JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 3, September 2011

EQUIVARIANT VECTOR BUNDLES AND CLASSIFICATION OF NONEQUIVARIANT VECTOR ORBIBUNDLES

Min Kyu Kim*

Abstract. Let a finite group R act smoothly on a closed manifold M. We assume that R acts freely on M except a union of closed submanifolds with codimension at least two. Then, we show that there exists an isomorphism between equivariant topological complex vector bundles over M and nonequivariant topological complex vector orbibundles over the orbifold M/R. By using this, we can classify nonequivariant vector orbibundles over the orbifold especially when the manifold is two-sphere because we have classified equivariant topological complex vector bundles over two sphere under a compact Lie group (not necessarily effective) action in [6]. This classification of orbibundles conversely explains for one of two exceptional cases of [6].

1. Introduction

Let M be a closed smooth manifold and R be a finite group acting smoothly on M. If M is orientable, let $R_{\text{rot}} \subset R$ be the subgroup of elements preserving an orientation of M. If the action is free, then there exists an isomorphism between equivariant vector bundles over M and nonequivariant vector bundles over M/R [1, Proposition 1.6.1], [11, p. 132]. This classical result can be slightly generalized. For this, we introduce some notations and an assumption on group action. Assume that the R-action satisfies the following condition:

A1. R acts freely on M except a union of submanifolds with codimension at least two.

Received July 04, 2011; Accepted August 13, 2011.
2010 Mathematics Subject Classification: Primary 57S25, 55P91; Secondary 20C99.
Key words and phrases: equivariant vector bundle, orbibundle.

We remark that the action satisfying Condition A1. is not necessarily orient preserving. By the assumption, the quotient by the action carries the orbifold structure by slice theorem. We use notations M/R and $|M/R|$ to denote the orbifold and its underlying topological space, respectively. Let $o : M \to M/R$ be the orbit map. And, if M is orientable, let

$$o_1 : M \to M/R_{\mathrm{rot}} \quad \text{and} \quad o_2 : M/R_{\mathrm{rot}} \to M/R$$

be two consecutive orbit maps so that $o = o_2 \circ o_1$ where M/R is diffeomorphic to $(M/R_{\mathrm{rot}})/(R/R_{\mathrm{rot}})$. For notational simplicity, we use the same notations to denote underlying continuous maps of o, o_1, o_2. Denote by $\mathrm{Vect}(M/R)$ and $\mathrm{Vect}_R(M)$ sets of the isomorphism classes of nonequivariant topological complex vector orbibundles over M/R and equivariant topological complex vector bundles over M, respectively. For orbifold and orbibundle, see [3]. Recalling that the pullback of an orbibundle need not always exist, we state our first result.

Proposition 1.1. *Assume that the R-action on M satisfies Condition A1. Then, two maps*

$$o_2^* : \mathrm{Vect}(M/R) \longrightarrow \mathrm{Vect}_{R/R_{\mathrm{rot}}}(M/R_{\mathrm{rot}}), \qquad [E] \longmapsto [o_2^* E],$$
$$o_1^* : \mathrm{Vect}_{R/R_{\mathrm{rot}}}(M/R_{\mathrm{rot}}) \longrightarrow \mathrm{Vect}_R(M), \qquad [F] \longmapsto [o_1^* F]$$

are well-defined isomorphisms.

By using this, we can classify nonequivariant topological complex vector orbibundles over M/R when equivariant topological complex vector bundles over M have been classified. In [6], Kim has classified equivariant topological complex vector bundles over S^2 under a compact Lie group (not necessarily effective) action. So, we can classify nonequivariant topological complex vector orbibundles over M/R when $M = S^2$ and the R-action on M satisfies Condition A1. In the remaining of Introduction, we deal with the case of $M = S^2$.

In [6], Theorem A, B, C classify three different types of equivariant vector bundles over S^2 by different invariants. Theorem C classifies equivariant vector bundles by their isotropy representations at (at most) three points of S^2. This seems similar to the classical result on classification of equivariant vector bundles over a transitive action [11, p. 130], [2, Proposition II.3.2] in which classification is done by an isotropy representation at one point of a base space. Theorem A classifies equivariant vector bundles by their nonequivariant Chern classes and their isotropy representations at (at most) three points of S^2, and lists possible Chern classes. Theorem B classifies two exceptional cases. These two cases are not classified even by nonequivariant Chern classes and isotropy representations. To give a precise explanation for one of these two exceptional cases, Kim has classified equivariant topological complex vector bundles over $\mathbf{RP}^2$ under a compact Lie group (not necessarily effective) action in [7]. By using results of this paper, we explain for possible Chern classes of Theorem A and one remaining case of Theorem B.

