JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 3, September 2011

COMPLETION OF STOCHASTIC HANKEL PARTIAL CONTRACTIONS OF EXTREMAL TYPE

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Abstract. We find concrete necessary and sufficient conditions for the existence of contractive completions of Stochastic Hankel partial contractions of size 4×4, extremal type.

1. Introduction

For 2×2 operator matrices (with no required Hankel condition), a solution to the completion problem

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}$$

has been given by G. Arsene and A. Gheondea [1], by C. Davis, W. Kahan and H. Weinberger [8] (see also [7] and [3]), by C. Foiaş and A. Frazho [9] (using Redheffer products), by S. Parrott [13], and by Y. L. Shmul'yan and R. N. Yanovskaya [15]; a solution is also implicit in the work of W. Arveson [2] (see also [14] and [11]).

A *Hankel matrix* is a square matrix with constant skew-diagonals. A *stochastic matrix* (or *transition matrix*) is a matrix used to describe the transitions of a Markov chain and whose rows and columns consists of nonnegative real numbers, with each row summing to 1 and each column summing to 1, respectively. A *Hankel stochastic partial contraction* is a Hankel stochastic matrix such that not all of its entries are determined, but in which every well-defined submatrix is a contraction. In this article, we study the problem of whether a Hankel stochastic partial

Received July 06, 2011; Accepted August 25, 2011.
2010 Mathematics Subject Classification: Primary 47B35, 15A60, 15A48, 15A83; Secondary 47A30, 15-04.
Key words and phrases: Hankel partial contraction, contractive completion, stochastic matrix .
This work was supported by Institute of Natural Science of SangMyung University in 2010.

contraction in which the upper left triangle is known can be completed to a contraction. Given real numbers $a_1, \cdots, a_n$, let

(1.1)

$$SH \equiv SH(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1}) := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix}$$

be a Hankel stochastic matrix, where $x_1, \cdots, x_{n-1}$ are real numbers to be determined. We say that SH is a *partial contraction* if all completely determined submatrices of SH are contractions (in the sense that their operator norms are at most 1).

PROBLEM 1.1. Given real numbers $a_1, a_2, \cdots, a_n$, find real numbers $x_1, \cdots, x_{n-1}$ such that

$$SH(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$$

is contractive.

We say that Problem 1.1 is *well-posed* if

$$SH(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$$

is partially contractive, and that it is *soluble* if

$$SH(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$$

is contractive for some $x_1, \cdots, x_{n-1}$. We also say that $H(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$ is *extremal* if $a_1^2 + \cdots + a_n^2 = 1$.

In [5, Section 4], the authors find necessary and sufficient conditions for the existence of contractive completions of Hankel partial contractions for extremal type of 4×4 case. In this paper, we find completely necessary and sufficient conditions for the existence of contractive completions of 4×4 Stochastic Hankel partial contractions for extremal type of 4×4 case by using a new technique, that is, the Moore-Penrose inverse of a matrix.

2. Some technical lemmas

For the reader's convenience, in this section we gather several auxiliary results which are needed for the proofs of the main results in this article. First, we begin by recalling that an $n \times n$ matrix M is a contraction if and only if the matrix

$$P \equiv P(M) := I - MM^*$$

is positive semi-definite (in symbols, $P \geq 0$), where I is the identity matrix and M^* is the adjoint of M. In order to check the positivity of P, we use the following version of Choleski's Algorithm.

