J. Korean Math. Soc. **48** (2011), No. 2, pp. 253–266
DOI 10.4134/JKMS.2011.48.2.253

STABLE MINIMAL HYPERSURFACES IN THE HYPERBOLIC SPACE

Keomkyo Seo

Abstract. In this paper we give an upper bound of the first eigenvalue of the Laplace operator on a complete stable minimal hypersurface M in the hyperbolic space which has finite L^2-norm of the second fundamental form on M. We provide some sufficient conditions for minimal hypersurface of the hyperbolic space to be stable. We also describe stability of catenoids and helicoids in the hyperbolic space. In particular, it is shown that there exists a family of stable higher-dimensional catenoids in the hyperbolic space.

1. Introduction

In [6], Cheng, Li, and Yau derived comparison theorems for the first eigenvalue of Dirichlet boundary problem on any compact domain in minimal submanifolds of the hyperbolic space by estimating the heat kernel of the compact domain. Recall that the first eigenvalue λ_1 of a complete non-compact Riemannian manifold M is defined by $\lambda_1 = \inf_\Omega \lambda_1(\Omega)$, where the infimum is taken over all compact domains in M. Throughout this paper, we shall denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space of constant sectional curvature -1. Recently Candel [2] gave an upper bound for the first eigenvalue of the universal cover of a complete stable minimal surface in $\mathbb{H}^3$. Indeed, he proved:

Theorem ([2]). *Let Σ be a complete simply connected stable minimal surface in the 3-dimensional hyperbolic space. Then the first eigenvalue of Σ satisfies*

$$\frac{1}{4} \leq \lambda_1(\Sigma) \leq \frac{4}{3}.$$

In Section 2, we extend this theorem to simply connected stable minimal surfaces in a Riemannian manifold whose sectional curvature is bounded below and above by negative constants (Theorem 2.1). For a complete stable minimal

Received September 2, 2009.
2010 *Mathematics Subject Classification.* 53C40, 53C42.
Key words and phrases. stable minimal hypersurface, hyperbolic space, first eigenvalue.
This research was supported by the Sookmyung Women's University Research Grants 2010.

hypersurface M in $\mathbb{H}^{n+1}$, Cheung and Leung [8] proved that

$$\frac{1}{4}(n-1)^2 \leq \lambda_1(M). \tag{1.1}$$

Here this inequality is sharp because equality holds when M is totally geodesic ([12]). In this paper, it is proved that if M is a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with finite L^2-norm of the second fundamental form A, then we have (Theorem 2.2)

$$\lambda_1(M) \leq n^2.$$

Recall that a minimal hypersurface is called *stable* if the second variation of its volume is always nonnegative for any normal variation with compact support. More precisely, an n-dimensional minimal hypersurface M in a Riemannian manifold N is called *stable* if it holds that for any compactly supported Lipschitz function f on M

$$\int_M |\nabla f|^2 - \Big(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\Big) f^2 dv \geq 0, \tag{1.2}$$

where ν is the unit normal vector of M, $\overline{\mathrm{Ric}}(\nu,\nu)$ denotes the Ricci curvature of N in the ν direction, $|A|^2$ is the square length of the second fundamental form A, and dv is the volume form for the induced metric on M. Note that when $N = \mathbb{H}^{n+1}$, $\overline{\mathrm{Ric}}(\nu,\nu)$ is equal to $-n$.

In Section 3, we give some conditions for complete minimal hypersurfaces in $\mathbb{H}^{n+1}$ to be stable as follows. If the L^∞-norm of the second fundamental form is sufficiently small at every point in a complete minimal hypersurface M, then M is stable (Theorem 3.1). Moreover if the L^n-norm of the second fundamental form is sufficiently small, then M is stable (Theorem 3.2).

In 1981, Mori [13] explicitly described a one-parameter family of complete stable minimal rotation surfaces in $\mathbb{H}^3$. This example shows that a theorem due to do Carmo and Peng [4] and Fischer-Colbrie and Schoen [10] which says that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane, does not hold in $\mathbb{H}^3$. Two years later, do Carmo and Dajczer [3] found a larger family of complete minimal rotation surfaces which are also stable. In Section 4, we study stability of catenoids in the hyperbolic space. In [3], it was shown that there exist a one-parameter family of unstable catenoids M_a in $\mathbb{H}^3$ for $1/2 < a < 0.69$. We improve the upper bound of a by estimating the L^2-norm of $|\nabla|A||$ in terms of the L^2-norm of the second fundamental form A (Theorem 4.1). We also prove that the above unstable catenoid in $\mathbb{H}^3$ should have index one (Theorem 4.3). Recall that for a compact subset Ω in a complete minimal hypersurface M in $\mathbb{H}^{n+1}$, the *index* of Ω is defined to be the number of negative eigenvalues of the stability operator $L := \Delta - |A|^2 + n$ on Ω, counting the multiplicity. The index of M is defined as the infimum of Index(Ω) for all compact subset Ω. Moreover we provide a family of complete minimal hypersurfaces in $\mathbb{H}^{n+1}$, which is an extension of Mori's result to higher dimensional cases (Theorem 4.4). Finally we investigate stability of helicoids in $\mathbb{H}^3$ in Section 5.