To state our results, we list all possible finite group actions on S^2 up to conjugacy which satisfy Condition A1. For this, we use the well-known fact that a topological action on S^2 by a compact Lie group is conjugate to a linear action [8, Theorem 1.2], [5]. Let O(3) and its finite subgroups act usually on S^2. We introduce some finite subgroups of O(3) as follows:

1. $Z = \{\mathrm{id}, -\mathrm{id}\}$,
2. $\mathbb{Z}_n$ generated by the rotation a_n through the angle $2\pi/n$ around z-axis,
3. D_n generated by a_n and the rotation b through the angle π around x-axis where we assume $n > 1$ because D_1 is conjugate to $\mathbb{Z}_2$,
4. T, O, I are (orient preserving) rotation groups of a regular tetrahedron, a regular octahedron, a regular icosahedron which have the origin as their centers, respectively. Here, we choose the regular tetrahedron to be inscribed to the regular octahedron as duality.

Note that $\mathrm{T} \subset \mathrm{O}$, and pick an element o_0 of $\mathrm{O} \setminus \mathrm{T}$ so that $\mathrm{O} = \langle \mathrm{T}, o_0 \rangle$. In Section 4, we show that the action of a finite subgroup R of O(3) on S^2 satisfies Condition A1. if and only if it is conjugate to one of

$$
\begin{array}{llll}
 & \mathrm{D}_n,\ n > 1, & \mathbb{Z}_n, & \mathrm{T}, \\
(1.1) & \mathrm{O}, & \mathrm{I}, & \mathbb{Z}_n \times Z,\ \text{odd } n, \\
 & \langle -a_n \rangle,\ \text{even } n/2 & &
\end{array}
$$

where the symbol $\times$ means the internal direct product of two subgroups in O(3). We observe that groups of (1.1) are finite groups appearing in [6, Theorem A, B].

In [6], a semigroup epimorphism

$$p_{\mathrm{vect}} : \mathrm{Vect}_R(S^2) \to A_R(S^2, \mathrm{id})$$

is defined where p_{vect} sends an equivariant vector bundle to its isotropy representations at (at most) three points and $A_R(S^2, \mathrm{id})$ is a semigroup defined by using these isotropy representations. Elements of $A_R(S^2, \mathrm{id})$ are pairs or triples of representations. Denote by p'_{vect} and p''_{vect} compositions $p_{\mathrm{vect}} \circ o_1*$ and $p_{\mathrm{vect}} \circ o^*$, respectively. Then, we have two classification results.

Theorem A. *Assume that R is equal to one of*

$$\mathrm{D}_n,\ n>1, \qquad \mathbb{Z}_n, \qquad \mathrm{T}, \qquad \mathrm{O}, \qquad \mathrm{I}$$

so that $|S^2/R|$ is homeomorphic to S^2. For each $\mathbf{W} \in A_R(S^2, \mathrm{id})$, the map o^* induces the bijective map between $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$ and $p^{-1}_{\mathrm{vect}}(\mathbf{W})$. The first Chern classes of different E's in $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$ are different, and consist of the set

$$\Big\{ m + k_0 \mid m \in \mathbb{Z} \Big\} \subset H^2\Big(|S^2/R|, \mathbb{Q}\Big)$$

for some $k_0 \in \mathbb{Q}$.

Theorem B. *Assume that R is equal to one of*

$$\mathbb{Z}_n \times Z,\ \textit{odd } n, \qquad \langle -a_n \rangle,\ \textit{even } n/2$$

so that $|S^2/R_{\mathrm{rot}}|$ and $|S^2/R|$ are homeomorphic to S^2 and $\mathbf{RP}^2$, respectively. For each $\mathbf{W} \in A_R(S^2, \mathrm{id})$, maps o^*, o_1^*, o_2^* induce bijective maps among $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$, $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$, $p^{-1}_{\mathrm{vect}}(\mathbf{W})$. The set $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$ has two elements, and the first Chern classes of two different E's in $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ are same in $H^2\Big(|S^2/R_{\mathrm{rot}}|, \mathbb{Q}\Big)$.

These results seem similar to usual classification of nonequivariant topological complex vector bundles over S^2 and $\mathbf{RP}^2$. And, since [6, Theorem A, B] are pullback versions of these two when R is finite and R acts effectively on S^2, readers would feel comfortable with statements of [6, Theorem A, B].