Lemma 2.1. ([4]) *Assume that*

$$P = \begin{pmatrix} u & \mathbf{t} \\ \mathbf{t}^* & P_0 \end{pmatrix},$$

where P_0 is an $(n-1) \times (n-1)$ matrix, $\mathbf{t}$ is a row vector, and u is a real number.
(i) If P_0 is invertible, then $\det P = \det P_0(u - \mathbf{t}P_0^{-1}\mathbf{t}^*)$.
(ii) If P_0 is invertible and positive, then $P \geq 0 \Leftrightarrow (u - \mathbf{t}P_0^{-1}\mathbf{t}^*) \geq 0 \Leftrightarrow$ $\det P \geq 0$.
(iii) If $u > 0$ then $P \geq 0 \Leftrightarrow P_0 - \mathbf{t}^* u^{-1}\mathbf{t} \geq 0$.
(iv) If $P \geq 0$ and $p_{ii} = 0$ for some i, $1 \leq i \leq n$, then $p_{ij} = p_{ji} = 0$ for all $j = 1, \cdots, n$.

We recall that for a $m \times n$ matrix A, a *Moore-Penrose inverse* of A is defined as a matrix as a $n \times m$ matrix $A^\dagger$ satisfying all of the following four creteria:
$AA^\dagger A = A$; $A^\dagger AA^\dagger = A^\dagger$; $\left(AA^\dagger\right)^* = AA^\dagger$ ($AA^\dagger$ is Hermitian); $\left(A^\dagger A\right)^* = A^\dagger A$ ($A^\dagger A$ is Hermitian).

The following result is a special form of Smul'jan's Lemma [16].

Lemma 2.2. ([16]) *Let* $P \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a finite matrix. Then $P \geq 0$ if and only if the following conditions hold:*
(i) $A \geq 0$;
(ii) ran $B \subseteq$ ran A; and
(iii) $C \geq B^* A^\dagger B$, where $A^\dagger$ is a Moore-Penrose inverse of A.

Lemma 2.3. (cf. [8], [13]) *If* $\begin{pmatrix} A \\ C \end{pmatrix}$ *and* $\begin{pmatrix} A & B \end{pmatrix}$ *are rectangular contractions, then there exists a matrix D such that the matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a contraction as well.*

For the study of the Stochastic Hankel contractive completions of the case of 3×3 matrices, we let

$$SH_3 := SH(a, b, c; x, y) = \begin{pmatrix} a & b & c \\ b & c & x \\ c & x & y \end{pmatrix} \in M_3(\mathbb{R}) \ \ (0 \leq a, b, c \leq 1)$$

and

$$P(SH_3)(x,y) = I - SH_3\,(SH_3)^* \equiv (p_{ij})_{i,j=1}^{3}$$
$$= \begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cx & -ac-bx-cy \\ -ab-bc-cx & 1-b^2-c^2-x^2 & -bc-cx-xy \\ -ac-bx-cy & -bc-cx-xy & 1-c^2-x^2-y^2 \end{pmatrix}.$$

We also let

$$P_{22} := I - \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1-a^2-b^2 & -ab-bc \\ -ab-bc & 1-b^2-c^2 \end{pmatrix}$$

and

$$\begin{aligned} P_{23}\,(x): \ &= \ I - \begin{pmatrix} a & b & c \\ b & c & x \end{pmatrix}\begin{pmatrix} a & b \\ b & c \\ c & x \end{pmatrix} \\ &= \begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cx \\ -ab-bc-cx & 1-b^2-c^2-x^2 \end{pmatrix}. \end{aligned}$$

3. Partially contractive stochastic Hankel matrices: the case 3×3

Since $\|S\| \leq \|T\|$ if S is a submatrix of the matrix T, it follows that each submatrix of a contraction is again a contraction. Thus, a necessary condition for a partial matrix T to be contraction is that each submatrix must be a contraction. We call a partial matrix meeting this necessary condition a *partial contraction* (*well-posed condition*).

We begin with known results for the reader's convenience. In Theorem 3.2 we formulate an improved version of a result in [5, Theorem 3.2] using the Moore-Penrose inverse of a matrix in Lemma 2.2. Theorem 3.2 is a crucial tool to prove our main results.

PROPOSITION 3.1. *Let* $SH_2 := SH(a,b;x) = \begin{pmatrix} a & b \\ b & x \end{pmatrix} \in M_2(\mathbb{R})$. *If* SH_2 *is well-posed then* SH_2 *admits a contractive completion. Moreover,* $x = a$.