2. First eigenvalue estimates

In this section we first extend Candel's result to a simply connected complete minimal surface in a Riemannian manifold. The proof is actually based on Candel's proof.

Theorem 2.1. *Let Σ be a simply connected stable minimal surface in a 3-dimensional simply connected Riemannian manifold N^3 with sectional curvature K_N satisfying $-b^2 \leq K_N \leq -a^2 < 0$ for $0 < a \leq b$. Then the first eigenvalue of Σ satisfies*

$$\frac{1}{4}a^2 \leq \lambda_1(\Sigma) \leq \frac{4}{3}b^2.$$

Proof. First we find an upper bound for $\lambda_1(\Sigma)$. Let $\{e_1, e_2, e_3\}$ be orthonormal frames in N such that the vectors $\{e_1, e_2\}$ are tangent to M and e_3 is normal to M. The Gauss curvature equation implies that the sectional curvature K_Σ of Σ satisfies

$$\begin{aligned} K_\Sigma &= R^1_{212} + h_{11}h_{22} - h^2_{12} \\ &= R^1_{212} - \frac{|A|^2}{2} \leq -a^2 - \frac{|A|^2}{2} < 0, \end{aligned} \tag{2.1}$$

where R^1_{212} is the sectional curvature of N for the section determined by e_1, e_2 and $h_{ij} = \langle \bar{\nabla}_{e_i} e_3, e_j \rangle$, $\bar{\nabla}$ denoting Riemannian connection of N. Since Σ is simply connected and has negative Gaussian curvature, there are global polar coordinates about any point in Σ. Using this polar coordinates, the metric tensor g of Σ can be written as

$$g = dr^2 + \phi(r, \theta)^2 d\theta^2,$$

where $\phi(0, \theta) = 0$ and $\frac{\partial \phi}{\partial r}\Big|_{(0,\theta)} := \phi_r(0, \theta) = 1$.

Using the equality (2.1) and $\overline{\mathrm{Ric}}(e_3) = R^3_{131} + R^3_{232}$, the stability inequality (1.2) becomes

$$\begin{aligned} 0 &\leq \int_\Sigma |\nabla f|^2 - (|A|^2 + \overline{\mathrm{Ric}}(e_3)) f^2 dv \\ &\leq \int_\Sigma |\nabla f|^2 - (R^3_{131} + R^3_{232} + 2R^1_{212} - 2K_\Sigma) f^2 dv \\ &\leq \int_\Sigma |\nabla f|^2 + 2K_\Sigma f^2 + 4b^2 f^2 dv. \end{aligned} \tag{2.2}$$

Since the inequality (2.2) holds for all compactly supported Lipschitz function f on Σ, we shall choose some specific functions which depend only on the distance r to the origin of the polar coordinates in Σ. More precisely, given $R > 0$, we consider a family $\mathcal{F}$ of radial functions f such that $f(0) = 0$, $f(r) = 0$ for $r \geq R > 0$ and $f(r)$ is piecewise linear in r, that is, $f''(r) = 0$ except for finitely many values of r. Note that the Gaussian curvature $K_\Sigma = -\frac{\phi_{rr}}{\phi}$. Thus the inequality (2.2) can be written as

$$2\int_0^{2\pi}\int_0^R f^2\phi_{rr}drd\theta \le \int_0^{2\pi}\int_0^R f_r^2\phi drd\theta + 4b^2\int_0^{2\pi}\int_0^R f^2\phi drd\theta.$$

Integrating the left side of the above inequality twice by parts and using the properties of the function f, we obtain

$$\begin{aligned}
-\int_\Sigma K_\Sigma f^2 dv &= \int_0^{2\pi}\int_0^R f^2\phi_{rr}drd\theta \\
&= \int_0^{2\pi}\Big[f^2\phi_r\Big]_0^R d\theta - 2\int_0^{2\pi}\int_0^R ff_r\phi_r drd\theta \\
&= \int_0^{2\pi}\Big[-2ff_r\phi\Big]_0^R d\theta + 2\int_0^{2\pi}\int_0^R (ff_r)_r\phi drd\theta \\
&= 2\int_0^{2\pi}\int_0^R (f_r^2\phi + ff_{rr})\phi drd\theta \\
&= 2\int_0^{2\pi}\int_0^R f_r^2\phi drd\theta = 2\int_\Sigma |\nabla f|^2 dv.
\end{aligned} \tag{2.3}$$

Combining the inequality (2.2) with the equation (2.3), we get

$$3\int_\Sigma |\nabla f|^2 dv \le 4b^2\int_\Sigma f^2 dv.$$

Hence it follows that

$$\lambda_1(\Sigma) \le \inf_{f\in\mathcal{F}} \frac{\int_\Sigma |\nabla f|^2 dv}{\int_\Sigma f^2 dv} \le \frac{4}{3}b^2. \tag{2.4}$$