Readers might ask whether we could interpret [6, Theorem C] similarly. To state one more result on this, we introduce the following condition on the R-action on M:

A2. R acts freely on M except a union of submanifolds with codimension at least one.

If the R-action on M does not satisfy Condition A1. but only Condition A2., then M/R can not deliver an orbifold structure because orbifold is defined by freeness except codimension at least two. So, we relax definition of orbifold, and define *orbifold with corner* in Section 5. In Section 5, we prove that the map

$$o^* : \mathrm{Vect}(M/R) \longrightarrow \mathrm{Vect}_R(M), \quad [E] \longmapsto [o^*E]$$

is a well-defined isomorphism where M/R delivers the orbifold with corner structure. Also in Section 5, we show that the action of a finite subgroup R of O(3) on S^2 does not satisfy Condition A1. but Condition A2. if and only if it is conjugate to one of

$$
\begin{array}{lll}
 & \mathrm{D}_n \times Z,\ \text{odd } n, & \langle a_n, -b\rangle,\ \text{odd } n,\\
 & \mathrm{D}_n \times Z,\ \text{even } n, & \langle a_n, -b\rangle,\ \text{even } n,\\
 & \langle -a_n, b\rangle,\ \text{odd } n/2,\ n>2, & \langle -a_n, b\rangle,\ \text{even } n/2,\\
(1.2) & \langle -a_n, -b\rangle,\ \text{odd } n/2,\ n>2, & \langle -a_n, -b\rangle,\ \text{even } n/2,\\
 & \mathbb{Z}_n \times Z,\ \text{even } n,\ n>2, & \langle -a_n\rangle,\ \text{odd } n/2,\ n>2,\\
 & \langle \mathrm{T}, -o_0\rangle, & \mathrm{T}\times Z,\\
 & \mathrm{O}\times Z, & \mathrm{I}\times Z.
\end{array}
$$

We observe that groups of (1.2) are finite groups appearing in [6, Theorem C]. Then, we can obtain the following result whose pullback is [6, Theorem C] when R is finite and R acts effectively on S^2:

THEOREM C. *Assume that R is equal to one of (1.2). Then, $|S^2/R|$ is homotopically trivial. For each $\mathbf{W} \in A_R(S^2, \mathrm{id})$, the map o^* induces a bijective map between $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$ and $p^{-1}_{\mathrm{vect}}(\mathbf{W})$. And, the set $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$ has only one element.*

By Theorem A, B, C, we can also observe that three main theorems of [6] classify three different types of equivariant vector bundles according to the topology of the orbit space $|S^2/R|$.

This paper is organized as follows. In Section 2, we prove Proposition 1.1. In Section 3, we deal with line orbibundles. In Section 4, we restrict our discussion to $M = S^2$, and prove Theorem A, B. In Section 5, we define orbifold with corner, and prove Theorem C.

2. Pullback of vector orbibundle

In this section, we introduce defining families of the orbifold M/R and orbibundles over it, and define the pullback of each vector orbibundle over M/R through o. Also, we endow the pullback with an R-action so that we prove Proposition 1.1. Let $\bar{\Sigma}$ be the subset of M on which R does not act freely, and let Σ be the set $o(\bar{\Sigma})$.

First, we give local uniformizing systems for open sets of M/R. We use notations of [9], [10]. Let $\dim_{\mathbb{R}} M = m_1$. For an arbitrary point $\bar{x}$ in M, put $x = o(\bar{x})$. Denote by d_x the cardinality of $o^{-1}(x)$. For a sufficiently small connected neighborhood U_x of x, let $\bar{U}_{\bar{x}}$ be the connected component containing $\bar{x}$ in $o^{-1}(U_x)$ which has d_x components. We call the local uniformizing system $\{\tilde{U}, \Lambda, \varphi\}$ of U defined by

$$\tilde{U} = \bar{U}_{\bar{x}}, \quad \Lambda = R_{\bar{x}}, \quad \varphi = o|_{\bar{U}_{\bar{x}}}$$

the *local uniformizing system of* U_x *at* $\bar{x}$ where

1. $\bar{U}_{\bar{x}}$ is contained in a coordinate chart of M, and moreover $\bar{U}_{\bar{x}}$ is regarded as contained in $T_{\bar{x}}M \cong \mathbb{R}^{m_1}$ where the origin of $T_{\bar{x}}M$ corresponds to $\bar{x}$,
2. $R_{\bar{x}}$ acts on $T_{\bar{x}}M$ as the isotropy representation, and o induces the homeomorphism from $\bar{U}_{\bar{x}}/R_{\bar{x}}$ to U.