Proof. Since SH_2 is a stochastic matrix, we must choose $x = a$. A straightforward calculation shows that

$$\|SH_2\| \leq 1 \Longleftrightarrow I - SH_2\,(SH_2)^* = \begin{pmatrix} 1-a^2-b^2 & -2ab \\ -2ab & 1-a^2-b^2 \end{pmatrix} \geq 0.$$

Since SH_2 is a stochastic matrix, we have $ab = 0$ and $a + b = 1$. Thus we have the desired result. $\square$

THEOREM 3.2. *Assume that $SH_3 = SH(a, b, c, x; y)$ is well-posed. Then SH_3 admits a contractive completion; in particular, $x = a$ and $y = b$.*

Proof. Since SH_3 is a stochastic matrix, we must choose $x = a$ and $y = b$. By Lemma 2.3, observe that SH_3 has a contractive completion if and only if $P(SH_3)(a, b) \geq 0$.
Case 1: $\{a, b, c\}$ is extremal. Since SH_3 is a stochastic matrix, we have $a + b + c = a^2 + b^2 + c^2$ and $ab + bc + ca = 0$. Thus we have

$$P(SH_3)(a, b) \geq 0.$$

Case 2: $\{a, b, c\}$ is not extremal. Note that

$$P(SH_3)(a, b) \geq 0 \Longleftrightarrow D \equiv \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \geq 0,$$

where $d_{11} := \frac{(1-a^2-b^2-c^2-(ab+bc+cd))(1-a^2-b^2-c^2+(ab+bc+cd))}{1-a^2-b^2-c^2}$,
$d_{12} = d_{21} := -\frac{(ab+bc+cd)(1-a^2-b^2-c^2+(ab+bc+cd))}{1-a^2-b^2-c^2}$ and
$d_{22} := \frac{(1-a^2-b^2-c^2-(ab+bc+cd))(1-a^2-b^2-c^2+(ab+bc+cd))}{1-a^2-b^2-c^2}$.

Since $a^2 + b^2 + c^2 < 1$ and SH_3 is a stochastic matrix, observe that $ab + bc + ca \neq 0$.
If $1 - a^2 - b^2 - c^2 = (ab + bc + cd)$. Then $P(SH_3)(a, b) \geq 0$.

If $1 - a^2 - b^2 - c^2 = -(ab + bc + cd)$. Then we have $ab + bc + ca = 0$ which drives a contradiction to the assumption.

If $1 - a^2 - b^2 - c^2 > (ab + bc + cd)$. Then we have

$$P(SH_3)(a, b) \geq 0 \Longleftrightarrow \det D \geq 0.$$

A direct calculation shows that $\det D = 0$. Thus we have the desired result. □

4. Partially contractive stochastic Hankel matrices of extremal type: the case 4×4

We now focus attention on the extremal case for 4×4 Hankel matrices, i.e., $a^2 + b^2 + c^2 + d^2 = 1$. Consider the solubility of Problem 1.1 for a Stochastic Hankel matrix $SH_4 := SH(a, b, c, d; x, y, z)$, which is well-posed and with $\{a, b, c, d\}$ extremal.

THEOREM 4.1. *Assume that SH_4 is well-posed. Then SH_4 admits a contractive completion if and only if $ac + bd = 0$. In particular, $x = a$, $y = b$ and $z = c$.*

Proof. Since SH_4 is a stochastic matrix, we must choose $x = a$, $y = b$ and $z = c$. By Lemma 2.3, observe that SH_4 has a contractive completion if and only if $P(SH_4)(a, b, c) \geq 0$. By a direct calculation, we have

