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ALGEBRAS IN $MC_n(k)$ WITH $dim(m_R^2) = 1$

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Abstract. We introduce a method to construct some algebras $R \in MC_n(k)$ with $dim(R) = n$ and $dim(m_R^2) = 1$ for each $n \geq 3$.

1. Introduction

Throughout this paper, (R, m_R, k) is a local maximal commutative subalgebra of matrix algebra $M_n(k)$ of size $n \times n$ with nilpotent maximal ideal m_R and residue class field k. The set of all local maximal commutative subalgebras (R, m_R, k) of $M_n(k)$ will be denoted by $MC_n(k)$. We assume the algebra $R \in MC_n(k)$ contains the multiplicative identity. The socle of the algebra R is denoted by $soc(R)$. Furthermore, I_t is the identity matrix of size $t \times t$ and $O_{t \times s}$ is the zero matrix of size $t \times s$.

The next theorems are known as the Kravchuk's theorem.

Theorem 1.1. ([5] Kravchuk's first theorem) *Let (R, m_R, k) be an algebra in $MC_n(k)$. Then, the matrix $r \in m_R$ can be assumed to be of the following form :*

$$r = \begin{pmatrix} O_{\ell \times \ell} & O & O \\ A(r) & B(r) & O \\ C(r) & D(r) & O_{q \times q} \end{pmatrix},$$

where $B(r) \in M_p(k)$, $n = \ell + p + q$, $\ell \neq 0, p \neq 0, q \neq 0$. Moreover, $soc(R)$ consists of all matrices of the form :

$$r = \begin{pmatrix} O_{(n-q) \times \ell} & O_{(n-q) \times (n-\ell)} \\ C(r) & O_{q \times (n-\ell)} \end{pmatrix},$$

where $C(r) \in M_{q \times \ell}(k)$.

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Theorem 1.2. ([5] Lemma 6) *Let (R, m_R, k) be an algebra in $MC_n(k)$. Suppose the matrices $r_i \in m_R$ which are of the form*

$$r_i = \begin{pmatrix} O_{\ell\times\ell} & O & O \\ A(r_i) & B(r_i) & O \\ C(r_i) & D(r_i) & O_{q\times q} \end{pmatrix}, \qquad i = 1, 2, \ldots, t$$

constitute a basis for m_R where $B(r_i) \in M_p(k)$. Then, the rank of the following $p \times \ell t$ matrix H is p :

$$H = \begin{pmatrix} A(r_1) & A(r_2) & \cdots & A(r_t) \end{pmatrix}.$$

Theorem 1.3. ([1] Theorem 4) *Let (R, m_R, k) be a commutative algebra. Then, R is a C_1-construction if and only if there is an ideal N of R satisfying the following conditions :*

(1) $Ann_R(N) = N$
(2) The exact sequence $0 \to N \to R \to R/N \to 0$ splits as k-algebras.
, where $Ann_R(N)$ is the annihilator of N.

Also, theorem 1.4 is an equivalent condition for a algebra R to be an algebra of the C_2-construction. The proof can be found in [3].

Theorem 1.4. ([3] Lemma 2.8) *Let (R, m_R, k) be a finite dimensional commutative algebra. Then, R is a C_2-construction if and only if R contains a subalgebra (B, m_B, k) and an element $x \in m_R$ satisfying the following conditions :*

(1) $x^\nu \neq 0 \in soc(B)$ for some positive integer $\nu > 1$
(2) $m_B x = \{0\}$
(3) $dim_k(R) = dim_k(B) + (\nu - 1)$

The following theorem 1.5 is an equivalent condition to be a C_2^t-construction that can be found in [9].