If we put

$$o^{-1}(x) = \left\{ \bar{x}_i \;\middle|\; i = 1, \cdots, d_x \right\},$$

then the preimage $o^{-1}(U_x)$ is expressed as the following disjoint union

$$\bar{U}_{\bar{x}_1} \cup \cdots \cup \bar{U}_{\bar{x}_{d_x}}.$$

We denote by $\mathfrak{F}$ the family

$$\Big\{\{\bar{U}_{\bar{x}}, R_{\bar{x}}, o|_{\bar{U}_{\bar{x}}}\} \;\Big|\; x \in |M/R|,\ \bar{x} \in o^{-1}(x),\ \text{and}$$
$$U_x \text{ is a sufficiently small connected neighborhood of } x\Big\}$$

which is a defining family for M/R.

Second, we describe local uniformizing systems of an arbitrary orbibundle $p : E \to M/R$ in $\mathrm{Vect}(M/R)$. For an open set $U \subset |M/R|$, denote by $E|_U$ the orbibundle E restricted to U. Let the fiber of E be $\mathbb{C}^{m_2}$ for some natural number m_2. For each local uniformizing system $\{\tilde{U}, \Lambda, \varphi\} \in \mathfrak{F}$ of a sufficiently small connected neighborhood U_x of $x \in |M/R|$, the local uniformizing system $\{\tilde{U}^*, \Lambda^*, \varphi^*\}$ of $E|_U$ is given by

$$\tilde{U}^* = \tilde{U} \times \mathbb{C}^{m_2}, \quad \Lambda^* = \Lambda, \quad \varphi^* : \tilde{U}^* \to \big|E|_U\big|$$

for some φ^* satisfying

1. $p\Big(\varphi^*(\tilde{x}, v)\Big) = \varphi(\tilde{x})$ for each $\tilde{x} \in \tilde{U}$ and $u \in \mathbb{C}^{m_2}$,
2. φ^* induces the homeomorphism from $\tilde{U}^*/\Lambda^*$ to $\big|E|_U\big|$.

We denote by $\mathfrak{F}^*$ the family of these systems which is a defining family for E. For sufficiently small connected neighborhoods $U_x \subset U'_x$ of $x \in |M/R|$ and an injection $\lambda : \{\tilde{U}, \Lambda, \varphi\} \to \{\tilde{U}', \Lambda', \varphi'\}$ between their local uniformizing systems, the injection

$$\lambda^* : \{\tilde{U}^*, \Lambda^*, \varphi^*\} \to \{\tilde{U}'^*, \Lambda'^*, \varphi'^*\}$$

satisfies

$$\varphi^* = \varphi'^* \circ \lambda^* \qquad \text{and} \qquad \lambda^*(\tilde{x}, v) = \Big(\ \lambda(\tilde{x}),\ g_\lambda(\tilde{x})v\ \Big)$$

for $\tilde{x} \in \tilde{U}$, $u \in \mathbb{C}^{m_2}$, and some transition map $g_\lambda : \tilde{U} \to \mathrm{GL}(m_2, \mathbb{C})$.

Now, we state a basic lemma:

Lemma 2.1. *For any orbibundle* $p : E \to M/R$ *in* $\mathrm{Vect}(M/R)$, *its pullback* o^*E *uniquely exists which satisfies the following commutative diagram:*

$$(2.1)\qquad \begin{array}{ccc} o^*E & \xrightarrow{\bar{o}} & E \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow{o} & M/R \end{array}$$

where $\bar{o}$ *is an orbibundle map.*

Proof. We show that o is a good map to obtain a proof. For good map, see [4, p. 5]. Let $\mathfrak{U}$ and $\bar{\mathfrak{U}}$ be open covers of $|M/R|$ and M defined by

$$\Big\{U_x \mid x \in |M/R|\Big\} \quad \text{and} \quad \{\bar{U}_{\bar{x}} \mid \bar{x} \in M\},$$

respectively. When $U_x = o(\bar{U}_{\bar{x}}) \in \mathfrak{U}$, put $\tilde{o} : \bar{U}_{\bar{x}} \to \bar{U}_{\bar{x}},\ \tilde{x} \mapsto \tilde{x}$ be the lifting of o. Then, we can check that o is a good map. So, we obtain a proof of the lemma.