Now we estimate a lower bound of $\lambda_1(\Sigma)$. The Laplacian of the distance function r on $\Sigma \subset N$ satisfies [9]

$$\Delta r \ge a(2 - |\nabla r|^2)\coth ar \ge a.$$

Integrating both sides over $\Omega \subset \Sigma$, we get

$$a\text{Area}(\Omega) \le \int_\Omega \Delta r dv = \int_{\partial\Omega}\frac{\partial r}{\partial \nu}ds \le \text{Length}(\partial\Omega). \tag{2.5}$$

Recall that the *Cheeger constant* of a Riemannian manifold M, $h(M)$ is defined by

$$h(M) := \inf_\Omega \frac{\text{Length}(\partial\Omega)}{\text{Area}(\Omega)},$$

where Ω ranges over all open submanifold of M, with compact closure in M, and smooth boundary. Then applying Cheeger's inequality [5] and inequality (2.5), we obtain

$$\lambda_1(\Sigma) \ge \frac{1}{4}h(\Sigma)^2 = \frac{1}{4}a^2. \tag{2.6}$$

Therefore the theorem follows from (2.4) and (2.6). □

The first eigenvalue of a complete minimal hypersurface in the hyperbolic space is bounded below by a constant $\frac{(n-1)^2}{4}$ as mentioned in the introduction. We give an upper bound for a stable minimal hypersurface with finite L^2-norm of the second fundamental form of M.

Theorem 2.2. *Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^2 dv < \infty$. Then we have*

$$\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq n^2.$$

Remark. There is no nontrivial example of such complete minimal hypersurfaces in $\mathbb{R}^{n+1}$, since do Carmo and Peng [4] proved that a complete stable minimal hypersurface M in $\mathbb{R}^{n+1}$ with $\int_M |A|^2 dv < \infty$ must be a hyperplane. However, there exist several examples of complete minimal hypersurfaces with finite L^2-norm of the second fundamental form in the hyperbolic space as we will see in Sections 4 and 5. Note that we do not assume that M is simply connected, which is different from Candel's result.

Proof. It is sufficient to show that $\lambda_1(M) \leq n^2$ by the inequality (1.1).

Take a function f as follows. For a fixed point $p \in M$ and a fixed $R > 0$, define a function $f : M \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & r(x) \leq R, \\ 2 - \dfrac{r(x)}{R}, & R \leq r(x) \leq 3R, \\ -1, & 3R \leq r(x) \leq 4R, \\ -5 + \dfrac{r(x)}{R}, & 4R \leq r(x) \leq 5R, \\ 0, & r(x) \geq 5R, \end{cases}$$

where $r(x)$ is the distance from p to x in M. Then it follows that $\int_M f < 0$. For $0 \leq t \leq R$, we now consider a family of functions $\{f_t\}$ defined by

$$f_t(x) = \begin{cases} 1, & r(x) \leq R, \\ 2 - \dfrac{r(x)}{R}, & R \leq r(x) \leq 2R + t, \\ -\dfrac{t}{R}, & 2R + t \leq r(x) \leq 4R + t, \\ -5 + \dfrac{r(x)}{R}, & 4R + t \leq r(x) \leq 5R, \\ 0, & r(x) \geq 5R. \end{cases}$$

Then it is easy to see that there exists t_0, $0 < t_0 < R$, such that $\int_M f_{t_0} = 0$. From the definition of $\lambda_1(M)$ and $\lambda_1(B_R)$ for a ball B_R of radius R centered at p, it follows

$$\lambda_1(M) \leq \lambda_1(B_R) \leq \frac{\int_{B_R} |\nabla \phi|^2}{\int_{B_R} \phi^2} \tag{2.7}$$

for any compactly supported Lipschitz function ϕ satisfying $\int_{B_R} \phi = 0$.

Now put $|A|f_{t_0}$ for ϕ in the inequality (2.7). Then

$$\begin{aligned} &\lambda_1(M) \int_{B_R} |A|^2 f_{t_0}^2 dv \\ \leq& \int_{B_R} |\nabla(|A| f_{t_0})|^2 dv \\ =& \int_{B_R} |\nabla|A||^2 f_{t_0}^2 dv + \int_{B_R} |A|^2 |\nabla f_{t_0}|^2 dv + 2\int_{B_R} |A| f_{t_0} \langle \nabla|A|, \nabla f_{t_0}\rangle dv. \end{aligned}$$

Moreover, using Schwarz inequality, for any positive number $\alpha > 0$, we have

$$2\int_{B_R} |A| f_{t_0} \langle \nabla|A|, \nabla f_{t_0}\rangle dv \leq \alpha \int_{B_R} |A|^2 |\nabla f_{t_0}|^2 dv + \frac{1}{\alpha}\int_{B_R} |\nabla|A||^2 f_{t_0}^2 dv.$$

Therefore we obtain

(2.8)

$$\lambda_1(M) \int_{B_R} |A|^2 f_{t_0}^2 dv \leq (1+\frac{1}{\alpha}) \int_{B_R} |\nabla|A||^2 f_{t_0}^2 dv + (1+\alpha)\int_{B_R} |A|^2 |\nabla f_{t_0}|^2 dv.$$

On the other hand, Chern, do Carmo, and Kobayashi [7] showed that

$$\sum h_{ij}\Delta h_{ij} = -\sum h_{ij}^2 h_{kl}^2 - n \sum h_{ij}^2. \tag{2.9}$$

