simeq S_{|F|+1}$.*

Proof. Let $N(R)$ be the set of all nonzero nilpotents in R. We choose $x_1, \ldots, x_{|F|+1} \in N(R)$ so that $X = o_\ell(x_1) \cup \cdots \cup o_\ell(x_{|F|+1})$ by Remark 1. Let $f \in \mathrm{Aut}(\Gamma(R))$ be arbitrary. By Lemma 3.1 and Theorem 3.5, for each $j = 1, \ldots, |F|+1$, $f(o_\ell(x_j)) = o_\ell(f(x_j)) = o_\ell(x_{i_j})$ for some i_j $(1 \leq i_j \leq |F|+1)$. Thus f is determined by the permutation

$$f_s = \begin{pmatrix} 1 & \cdots & |F|+1 \\ i_1 & \cdots & i_{|F|+1} \end{pmatrix} \in S_{|F|+1}.$$

Since $S_{|F|+1}$ is generated by the transpositions $s_2 = (1, 2), \ldots, s_{|F|+1} = (1, |F|+1)$, and each $f_{s_j} = (x_1, x_j)$, a transposition in S_X, is a graph automorphism in $\Gamma(R)$ by Corollary 3.7, f is generated by $f_{s_1}, \ldots, f_{s_{|F|+1}}$. Hence the map $\sigma : \mathrm{Aut}(\Gamma(R)) \longrightarrow S_{|F|+1}$ by $\sigma(f) = f_s$ is bijective. Also σ is a group homomorphism by observing that for all $s_i, s_j \in S_{|F|+1}$ $(i, j = 2, \ldots, |F| + 1)$, $(f_{s_i} \circ f_{s_j}) = f_{s_is_j}$. Therefore, $\mathrm{Aut}(\Gamma(R)) \simeq S_{p+1}$. □

Acknowledgements. The authors thank Prof. J. Park at Pusan National University for reading this paper and kind suggestions. The authors would like to thank the referee for a careful checking of the details and helpful comments about some references for making the paper more readable.

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