For reader's convenience, we explain for the definition of o^*E. For sufficiently small connected neighborhoods $U_x \subset U'_x$ of each $x \in |M/R|$ and their local uniformizing systems $\{\tilde{U}, \Lambda, \varphi\}$, $\{\tilde{U}', \Lambda', \varphi'\} \in \mathfrak{F}$ such that $\tilde{U} \subset \tilde{U}' \subset M$, the trivializations $\tilde{U} \times \mathbb{C}^{m_2}$, $\tilde{U}' \times \mathbb{C}^{m_2}$ and the transition map $g_\lambda : \tilde{U} \to \mathrm{GL}(m_2, \mathbb{C})$ define the pullback o^*E. And, define $\bar{o}$ to satisfy $\bar{o}|_{\tilde{U}^*} = \varphi^*$. □

Remark 2.2. The pullback of an orbibundle through a smooth map between orbifolds need not exist as noted in [4, p. 4].

Next, we state a key lemma.

Lemma 2.3. *The map*

$$o^* : \mathrm{Vect}(M/R) \to \mathrm{Vect}_R(M), \quad [E] \longmapsto [o^*E]$$

is isomorphic.

Proof. We use notations of Lemma 2.1. We will show that the pullback o^*E of E in $\mathrm{Vect}(M/R)$ carries the unique R-bundle structure up to isomorphism to satisfy

$$\big(o^*E\big)/R \cong E.$$

The bundle o^*E satisfies the diagram (2.1). It is well-known that the pullback $o^*E\big|_{M\setminus\bar\Sigma}$ carries the unique R-bundle structure to satisfy

$$\bar{o}(r\cdot u)=\bar{o}(u) \tag{2.2}$$

for $r\in R$, $u\in o^*E\big|_{M\setminus\bar\Sigma}$ which satisfies

$$\Big(o^*E\big|_{M\setminus\bar\Sigma}\Big)\Big/R\;\cong\;E\big|_{o(M\setminus\bar\Sigma)}. \tag{2.3}$$

For a sufficiently small connected neighborhood U_x of $x\in|M/R|$, put

$$o^{-1}(x)=\big\{\bar{x}_i\big|i=1,\cdots,d_x\big\}\quad\text{ and }\quad o^{-1}(U_x)=\bar{U}_{\bar{x}_1}\cup\cdots\cup\bar{U}_{\bar{x}_{d_x}}.$$

First, we give an R-action on $o^*E\big|_{o^{-1}(U_x)}$. Pick an arbitrary point $\tilde{x}\in o^{-1}(U_x)$ and an element r of R. Put $\tilde{x}\in\bar{U}_{\bar{x}_i}$ and $r\cdot\tilde{x}\in\bar{U}_{\bar{x}_{i'}}$ for some i, i'. We will define $r\cdot(\tilde{x},u)$ for arbitrary $u\in\mathbb{C}^{m_2}$. For these i, i', r, define the injection

$$\lambda_{i,i',r}:\bar{U}_{\bar{x}_i}\to\bar{U}_{\bar{x}_{i'}},\quad \tilde{y}\mapsto r\cdot\tilde{y}$$

for $\tilde{y}\in\bar{U}_{\bar{x}_i}$. By using this, define $r\cdot(\tilde{x},u)$ as

$$\lambda^*_{i,i',r}(\tilde{x},u).$$

Note that if $\tilde{x}\in M\setminus\bar\Sigma$, then

$$\bar{o}\Big(\lambda^*_{i,i',r}(\tilde{x},u)\Big)=\bar{o}(\tilde{x},u). \tag{*}$$

If we restrict so defined action to $o^*E\big|_{o^{-1}(U_x\setminus\Sigma)}$, then (*) means that it is equal to the R-action on $o^*E\big|_{o^{-1}(U_x\setminus\Sigma)}$ defined by (2.2). So, the action is actually an action and is uniquely defined because $\bar\Sigma$ is codimension at least two. That is, the R-action on $o^*E\big|_{M\setminus\bar\Sigma}$ can be extended uniquely to the whole o^*E. Therefore, we obtain a proof. □

In a similar way, we can obtain similar results for o_1, o_2 instead of o, so we obtain Proposition 1.1.

3. Line orbibundles over M/R

In this section, we deal with line orbibundle. For this, we introduce some terminologies. Pick an arbitrary point $x\in|M/R|$, and a point $\bar{x}\in o^{-1}(x)$. We call $R_{\bar{x}}$ the *structure group* of M/R at $\bar{x}$, and call the isotropy $R_{\bar{x}}$-representation on the tangent space $T_{\bar{x}}M$ the *structure representation* of M/R at $\bar{x}$. Similarly, we call the isotropy $R_{\bar{x}}$-representation $(o^*E)_{\bar{x}}$ the *structure representation* of E at $\bar{x}$. In this section, we assume that the R-action satisfies the following condition:

A3. R acts freely on M except finite set $\bar{\Sigma}$.