Theorem 1.5. ([9] Theorem 3.1) *Let (R, m_R, k) be a finite dimensional commutative algebra and let t be a positive integer. Then, R is a C_2^t-construction if and only if there exist a subalgebra (B, m_B, k) of R and elements $x_i \in m_R$, $i = 1, 2, \ldots, t$ satisfying the following properties :*

(1) $x_i^2 = x_j^2 \in soc(B) - \{0\}$ *for all* $1 \leq i, j \leq t$
(2) $x_i x_j = 0$ *for all* $1 \leq i \neq j \leq t$
(3) $m_B x_i = \{0\}$ *for all* $1 \leq i \leq t$
(4) $dim_k(R) = dim_k(B) + t$

2. Algebras in $MC_n(k)$ with $dim(R) = n$ and $dim(m_R^2) = 1$

In this section, for each positive integer $n \geq 3$, we will introduce a construction to produce some algebras $R \in MC_n(k)$ with $dim(R) = n$ and $dim(m_R^2) = 1$.

From now on, we will consider the case of $\ell = 1$ in theorem 1.1. That is, we assume the matrices r_i in m_R are of the form

$$r_i = \begin{pmatrix} O_{1\times 1} & O & O \\ A(r_i) & B(r_i) & O \\ C(r_i) & D(r_i) & O_{q\times q} \end{pmatrix}, \qquad i = 1, 2, \ldots, p+q$$

constitute a basis for m_R.

Now, we define θ-relation as follows :

Definition 2.1. Let θ be a subset of a set $\{1, 2, \ldots, q\}$. We say the pair $(A(r_i), D(r_i))$ are in θ-relation if for each $A(r_i) \in M_{p\times 1}(k)$, the matrix $D(r_i) \in M_{q\times p}(k)$ are defined as follows :

$$D(r_i) = \begin{pmatrix} D_1(r_i) \\ D_2(r_i) \\ \vdots \\ D_q(r_i) \end{pmatrix}, \qquad D_j(r_i) = \begin{cases} A(r_i)^T, & \text{if } j \in \theta, \\ O_{1\times p}, & \text{otherwise} \end{cases}$$

Here, $A(r_i)^T$ is the transpose of $A(r_i)$.

Now, we can construct algebras $R \in MC_n(k)$ with $dim(R) = n$ and $dim(m_R^2) = 1$ for each $n \geq 3$ as the following theorem.

Theorem 2.2. *Let R be a subalgebra of $M_n(k)$ and let m_R have a basis consisting of following form of matrices :*

$$r_i = \begin{pmatrix} O_{1\times 1} & O & O \\ A(r_i) & O_{p\times p} & O \\ C(r_i) & D(r_i) & O_{q\times q} \end{pmatrix}, \qquad i = 1, 2, \ldots, p+q$$

,where $A(r_i) \in M_{p\times 1}(k)$, $C(r_i) \in M_{q\times 1}(k)$, $D(r_i) \in M_{q\times p}(k)$, and $n = p + q + 1$. If the pairs $(A(r_i), D(r_i))$ are in θ-relation for all $i = 1, 2, \ldots, p+q$, then R is an algebra in $MC_n(k)$.

Proof. Note that by theorem 1.1 and theorem 1.2, we may assume $A(r_i)$, $A(r_j)$ and $C(r_i)$, $C(r_j)$ are of the following form for $i = 1, 2, \ldots, p$, $j = p+1, p+2, \ldots, p+q$:

$$A(r_i) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}\ row, \qquad C(r_j) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th}\ row$$

$$A(r_j) = O_{p\times 1}, \qquad C(r_i) = O_{q\times 1}$$

Now let

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \in M_n(k)$$

Here, $S_{11} \in k,\ S_{22} \in M_p(k),\ S_{33} \in M_q(k)$.

Then, we have the following equations from the equation $r_iS = Sr_i$ for all i :

(1) $S_{12}A(r_i) + S_{13}C(r_i) = 0$
(2) $S_{13}D(r_i) = 0$
(3) $S_{22}A(r_i) + S_{23}C(r_i) = A(r_i)S_{11}$
(4) $S_{23}D(r_i) = A(r_i)S_{12}$
(5) $A(r_i)S_{13} = O_{p\times q}$
(6) $S_{32}A(r_i) + S_{33}C(r_i) = C(r_i)S_{11} + D(r_i)S_{21}$
(7) $S_{33}D(r_i) = C(r_i)S_{12} + D(r_i)S_{22}$
(8) $C(r_i)S_{13} + D(r_i)S_{23} = O_{q\times q}$