Furthermore, we have

$$|A|\Delta|A| + |\nabla|A||^2 = \frac{1}{2}\Delta|A|^2 = \sum h_{ijk}^2 + \sum h_{ij}\Delta h_{ij}. \tag{2.10}$$

Combining (2.9) with (2.10), we get

$$|A|\Delta|A| + |A|^4 + n|A|^2 = |\nabla A|^2 - |\nabla|A||^2.$$

However the curvature estimate by Xin [16] says that

$$|\nabla A|^2 - |\nabla|A||^2 \geq \frac{2}{n}|\nabla|A||^2,$$

and hence we have

$$|A|\Delta|A| + |A|^4 + n|A|^2 \geq \frac{2}{n}|\nabla|A||^2.$$

Multiplying both sides by a Lipschitz function f^2 with compact support in $B_R \subset M$ and integrating over B_R, we have

$$\int_{B_R} f^2|A|\Delta|A| dv + \int_{B_R} f^2|A|^4 dv + n\int_{B_R} f^2|A|^2 dv \geq \frac{2}{n}\int_{B_R} f^2|\nabla|A||^2 dv.$$

The divergence theorem yields that

$$\begin{aligned} 0 &= \int_{B_R} \operatorname{div}(|A| f^2 \nabla|A|) dv \\ &= \int_{B_R} f^2|A|\Delta|A| dv + \int_{B_R} |\nabla|A||^2 f^2 dv + 2\int_{B_R} |A| f \langle \nabla|A|, \nabla f\rangle dv. \end{aligned}$$

Therefore

$$\int_{B_R} f^2|A|^4 dv + n\int_{B_R} f^2|A|^2 dv - \int_{B_R} |\nabla|A||^2 f^2 dv - 2\int_{B_R} |A|f\langle\nabla|A|, \nabla f\rangle dv \tag{2.11}$$
$$\geq \frac{2}{n}\int_{B_R} f^2|\nabla|A||^2 dv.$$

Since M is stable, we have

$$\int_M |\nabla\phi|^2 - (|A|^2 - n)\phi^2 dv \geq 0$$

for any compactly supported function ϕ on M. Substituting $|A|f$ for ϕ gives

$$\int_{B_R} |\nabla(|A|f)|^2 - (|A|^2 - n)|A|^2 f^2 dv \geq 0.$$

Thus

$$\int_{B_R} |A|^2|\nabla f|^2 dv + \int_{B_R} |\nabla|A||^2 f^2 dv + 2\int_{B_R} |A|f\langle\nabla|A|, \nabla f\rangle dv \tag{2.12}$$
$$\geq \int_{B_R} |A|^4 f^2 dv - n\int_{B_R} |A|^2 f^2 dv.$$

By (2.11) and (2.12), we obtain, for any compactly supported Lipschitz function f

$$\int_{B_R} |A|^2|\nabla f|^2 dv + 2n\int_{B_R} |A|^2 f^2 dv \geq \frac{2}{n}\int_{B_R} |\nabla|A||^2 f^2 dv. \tag{2.13}$$

Combining (2.8) with the inequality obtained by substituting f_{t_0} for f in (2.13), we get

(2.14)

$$\left\{1 + \frac{2n(1+\alpha)}{\lambda_1(M)}\right\}\int_{B_R} |A|^2|\nabla f_{t_0}|^2 dv \geq \left\{\frac{2}{n} - \frac{2n(1+\frac{1}{\alpha})}{\lambda_1(M)}\right\}\int_{B_R} |\nabla|A||^2 f_{t_0}^2 dv.$$

Now suppose that $\lambda_1(M) > n^2$. Choosing $\alpha > 0$ sufficiently large and letting $R \to \infty$, we obtain $\nabla|A| \equiv 0$, i.e., $|A|$ is constant. However, since $\int_M |A|^2 < \infty$ and the volume of M is infinite, it follows from the above inequality (2.14) that $|A| \equiv 0$ which means that M is a totally geodesic hyperplane. Since the first eigenvalue of totally geodesic hyperplane in $\mathbb{H}^{n+1}$ is equal to $\frac{(n-1)^2}{4}$, this is a contradiction. Therefore we get $\lambda_1(M) \leq n^2$. □

3. Sufficient conditions for stability of minimal hypersurfaces in $\mathbb{H}^{n+1}$

In this section we prove that if $|A|$ is bounded by a sufficiently small constant at every point in a complete minimal hypersurface M in the hyperbolic space, then M must be stable. More precisely,

Theorem 3.1. *Let M be a complete minimal hypersurface in $\mathbb{H}^{n+1}$. If $|A| \leq \frac{(n+1)^2}{4}$ at every point in M, then M is stable.*

Proof. Since the lower bound of the first eigenvalue of M is $\frac{(n-1)^2}{4}$ by the inequality (1.1), we have

$$\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

for every compactly supported Lipschitz function f on M. Hence the assumption that $|A|^2 \leq \frac{(n+1)^2}{4}$ implies

$$\int_M |\nabla f|^2 - (|A|^2 - n) f^2 dv \geq \int_M (\lambda_1(M) + n - |A|^2) f^2 dv \geq 0,$$

which completes the proof. □

It is well-known that the following Sobolev inequality [11] on a minimal hypersurface M in $\mathbb{H}^{n+1}$ holds

$$\Big(\int_M |f|^{\frac{2n}{n-2}} dv\Big)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla f|^2 dv, \tag{3.1}$$

where C_s is the Sobolev constant which dependent only on $n \geq 3$. Using this inequality one obtains another sufficient condition for minimal hypersurfaces to be stable.