Two orbibundles E, E' over M/R are called *isomorphic over* Σ if their structure representations at $\bar{s}$ are $R_{\bar{s}}$-isomorphic for each $\bar{s} \in \bar{\Sigma}$. We denote by $\mathrm{Vect}^1(M/R)$ and $\mathrm{Vect}^1\Big(|M/R|\Big)$ subsets of rank 1 elements of $\mathrm{Vect}(M/R)$ and $\mathrm{Vect}\Big(|M/R|\Big)$, respectively.

Lemma 3.1. *If two line orbibundles L, L' in $\mathrm{Vect}^1(M/R)$ are isomorphic over Σ, then L' is isomorphic to $L \otimes L_0$ for some (usual) line bundle L_0 in $\mathrm{Vect}^1(|M/R|)$.*

Proof. Since L and L' are isomorphic over Σ, we can observe that $\mathrm{Hom}_{\mathbb{C}}(L, L')$ is an line orbibundle over M/R. Also, observe that the structure representation of $\mathrm{Hom}_{\mathbb{C}}(L, L')$ at each $\bar{s} \in \bar{\Sigma}$ is the trivial representation. So, $\mathrm{Hom}_{\mathbb{C}}(L, L')$ becomes a usual vector bundle over $|M/R|$. And, the map

$$L \otimes \mathrm{Hom}_{\mathbb{C}}(L, L') \to L', \quad u \otimes f \mapsto f(u)$$

is a well-defined isomorphism. Therefore, we obtain a proof if we put $L_0 = \mathrm{Hom}_{\mathbb{C}}(L, L')$. □

When $M = S^2$, the terminology 'isomorphic over Σ' can be described in terms of p_{vect}.

Lemma 3.2. *Assume that R is a finite group appearing in (1.1). Two orbibundles E, E' in $\mathrm{Vect}(S^2/R)$ are isomorphic over Σ if and only if*

$$p_{\mathrm{vect}}\Big(o^*(E)\Big) = p_{\mathrm{vect}}\Big(o^*(E')\Big).$$

Proof. We can check that a point $\bar{x}$ in S^2 with nontrivial $R_{\bar{x}}$ is in the same orbit with some d^i or S, N by definition of p_{vect}, [6, Table 1.1], [6, Table 3.4], and that all structure representations of E and E' are determined by structure representations at these points. □

The dimension of an element $\mathbf{W} \in A_R(S^2, \mathrm{id})$ is defined as d if its entries are all d-dimensional. For one-dimensional $\mathbf{W}$, if we apply Lemma 3.2 to Lemma 3.1, we can obtain more precise result.

Proposition 3.3. *Assume that R is a finite group appearing in (1.1). For each one-dimensional $\mathbf{W} \in A_R(S^2, \mathrm{id})$ and some L_0 in $p''^{-1}_{\mathrm{vect}}(\mathbf{W})$, the map*

$$\mathrm{Vect}^1(|S^2/R|) \longrightarrow p''^{-1}_{\mathrm{vect}}(\mathbf{W}) \subset \mathrm{Vect}^1(S^2/R), \quad L \mapsto L_0 \otimes L$$

is bijective.

Proof. We obtain surjectivity of the map by Lemma 3.1, 3.2. Since

$$c_1(L \otimes L_0) = c_1(L) + c_1(L_0),$$

we obtain injectivity. □

4. Orbibundles over S^2/R

In this section, we deal with $M = S^2$, and prove Theorem A, B. First, we list all possible finite group actions on S^2 which satisfies Condition A1.

LEMMA 4.1. *The action of a finite subgroup R of* O(3) *on S^2 satisfies Condition A1. if and only if R is conjugate to one of (1.1).*

Proof. In [6, Table 1.1], we have listed all closed subgroups R of O(3) up to conjugacy. In [6, Table 3.4], isotropy subgroups of each R in [6, Table 1.1] are calculated. If an isotropy subgroup $R_{\bar{x}}$ does not preserve orientation of $T_{\bar{x}}S^2$ for some $\bar{x} \in S^2$, then R-action does not satisfy Condition A1. Those of (1.1) are finite R's in [6, Table 1.1] whose isotropy subgroup $R_{\bar{x}}$ preserves an orientation of $T_{\bar{x}}S^2$ for each $\bar{x}$. And, it is easy that these satisfy Condition A1. □

Now, we prove Theorem A.