From the equation (1) and (3), $S_{12} = O_{1\times p},\ S_{13} = O_{1\times q},\ S_{23} = O_{p\times p}$. Thus, we have the following equations :

(3-1) $S_{22}A(r_i) = A(r_i)S_{11}$
(6-1) $S_{33}C(r_i) = C(r_i)S_{11}$
(6-2) $S_{32}A(r_i) = D(r_i)S_{21}$

From the equation (3-1),

$$S_{22}A(r_i) = Col_i(S_{22}) = \begin{pmatrix} 0 \\ \vdots \\ s_{11} \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \ = A(r_i)S_{11},$$

where $S_{11} = (s_{11})_{1\times 1}$ and $Col_i(S_{22})$ is the i-th column of S_{22}. Thus, $S_{22} = s_{11}I_p$.

From the equation (6-1),

$$S_{33}C(r_i) = Col_i(S_{33}) = \begin{pmatrix} 0 \\ \vdots \\ s_{11} \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \; = C(r_i)S_{11}$$

Thus, $S_{33} = s_{11}I_q$.

From the equation (6-2), we have $Col_i(S_{32}) = S_{32}A(r_i) = D(r_i)S_{21}$.

If we let $S_{21} = (d_1, d_2, \ldots, d_p)^T$ for some $d_i \in k,\ i = 1, 2, \ldots, p$, then

$$D(r_i)S_{21} = \begin{pmatrix} D_1(r_i) \\ D_2(r_i) \\ \vdots \\ D_q(r_i) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{pmatrix} = \begin{pmatrix} d_{1i} \\ d_{2i} \\ \vdots \\ d_{qi} \end{pmatrix},$$

where $d_{ti} = \begin{cases} d_i, & \text{if } t \in \theta, \\ 0, & \text{otherwise} \end{cases}$

Thus,

$$Col_i(S_{32}) = \begin{pmatrix} d_{1i} \\ d_{2i} \\ \vdots \\ d_{qi} \end{pmatrix}$$

and so the matrix S is of the form

$$S = \begin{pmatrix} a & O & O \\ S_{21} & aI_p & O \\ S_{31} & S_{32} & aI_q \end{pmatrix}.$$

for some $a \in k$, where

$$S_{21} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{pmatrix}, \qquad S_{32} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_q \end{pmatrix},$$

where

$$S_j = \begin{cases} S_{21}^T, & \text{if } j \in \theta, \\ O_{1\times p}, & \text{otherwise} \end{cases}$$

Here, S_{21}^T is the transpose of S_{21}. This implies $S \in R$ and therefore, we can conclude that $R \in MC_n(k)$. $\square$

Example 2.3. *Let R be a subalgebra of $M_6(k)$ defined as following:*

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ a_1 & a & 0 & 0 & 0 & 0 \\ a_2 & 0 & a & 0 & 0 & 0 \\ a_3 & a_1 & a_2 & a & 0 & 0 \\ a_4 & 0 & 0 & 0 & a & 0 \\ a_5 & a_1 & a_2 & 0 & 0 & a \end{pmatrix} \mid a, a_1, a_2, a_3, a_4, a_5, \alpha \in k \right\}$$

Then, we may consider as $p = 2, q = 3, \theta = \{1, 3\}$,

$$A(r_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A(r_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; A(r_3) = A(r_4) = A(r_5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$D(r_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A(r_1)^T \\ O \\ A(r_1)^T \end{pmatrix}, \; D(r_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A(r_2)^T \\ O \\ A(r_2)^T \end{pmatrix},$$

$$D(r_3) = D(r_4) = D(r_5) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C(r_1) = C(r_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad C(r_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$C(r_4) = C(r_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad C(r_5) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, by the theorem 2.2, R should be a subalgebra in $MC_6(k)$ with $dim(R) = 6$ and $dim(m_R^2) = 1$.

The following lemma 2.4 and theorem 2.5 provides some properties of an algebra $R \in MC_n(k)$ in theorem 2.2 and the proof can be obtained by straightforward calculations. The $(i,j)^{th}$ matrix unit will be denoted by E_{ij}.