Theorem 3.2. *Let M be a complete minimal hypersurface in $\mathbb{H}^{n+1}$, $n \geq 3$. If $\int_M |A|^n dv \leq (\frac{1}{C_s})^{\frac{n}{2}}$, then M is stable.*

Proof. It suffices to show that

$$\int_M |\nabla f|^2 - (|A|^2 - n) f^2 dv \geq 0$$

for all compactly supported Lipschitz function f. By Sobolev inequality (3.1), we have

$$\int_M |\nabla f|^2 - (|A|^2 - n) f^2 dv \geq \frac{1}{C_s}\Big(\int_M |f|^{\frac{2n}{n-2}} dv\Big)^{\frac{n-2}{n}} - \int_M |A|^2 f^2 dv. \tag{3.2}$$

On the other hand, applying Hölder inequality, we get

$$\int_M |A|^2 f^2 dv \leq \Big(\int_M |A|^n dv\Big)^{\frac{2}{n}} \Big(\int_M |f|^{\frac{2n}{n-2}} dv\Big)^{\frac{n-2}{n}}. \tag{3.3}$$

Combining (3.2) with (3.3) we have

$$\begin{aligned} \int_M |\nabla f|^2 - (|A|^2 - n) f^2 dv &\geq \Big\{\frac{1}{C_s} - \Big(\int_M |A|^n dv\Big)^{\frac{2}{n}}\Big\}\Big(\int_M |f|^{\frac{2n}{n-2}} dv\Big)^{\frac{n-2}{n}} \\ &\geq 0, \qquad \text{(by assumption)} \end{aligned}$$

which completes the proof. □

4. Catenoids in $\mathbb{H}^{n+1}$

In [3], do Carmo and Dajczer proved that there exist three types of rotationally symmetric minimal hypersurfaces in $\mathbb{H}^{n+1}$. Following [3], we say that a rotationally symmetric minimal hypersurface M is a *spherical* catenoid, if M is foliated by spheres, a *hyperbolic* catenoid, if it is foliated by hyperbolic spaces, and a *parabolic* catenoid, if it is foliated by horospheres. Do Carmo and Dajczer showed that the complete hyperbolic and parabolic catenoids in $\mathbb{H}^3$ are all globally stable. Furthermore they also proved that there exist some unstable spherical catenoids in $\mathbb{H}^3$. In what follows, we shall denote by $\mathbb{L}^{n+1}$ the space of $(n+1)$-tuples $x = (x_1, \ldots, x_{n+1})$ with Lorentzian metric $\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1}$, where $y = (y_1, \ldots, y_{n+1})$. The hyperbolic space $\mathbb{H}^n$ is the simply connected hypersurface of $\mathbb{L}^{n+1}$ defined by $\mathbb{H}^n = \{x \in \mathbb{L}^{n+1} : \langle x, x\rangle = -1, x_1 \geq 1\}$.

To state their result for unstable spherical catenoids in $\mathbb{H}^3$, we parametrize a spherical catenoid in $\mathbb{H}^3$ as follows (See [3] and [13]). For each constant $a > 1/2$, define the mapping $f_a : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{H}^3$ by

$$f_a(s,\theta) = \Big(\sqrt{a\cosh(2s) + \frac{1}{2}}\cosh\phi(s), \sqrt{a\cosh(2s) + \frac{1}{2}}\sinh\phi(s),$$
$$\sqrt{a\cosh(2s) - \frac{1}{2}}\cos\theta, \sqrt{a\cosh(2s) - \frac{1}{2}}\sin\theta\Big),$$

where $\phi(s) = (a^2 - \frac{1}{4})^{1/2}\int_0^s \frac{1}{(a\cosh(2t)+\frac{1}{2})(a\cosh(2t)-\frac{1}{2})^{1/2}}dt.$

Do Carmo and Dajczer observed that if $1/2 < a < c_0, c_0 \simeq 0.69$, then the spherical catenoids M_a's are unstable. We shall improve the upper bound c_0 by using the inequality (2.13), which is different from their method. Letting $R \to \infty$ in (2.13), one can immediately obtain the following.

Theorem 4.1. *Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^2 dv < \infty$. Then we have*

$$\int_M |\nabla|A||^2 dv \leq n^2 \int_M |A|^2 dv,$$

and hence the L^2-norm of $|\nabla|A||$ is finite.

As a consequence of this Theorem 4.1, the upper bound c_0 due to do Carmo and Dajczer can be improved as follows.

Corollary 4.2. *Spherical catenoid M_a in $\mathbb{H}^3$ is unstable for* $1/2 < a < c_0,\ c_0 \simeq 0.73$.