Proof of Theorem A. By bijectivity of o^* and definition of p''_{vect}, we obtain the first statement. By [6, Theorem A], the first Chern classes of different bundles in $p_{\mathrm{vect}}^{-1}(\mathbf{W})$ are all different, and consist of the set

$$\Big\{ |R|m + k'_0 \ \Big|\ m \in \mathbb{Z} \Big\} \subset H^2(S^2, \mathbb{Z})$$

for some $k'_0 \in \mathbb{Z}$. Since $o^* c_1(E) = c_1\Big(o^* E\Big)$ for $E \in \mathrm{Vect}(S^2/R)$ and the degree of the map

$$o^* : H^2\Big(|S^2/R|, \mathbb{Q}\Big) \to H^2(S^2, \mathbb{Q})$$

is equal to $|R|$ up to sign, we obtain the second statement. □

Next, we prove Theorem B.

Proof of Theorem B. The first statement is easy. Before we prove the second statement, we deal with one-dimensional $\mathbf{W}$. By bijectivity of o_2^* and definition of p'_{vect} and p''_{vect}, two elements L_1 and L_2 of $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ are expressed as $o_2^* L'_1$ and $o_2^* L'_2$ for two elements L'_1 and L'_2

of $p''^{-1}_{\text{vect}}(\mathbf{W})$, respectively. By Proposition 3.3, $L'_2 \cong L'_1 \otimes L_0$ for some $L_0 \in \text{Vect}(|M/R|)$. So, $L_2 \cong o_2^* L'_1 \otimes o_2^* L_0$ by Proposition 3.3. From this,

$$c_1(L_2) = c_1(o_2^* L'_1 \otimes o_2^* L_0) = c_1(o_2^* L'_1) + c_1(o_2^* L_0) = c_1(L_1)$$

because $o_2^* : H^2(|S^2/R|, \mathbb{Z}) \to H^2(|S^2/R_{\text{rot}}|, \mathbb{Z})$ is trivial. So, we obtain a proof for one-dimensional $\mathbf{W}$.

For m_2-dimensional $\mathbf{W}$, we can show that two elements of $p'^{-1}_{\text{vect}}(\mathbf{W})$ is expressed as $E \oplus L_1$ and $E \oplus L_2$ for some rank $(m_2 - 1)$ bundle E and line orbibundles L_1 and L_2 by [6, Theorem B, D]. Since $p'^{-1}_{\text{vect}}(L_1) = p'^{-1}_{\text{vect}}(L_2)$, we have $c_1(L_1) = c_1(L_2)$ by the above arguments. So, we have $c_1(E \oplus L_1) = c_1(E \oplus L_2)$. □

5. Orbifold with corner

We define *orbifold with corner* by admitting a local uniformizing system $\{\tilde{U}, \Lambda, \varphi\}$ such that Λ acts freely on $\tilde{U}$ except a finite union of submanifolds of $\tilde{U}$ with codimension at least "one." If an orbifold with corner is also an orbifold, then we call it just an orbifold. To deal with orbifold with corner, we prove a basic lemma which holds for orbifold by [9, Lemma 1].

LEMMA 5.1. *Let λ, $\mu : \{\tilde{U}, \Lambda, \varphi\} \to \{\tilde{U}', \Lambda', \varphi'\}$ be two injections for two local uniformizing systems of two open sets of the underlying topological space of an orbifold with corner. Then there exists the uniquely determined $\sigma' \in \Lambda'$ such that $\mu = \sigma' \circ \lambda$.*

Proof. Let S be the subset of points $\tilde{x} \in \tilde{U}$ with nontrivial $\Lambda_{\tilde{x}}$. Pick a point $\tilde{x}_0 \in S$ such that $V \cap S$ is a nonempty codimension one submanifold in V for some sufficiently small connected neighborhood V of $\tilde{x}_0$. Pick a nontrivial $b \in \Lambda_{\tilde{x}_0}$ fixing S so that b is orientation reversing. For each orientation preserving element $a \in \Lambda_{\tilde{x}_0}$, since

$$aba^{-1}a(s) = a(s)$$

for each $s \in V \cap S$, the element a preserve S. If we denote by $\Lambda^0_{\tilde{x}_0}$ the subgroup of orientation preserving elements in $\Lambda_{\tilde{x}_0}$, then we can assume that $V \setminus S$ has two components V_1, V_2 so that V is $\Lambda_{\tilde{x}_0}$-invariant and V_1, V_2 are $\Lambda^0_{\tilde{x}_0}$-invariant. And, restrictions of λ and μ to V, V_1, V_2 give injections. By [9, Lemma 1],

