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FRACTIONAL GAGLIARDO-NIRENBERG INEQUALITY

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ABSTRACT. A fractional Gagliardo-Nirenberg inequality is established. A sharp fractional Sobolev inequality is discussed as a direct consequence.

1. Introduction

Fractional differential equations have been of increasing importance for the past decades due to their diverse applications in science and engineering. Some of such areas are fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport, material viscoelastic theory, electromagnetic theory, optics and signal processing, bio-sciences, economics, geology, and astrophysics. Fractional integral inequalities certainly provide fundamental tools for the study of fractional differential equations and optimizing problems including fractional Laplacian.

In this paper, we establish a fractional version of Gagliardo-Nirenberg inequality(Theorem 2.1). We make an effort to deliver a *simple and direct proof* in order to present a useful upper bound of the constant based on the effective upper bound of the constant in Hardy-Littlewood-Sobolev inequality. In fact, when we set up the fractional Gagliardo-Nirenberg inequalities, we found Wang's preprint [9] in which fractional Gagliardo-Nirenberg inequalities have been built via Littlewood-Paley decomposition. We present a totally different way of proof - we exhibit a self-contained simple and elegant proof, and provide a *useful upper bound* for the constant which was not accessible in Wang's proof. This upper bound plays an important role in actual applications [5]. A long

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standing problem of the optimality of the constants in the (classical) Gagliardo-Nirenberg inequalities is still open. Using this line of the proof, however, we could establish a *sharp* form of a fractional Sobolev inequality(Corollary 2.2).

Among some equivalent definitions of the fractional Laplacian, we employ it as

$$\sqrt{-\Delta}^s \phi := \mathcal{F}^{-1}(|\cdot|^s \mathcal{F}(\phi)),$$

where $\hat{u} = \mathcal{F}(u)$ represents the Fourier transform of u on $\mathbb{R}^n$ defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$$

for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

2. Fractional Gagliardo-Nirenberg inequality

Gagliardo-Nirenberg inequality for fractional Laplacian is presented, and a sharp form of the fractional Sobolev inequality is obtained as a corollary.

THEOREM 2.1. *Let $m, q, \theta \in \mathbb{R} \setminus \{0\}$ with $q \neq m\theta > 0$, $0 < s < n$, $1 < p < \frac{n}{s}$ and $1 < \frac{r}{q-m\theta}$. Then the inequality*

(2.1)

$$\int_{\mathbb{R}^n} |u(x)|^q dx \le C_0 \left(\int_{\mathbb{R}^n} |\sqrt{-\Delta}^s u(x)|^p dx \right)^{\frac{m\theta}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^r dx \right)^{\frac{q-m\theta}{r}}$$

holds for the indices with the relation

$$m\theta \left(\frac{1}{p} - \frac{s}{n} \right) + \frac{q-m\theta}{r} = 1. \tag{2.2}$$

The sharp constant C_0 satisfies

$$C_0^{\frac{1}{m\theta}} \le \frac{2^{1-s}}{\pi^{\frac{s}{2}}} \frac{\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{\frac{s}{n}} \Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{1-\frac{1}{p}+\frac{s}{n}}{sp} \left(\left[\frac{1-\frac{s}{n}}{1-\frac{1}{p}} \right]^{1-\frac{s}{n}} + \left[\frac{1-\frac{s}{n}}{\frac{1}{p}-\frac{s}{n}} \right]^{1-\frac{s}{n}} \right).$$

In particular, when $m = q$, we have a fractional version of Gagliardo-Nirenberg inequality:

(2.3)

$$\left(\int_{\mathbb{R}^n} |u(x)|^q dx \right)^{\frac{1}{q}} \le C_0^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |\sqrt{-\Delta}^s u(x)|^p dx \right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^r dx \right)^{\frac{1-\theta}{r}},$$

where the indices satisfy the relation

$$\theta\left(\frac{1}{p}-\frac{s}{n}\right)+\frac{1-\theta}{r}=\frac{1}{q}. \tag{2.4}$$