Proof. We first observe that the spherical catenoid M_a satisfies $\int_{M_a} |A|^2 dv < \infty$. To see this, we note that for $a > 1/2$

$$I \quad = \quad ds^2 + (a\cosh 2s - \frac{1}{2})dt^2,$$

$$
\begin{aligned}
|A|^2 &= \frac{2(a^2-\frac{1}{4})}{(a\cosh 2s-\frac{1}{2})^2}, \\
dv &= (a\cosh 2s - \frac{1}{2})^{\frac{1}{2}} ds dt \quad \text{for } a > \frac{1}{2} \text{ and } 0 \le t \le 2\pi.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_M |A|^2 dv &= 8\pi(a^2 - \frac{1}{4}) \int_0^\infty \frac{1}{(a\cosh 2s - \frac{1}{2})^{\frac{3}{2}}} ds \\
&< 8\pi(a^2 - \frac{1}{4}) \int_0^\infty \frac{1}{(a + as^2 - \frac{1}{2})^{\frac{3}{2}}} ds < \infty.
\end{aligned}
$$

Now define a function $F(a)$ by

$$
F(a) := 4\int_M |A|^2 dv - \int_M |\nabla |A||^2 dv.
$$

Using $|\nabla|A|| = \Big| - \sqrt{2(a^2-\frac{1}{4})}\frac{2a\sinh 2s}{(a\cosh 2s - \frac{1}{2})^2}\Big|$, we have

$$
F(a) = 32\pi(a^2-\frac{1}{4})\int_0^\infty \Big\{ \frac{1}{(a\cosh 2s - \frac{1}{2})^{\frac{3}{2}}} - a^2 \frac{\sinh^2 2s}{(a\cosh 2s - \frac{1}{2})^{\frac{7}{2}}} \Big\} ds.
$$

By Theorem 4.1, we see that if M_a is stable for some a, then $F(a) \ge 0$. However a straightforward computation by using a computer shows that $F(a) < 0$ for $1/2 < a < c_0$, $c_0 \simeq 0.73$. Therefore we get the conclusion. □

As we have seen before, there exist some unstable catenoids in $\mathbb{H}^3$. Hence it is interesting to find the index of such catenoids which measures the degree of instability. It is well-known that catenoids have index 1 in $\mathbb{R}^3$. Very recently, Tam and Zhou [15] proved that higher dimensional catenoids in $\mathbb{R}^{n+1}$ with $n \ge 3$ have index one. Motivated by this, we shall prove the following result using the similar arguments as in [15].

Theorem 4.3. *Let M be a spherical catenoid in $\mathbb{H}^{n+1}$. Then the index of M is at most* 1.

Proof. We may assume that M is unstable. It suffices to show that the second eigenvalue $\lambda_2(D) \ge 0$ of the stability operator $L = \Delta + |A|^2 - n$ on some bounded domain $D \subset M$. We prove this theorem by contradiction. For this purpose, suppose that the index of M is at least 2. Then there exists a domain $D(R) = (-R, R) \times \mathbb{S}^{n-1}$ such that $\lambda_2(D(R)) < 0$ for $R > 0$.

Let f be the second eigenfunction satisfying

$$
\begin{aligned}
Lf &= -\lambda_2(D(R))f \quad \text{in} \quad D(R) \\
f &= 0 \quad \text{on} \quad \partial D(R).
\end{aligned}
$$

We claim that f is rotationally symmetric, that is, $f(t_1, \ldots, t_{n-1}, s) = f(s)$.

To see this, consider a generating curve $\alpha(s) := (x(s), y(s), z(s), 0, \ldots, 0) \subset \mathbb{H}^{n+1}$ and its rotation axis $\{(\cosh u, \sinh u, 0, \ldots, 0)\} \subset \mathbb{H}^{n+1}$. Let P_0 be the totally geodesic hyperplane such that $P_0 \perp \alpha'(0)$ and $\alpha(0) = (1, 0, \ldots, 0) \in P_0$. For any vector $v \in S_{\alpha(0)}P_0 := \{v \in T_{\alpha(0)}P_0 : |v| = 1\}$, denote by P_v the (unique) totally geodesic hyperplane such that $\alpha(0) \in P_v$ and $P_v \perp v$ at $\alpha(0)$.

Let σ_v be the reflection across the hyperplane P_v. For any point $p \in D(R)$, define the difference function $\varphi_v(t_1, \ldots, t_{n-1}, s)$ by

$$\varphi_v(t_1, \ldots, t_{n-1}, s) := f(t_1, \ldots, t_{n-1}, s) - f_v(t_1, \ldots, t_{n-1}, s),$$

where $f_v(p) := f(\sigma_v(p))$. Then it follows that $\Delta f = \Delta f_v$. Thus

$$\begin{cases} L\varphi_v & = -\lambda_2(D(R)) \quad \text{in} \quad D(R) \\ \varphi_v & = 0 \quad \text{on} \quad \partial D(R) \cap P_v. \end{cases} \tag{4.1}$$

Since P_v divides $D(R)$ into two parts, we choose one of them and denote by $D_v^+(R)$. Note that $D_v^+(R)$ is a minimal graph over a domain P_v. Hence $D_v^+(R)$ is stable. However from (4.1) and the assumption that $\lambda_2 < 0$, it follows that $\varphi_v \equiv 0$. As in the Euclidean space, any rotation around the axis $\{(\cosh u, \sinh u, 0, \ldots, 0)\} \subset \mathbb{H}^{n+1}$ can be expressed as a composition of finite number of reflections. Since v was arbitrarily chosen, the claim is obtained.