(4.1)

$$P(SH_4)(a, b, c)$$

$$= \begin{pmatrix} p_{11} & -(a+c)(b+d) & -2(ac+bd) & -(a+c)(b+d) \\ -(a+c)(b+d) & p_{11} & -(a+c)(b+d) & -2(ac+bd) \\ -2(ac+bd) & -(a+c)(b+d) & p_{11} & -(a+c)(b+d) \\ -(a+c)(b+d) & -2(ac+bd) & -(a+c)(b+d) & p_{11} \end{pmatrix},$$

where $p_{11} := 1 - a^2 - b^2 - c^2 - d^2$. Since SH_4 is a stochastic matrix, we have $a + b + c + d = a^2 + b^2 + c^2 + d^2$ and $ab + bc + cd + da = 0$.

$(\Longrightarrow)$: We assume that SH_4 admits a contractive completion. Since $\{a, b, c, d\}$ is extremal. Then we have $p_{11} = 0$ and $ac + bd = 0$. Thus we have $P(SH_4)(a, b, c) \geq 0$.

$(\Longleftarrow)$: We assume that $ac + bd = 0$. Then by (4.1), SH_4 admits a contractive completion. $\square$

References

[1] G. Arsene and A. Gheondea, Completing matrix contractions, *J. Operator Th.* **7** (1982), 179-189.

[2] W. Arveson, Interpolation problems in nest algebra, *J. Funct. Anal.* **20** (1975), 208-233.

[3] M. G. Crandall, Norm preserving extensions of linear transformations on Hilbert space, *Proc. Amer. Math. Soc.* **21** (1969), 335-340.

[4] R. Curto, C. Hernández and E. de Oteyza, Contractive completions of Hankel partial contractions, *J. Math. Anal. Appl.* **203** (1996), 303-332.

[5] R. Curto, S. H. Lee and J. Yoon, Completion of Hankel partial contractions of extremal type, *(preprint 2011).*

[6] R. E. Curto and W. Y. Lee, Joint hyponormality of Toeplitz pairs, *Memoirs Amer. Math. Soc.*, vol. 712, 2001.

[7] C. Davis, An extremal problem for extensions of a sesquilinear form, *Linear Alg. Appl.* **13** (1976), 91-102.

[8] C. Davis, W. M. Kahan, and H. F. Weinberger, Norm-preserving dilations and their application to optimal error bounds, *SIAM J. Numer. Anal* **19** (1982), 445-469.

[9] C. Foiaş and A. E. Frazho, Redheffer products and the lifting of contractions on Hilbert space, *J. Operator Th.* **11** (1984), 193-196.

[10] I. S. Iohvidov, *Hankel and Toeplitz Matrices and Forms: Algebraic Theory*, Birkhäuser-Verlag, Boston, 1982.

[11] C. R. Johnson and L. Rodman, Completion of partial matrices to contractions, *J. Funct. Anal.* **69** (1986), 260-267.

[12] C. R. Johnson and L. Rodman, Completion of Toeplitz partial contractions, *SIAM J. Matrix Anal. Appl.* **9** (1988), no. 2, 159-167.

[13] S. Parrott, On a quotient norm and Sz.-Nagy-Foiaş lifting theorem, *J. Funct. Anal.* **30** (1978), 311-328.

[14] S. Power, The distance to upper triangular operators, *Math. Proc. Cambridge Phil. Soc.* 88(198), 327-329.

[15] Y. L. Shmul'yan and R. N. Yanovskaya, Blocks of a contractive operator matrix (Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.* 25(1981), 72-75; (English translation) *Soviet Math. (Iz. VUZ)* **25** (1981), 82-86.

[16] J. L. Smul'jan, An operator Hellinger integral, *Mat. Sb. (N.S.)* **49** (1959), 381–430 (in Russian).

[17] H. J. Woerdeman, Strictly contractive and positive completions for block matrices, *Linear Alg. and Its Appl.* **136** (1990), 63-105.

[18] Wolfram Research, Inc., *Mathematica*, Version 4.2, *Wolfram Research Inc.*, Champaign, IL, 2002.

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