Lemma 2.4. *Suppose $R \in MC_n(k)$ is an algebra as in theorem 2.2. Then, $m_R^2 = (E_{1+p+\ell_1,1} + \cdots + E_{1+p+\ell_\mu,1})$, where $\ell_1, \ldots, \ell_\mu \in \theta$ with $\ell_1 < \cdots < \ell_\mu$.*

Theorem 2.5. *Suppose $R \in MC_n(k)$ is an algebra as in theorem 2.2. Then, the following properties hold :*

(1) *$dim(R) = n$*

(2) $dim(m_R^2) = 1$
(3) $m_R^2 \subseteq soc(R)$
(4) $dim(soc(R)) = q$
(5) $dim(soc(R)/m_R^2) = q - 1$
(6) $i(m_R) = 3$, *where* $i(m_R)$ *is the index of the nilpotency of* m_R.

3. Relation with C_i-constructions

In this section, we want to prove if the construction in section 2 imply the C_2-construction and the C_2^t-construction but not the C_1-construction.

THEOREM 3.1. *Suppose* $R \in MC_n(k)$ *is an algebra in theorem 2.2. Then,* R *is not a* C_1*-construction.*

Proof. Suppose R is a C_1-construction. Then, R should contain an ideal N satisfying $Ann_R(N) = N$. Let $r \in Ann(N)$. Then $r = a_1r_1 + a_2r_2 + \cdots + a_pr_p + bs$ for some $a_i, b \in k,\ i = 1, 2, \ldots, p$ and $s \in soc(R)$. Note that $r^2 = (a_1^2 + a_2^2 + \cdots + a_p^2)s_1$ for some $s_1 \neq O_{n\times n} \in soc(R)$. Since $r \in Ann_R(N)$, $r^2 = O_{n\times n}$ and so $r^2 = (a_1^2 + a_2^2 + \cdots + a_p^2)s_1 = O_{n\times n}$ which implies $a_i = 0$ for all $i = 1, 2, \ldots, p$. Thus, $r = bs \in soc(R)$ and so $Ann_R(N) \subseteq soc(R)$. Since $soc(R) \subseteq Ann_R(N)$, we have $N = Ann_R(N) = soc(R)$. But then $N = Ann_R(soc(R)) = m_R$ and $m_R^2 = N^2 = \{O_{n\times n}\}$ which is impossible since $dim(m_R^2) = 1$. Therefore, there doesn't exist an ideal N satisfying $Ann_R(N) = N$ and we can conclude that R is not a C_1-construction. □

THEOREM 3.2. *Suppose* $R \in MC_n(k)$ *is an algebra in theorem 2.2. Then,* R *is a* C_2*-construction.*

Proof. Let $B = k[r_2, \ldots, r_p] \oplus soc(R)$. Then B is a subalgebra of R and for the element $x = r_1$, the following properties holds :
(1) $x^2 \neq O_{n\times n} \in soc(B)$
(2) $m_Bx = \{O_{n\times n}\}$
(3) $dim(R) = dim(B) + 1$.

Thus, the algebra R satisfies the conditions in theorem 1.4 and so R is a C_2-construction. □

THEOREM 3.3. *Suppose* $R \in MC_n(k)$ *is an algebra in theorem 2.2. Then,* R *is a* C_2^t*-construction.*

Proof. Let $B = k[soc(R)]$ and let $x_i = r_i$ for all $i = 1, 2, \ldots, p$. Then B and x_i satisfies the following conditions :

(1) $x_i^2 = x_j^2 \in soc(B) - \{O_{n\times n}\}$ for all $1 \le i, j \le p$
(2) $x_i x_j = O_{n\times n}$ for all $1 \le i \ne j \le p$
(3) $m_B x_i = \{O_{n\times n}\}$ for all $1 \le i \le p$
(4) $dim_k(R) = dim_k(B) + p$

Thus, by the theorem 1.5, R is a C_2^t-construction for $t = p$. □

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