Proof. For convenience, we use the notation $\|u\|_{L^t} := \left(\int_{\mathbb{R}^n}|u(x)|^t dx\right)^{\frac{1}{t}}$ for *any* $t \in \mathbb{R}\backslash\{0\}$. First we point out by the standard *dilation argument* that the index relation (2.2) is necessary. In fact, by replacing $u(\cdot)$ with $u(\delta\ \cdot)$, we can observe that

$$\delta^{-n}\|u\|_{L^q}^q \le C_0\delta^{(s-\frac{n}{p})m\theta+\frac{n(m\theta-q)}{r}}\|\sqrt{-\Delta}^s u\|_{L^p}^{m\theta}\|u\|_{L^r}^{q-m\theta},$$

for all $\delta > 0$, which forces the relation (2.2).

Now, for any $u \in \mathcal{S}(\mathbb{R}^n)$(the Schwartz class), we have

$$\begin{aligned}\int_{\mathbb{R}^n}|u(x)|^q dx &= \int_{\mathbb{R}^n}|u(x)|^{m\theta}|u(x)|^{q-m\theta}dx\\ &\le \||u|^{m\theta}\|_{L^{\bar{p}}}\||u|^{q-m\theta}\|_{L^{\bar{r}}}, \qquad \frac{1}{\bar{p}}+\frac{1}{\bar{r}}=1\\ &= \|u\|_{L^{m\theta\bar{p}}}^{m\theta}\|u\|_{L^{(q-m\theta)\bar{r}}}^{q-m\theta}.\end{aligned} \tag{2.5}$$

We set $m\theta\bar{p} := p_0$ and $(q-m\theta)\bar{r} := r$ to have

$$\int_{\mathbb{R}^n}|u(x)|^q dx \le \|u\|_{L^{p_0}}^{m\theta}\|u\|_{L^r}^{q-m\theta}, \tag{2.6}$$

and $\frac{m\theta}{p_0}+\frac{q-m\theta}{r}=1$. Let $\sqrt{-\Delta}^s u := f$, and we have

$$\begin{aligned}u(x) &= \frac{1}{2^s\pi^{n/2}}\frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-s}}dy\\ &= \frac{1}{2^s\pi^{n/2}}\frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})}\int_{\mathbb{R}^n}\frac{\sqrt{-\Delta}^s u(y)}{|x-y|^{n-s}}dy.\end{aligned}$$

Indeed, we may take the Fourier transform on $\sqrt{-\Delta}^s u = f$, and take it back to have u after solving for $\widehat{u}$. Therefore Hardy-Littlewood-Sobolev inequality yields

$$\|u\|_{L^{p_0}} \le \frac{1}{2^s\pi^{n/2}}\frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})}C_1\left\|\sqrt{-\Delta}^s u\right\|_{L^p}, \tag{2.7}$$

where C_1 is a positive constant(see the comments after the proof) and p satisfies

$$\frac{1}{p}+\frac{n-s}{n}=1+\frac{1}{p_0}. \tag{2.8}$$

This index relation combined with the index relation appeared at (2.6) implies (2.2), and (2.6) together with (2.7) implies (2.1). □

The extremals and the best constant C_1 of Hardy-Littlewood-Sobolev inequality for some special cases are known (see [7] or Section 4.3 in [8]). Thanks to those cases, we have a sharp form of a fractional Sobolev inequality:

Corollary 2.2 (Fractional Sobolev inequality). *For* $0 < s < n$, $1 < p < \frac{n}{s}$ *and* $q = \frac{np}{n-ps}$, *we have*

$$\|u\|_{L^q} \leq C_0^{1/q} \left\| \sqrt{-\Delta}^{\,s} u \right\|_{L^p}.$$

The sharp constant for the inequality for the case $p = \frac{2n}{n+s}$ *and* $q = \frac{2n}{n-s}$ *is*

$$\left[\frac{\pi^{\frac{n}{2}-s}}{2^s} \frac{\Gamma(\frac{n}{q})}{\Gamma(\frac{n}{p})} \left\{ \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right\}^{s/n} \right]^{1/q}.$$

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