$$\begin{aligned} \mu &= \sigma'_1 \circ \lambda \quad \text{on } V_1, \\ \mu &= \sigma'_2 \circ \lambda \quad \text{on } V_2 \end{aligned} \tag{5.1}$$

for some σ_1, $\sigma_2 \in \Lambda'$. It suffices to show $\sigma'_1 = \sigma'_2$ to prove the lemma. So, assume that $\sigma'_1 \neq \sigma'_2$. By continuity of μ,

$$\sigma'_1 \circ \lambda = \sigma'_2 \circ \lambda \text{ on } V \cap S,$$

i.e.

$$(\sigma_2'^{-1}\sigma'_1) \circ \lambda = \lambda \text{ on } V \cap S.$$

So, $\sigma_2'^{-1}\sigma'_1$ fixes $\lambda(V \cap S)$, and we have

$$\Big(\sigma_2'^{-1}\sigma'_1\lambda(V_1)\Big) \cap \lambda(V_2) \neq \varnothing$$

because $\sigma_2'^{-1}\sigma'_1$ is nontrivial. But, (5.1) gives

$$\Big(\sigma_2'^{-1}\sigma'_1\lambda(V_1)\Big) \cap \lambda(V_2) = \varnothing$$

so that we obtain a contradiction. Therefore, $\sigma'_1 = \sigma'_2$ and we obtain a proof. □

By using this lemma, we can generalize basic lemmas on orbifold to orbifold with corner. Orbibundle over orbifold with corner is also defined. And, we can obtain the following isomorphism which is a slight generalization of Proposition 1.1:

Proposition 5.2. *Assume that the R-action on M satisfies Condition E2. Then, the map*

$$o^* : \mathrm{Vect}(M/R) \longrightarrow \mathrm{Vect}_R(M), \quad [E] \longmapsto [o^*E]$$

is isomorphic.

Proof. Similarly to Lemma 2.1, we can prove that the pullback bundle uniquely exists. Since M is a smooth manifold, it is easy.

Next, we endow each o^*E with the unique R-action. Let $\bar{\Sigma}$ be the closure of the union of codimension one submanifolds of M on which R does not act freely, and let Σ be the set $o(\bar{\Sigma})$. By Lemma 2.3, each pullback bundle o^*E delivers the R-action on $o^*E|_{M\setminus\bar{\Sigma}}$ to satisfy (2.2) and (2.3). For a sufficiently small neighborhood V of an arbitrary point $\bar{x}$ in $\bar{\Sigma}$, we can define an R-action on $o^*E|_V$ as in the proof of Lemma 2.3 which is equal to the action on $o^*E|_{M\setminus\bar{\Sigma}}$. By continuity, the action on $o^*E|_V$ is unique, and we obtain the R-action on the whole o^*E. □

Last, we prove Theorem C.

Proof of Theorem C. For each R, we can check case by case that $|S^2/R|$ is homotopically trivial by [6, Table 1.1] and [6, Table 3.4]. By bijectivity of o^* and definition of p''_{vect}, the second statement is easy. The third statement is obtained by [6, Theorem C]. □

References

[1] M. F. Atiyah, *K-Theory*, Addison-Wesley, 1989.

[2] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.

[3] W. Chen, Y. Ruan, *A new cohomology theory for orbifolds*, Commun. Math. Phys. **248** (2004), 1-31.

[4] W. Chen, Y. Ruan, *Orbifold quantum cohomology*, arXiv:math/0005198v2

[5] A. Constantin, B. Kolev, *The theorem of Kerekjarto on periodic homeomorphisms of the disk and the sphere*, Enseign. Math. (2) **40** (1994), 193-204.

[6] M. K. Kim, *Classification of equivariant vector bundles over two-sphere*, submitted to Transform. Groups, arXiv:1005.0681v2.

[7] M. K. Kim, *Classification of equivariant vector bundles over real projective plane*, J. Chungcheong Math. Soc. **24** (2011), no. 2, 319-335.

[8] B. Kolev, *Sous-groupes compacts d'homeomorphismes de la sphere*, Enseign. Math. (2) **52** (2006), no. 3-4, 193-214.

[9] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359-363.

[10] I. Satake, *The Gauss-Bonnet Theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), no. 4, 464-492.

[11] G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129-151.

*

Department of Mathematics Education
Gyeongin National University of Education
Incheon 407-753, Republic of Korea
E-mail: mkkim@kias.re.kr