On the other hand, since the second eigenfunction of the operator L changes sign, there exists a number $r_0 \in (-R, R)$ satisfying $f(r_0) = 0$. We may assume that $r_0 \geq 0$ and the second eigenfunction $f(s) > 0$ on the domain $D(r_0, R) = \{(t_1, \ldots, t_{n-1}, s) \in D(R) : s \in (r_o, R)\}$. The function f is still an eigenfunction of L on $D(r_0, R)$. Moreover it is easy to see that $D(r_0, R)$ is a minimal graph over the hyperplane P_0, which means that $D(r_0, R)$ is stable. This is a contradiction to the assumption that $\lambda_2 < 0$. Therefore we get the conclusion. □

Remark. When $n = 2$, we observed that a spherical catenoid $M_a \subset \mathbb{H}^3$ is unstable if $1/2 < a < 0.73$ in Corollary 4.2. It follows from the above theorem that these spherical catenoids must have index 1.

We now describe stability of hyperbolic catenoids in the hyperbolic space $\mathbb{H}^{n+1}$. For that purpose, we give a parametrization of a hyperbolic catenoid generated by a curve $(x(s), y(s), z(s))$ in the hyperbolic plane $\mathbb{H}^2$ which is parametrized by arclength. It follows that

$$-x(s)^2 + y(s)^2 + z(s)^2 = -1, \qquad x(s) \geq 1, \tag{4.2}$$

$$-x'(s)^2 + y'(s)^2 + z'(s)^2 = 1, \tag{4.3}$$

$$\begin{gathered} f(t_1, \ldots, t_{n-1}, s) = (x(s)\varphi_1, \ldots, x(s)\varphi_n, y(s), z(s)), \\ \varphi_i = \varphi_i(t_1, \ldots, t_{n-1}), \quad -\varphi_1^2 + \varphi_2^2 + \cdots + \varphi_n^2 = -1, \end{gathered} \tag{4.4}$$

where $(\varphi_1, \ldots, \varphi_n)$ is an orthogonal parametrization of the hyperbolic space $\mathbb{H}^{n-1}$. From (4.2) and (4.3), $y(s)$ and $z(s)$ are determined by

$$\begin{aligned} y(s) &= \sqrt{x(s)^2 - 1}\sin\phi(s), \\ z(s) &= \sqrt{x(s)^2 - 1}\cos\phi(s), \end{aligned}$$

where $\phi(s) = \int_0^s \frac{\sqrt{x^2 - x'^2 - 1}}{x^2 - 1} dt$.

Using minimality and rotationally symmetric property of a catenoid, one can see that the direction of the parameters are principal directions and the principal curvatures are given by

$$\lambda_1 = \cdots = \lambda_{n-1} = -\frac{\sqrt{x^2 - x'^2 - 1}}{x},$$
$$\lambda_n = \frac{x'' - x}{\sqrt{x^2 - x'^2 - 1}} = (n-1)\frac{\sqrt{x^2 - x'^2 - 1}}{x}$$

(See [3, Proposition 3.2]). Furthermore we can write down an ordinary differential equation as follows [3, Lemma 3.15]:

$$x' = \sqrt{x^2 - 1 - a^2 x^{2(1-n)}}, \qquad a = \text{const.} \tag{4.5}$$

To find a unique solution of (4.5), we fix initial data as follows:

$$\begin{aligned} x(0) &= t \geq 1, \\ x'(0) &= 0. \end{aligned}$$

Then from the initial data it follows that

$$a = t^{n-1}\sqrt{t^2 - 1} \geq 0. \tag{4.6}$$

Moreover in order to have a nontrivial parametrization of a hyperbolic catenoid we see that $a > 0$. Therefore for each constant $t > 1$ the parametrization $f(t_1, \ldots, t_{n-1}, s)$ defines a hyperbolic catenoid M_t in $\mathbb{H}^{n+1}$. We now state a our result about stability of hyperbolic catenoids in the hyperbolic space.

Theorem 4.4. *Let M_t be a family of hyperbolic catenoids in $\mathbb{H}^{n+1}$ defined as in* (4.4). *Then M_t is a complete stable hypersurface in $\mathbb{H}^{n+1}$ for $1 < t < 1 + \frac{(n+1)^2}{4n(n-1)}$.*

Proof. Observe that

$$\begin{aligned} |A|^2 &= \sum \lambda_i^2 = (n-1)\lambda_1^2 + \lambda_n^2 = (n-1)\lambda_1^2 + (n-1)^2\lambda_1^2 \\ &= n(n-1)\lambda_1^2 \\ &= n(n-1)\frac{x^2 - x'^2 - 1}{x^2} \\ &= n(n-1)\frac{a^2}{x^{2n}} \qquad \text{(by (4.5))} \\ &= n(n-1)\frac{t^{2(n-1)}(t^2 - 1)}{x^{2n}}. \qquad \text{(by (4.6))} \end{aligned}$$

Since $x(s)$ is monotonically increasing by (4.5), we get $x(s) \geq x(0) = t > 1$. Therefore $|A|^2 \leq n(n-1)(t^2-1)$. The assumption on t implies that $|A|^2 \leq \frac{(n+1)^2}{4}$. The conclusion follows from Theorem 3.1. □

Remark. It is not hard to see that the family $\{M_t\}$ of hyperbolic catenoids in the above theorem satisfy $\int_{M_t} |A|^2 dv < \infty$ by using $\sqrt{x^2 - 1 - a^2} < x' < \sqrt{x^2 - 1}$, which is obtained from the equality (4.5) and the fact that $x > 1$.

5. Helicoids in $\mathbb{H}^3$

Let l be a geodesic in $\mathbb{H}^3$. Let $\{\psi_t\}$ be the translation of distance t along l and let $\{\varphi_t\}$ be the rotation of angle t around l. Given any $\alpha \in \mathbb{R}$, one can see that $\lambda = \{\lambda_t\} = \{\psi_t \circ \varphi_{\alpha t}\}$ is a one-parameter subgroup of isometries of $\mathbb{H}^3$ which is called a *helicoidal* group of isometries with angular pitch α. A *helicoid* in $\mathbb{H}^3$ is a λ-invariant surface (See [14]). In 1989, Ripoll [14] proved that a helicoid M_α with angular pitch $|\alpha| < 1$ is stable by showing that such M_α foliates $\mathbb{H}^3$. In this section, we improve the upper bound of angular pitch $|\alpha|$ by simple arguments.

A helicoid $M_\alpha \subset \mathbb{H}^3 \subset \mathbb{L}^4$ can be written explicitly as follows [1] :

$$X(s,t) = (\cosh s \cosh t, \sinh s \cosh t, \cos \alpha s \sinh t, \sin \alpha s \sinh t).$$

A little computation shows that the first and second fundamental forms of M_α are given by

$$\begin{aligned} I &= (\cosh^2 t + \alpha^2 \sinh^2 t) ds^2 + dt^2, \\ II &= -2 \frac{\alpha}{\sqrt{\cosh^2 t + \alpha^2 \sinh^2 t}}\, ds dt. \end{aligned}$$

Since $\cosh^2 t + \alpha^2 \sinh^2 t \geq 1$, it follows

$$|A|^2 = \frac{\alpha^2}{\cosh^2 t + \alpha^2 \sinh^2 t} + \frac{\alpha^2}{(\cosh^2 t + \alpha^2 \sinh^2 t)^3} \leq 2\alpha^2.$$

The following theorem is an immediate consequence of Theorem 3.1.

Theorem 5.1. *A helicoid M_α with angular pitch $|\alpha|^2 \leq \frac{9}{8}$ is stable.*

References

[1] J. Barbosa, M. Dajczer, and L. Jorge, *Minimal ruled submanifolds in spaces of constant curvature*, Indiana Univ. Math. J. **33** (1984), no. 4, 531–547.

[2] A. Candel, *Eigenvalue estimates for minimal surfaces in hyperbolic space*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3567–3575.

[3] M. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. **277** (1983), no. 2, 685–709.

[4] M. do Carmo and C. K. Peng, *Stable complete minimal hypersurfaces*, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), 1349–1358, Science Press, Beijing, 1982.

[5] I. Chavel, *Isoperimetric Inequalities*, Cambridge Tracts in Mathematics, 145. Cambridge University Press, Cambridge, 2001.

[6] S. Y. Cheng, P. Li, and S.-T. Yau, *Heat equations on minimal submanifolds and their applications*, Amer. J. Math. **106** (1984), no. 5, 1033–1065.

[7] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, 1970 Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968) pp. 59–75 Springer, New York.

[8] L. F. Cheung and P. F. Leung, *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*, Math. Z. **236** (2001), no. 3, 525–530.

[9] J. Choe, *The isoperimetric inequality for minimal surfaces in a Riemannian manifold*, J. Reine Angew. Math. **506** (1999), 205–214.

[10] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211.

[11] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727.

[12] H. P. McKean, *An upper bound to the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.

[13] H. Mori, *Minimal surfaces of revolution in H^3 and their global stability*, Indiana Univ. Math. J. **30** (1981), no. 5, 787–794.

[14] J. Ripoll, *Helicoidal minimal surfaces in hyperbolic space*, Nagoya Math. J. **114** (1989), 65–75.

[15] L.-F. Tam and D. Zhou, *Stability properties for the higher dimensional catenoid in $\mathbb{R}^{n+1}$*, Proc. Amer. Math. Soc. **137** (2009), no. 10, 3451–3461.

[16] Y. L. Xin, *Bernstein type theorems without graphic condition*, Asian J. Math. **9** (2005), no. 1, 31–44.

Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail address: kseo@sookmyung.ac.kr