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A CHANGE OF SCALE FORMULA FOR GENERALIZED WIENER INTEGRALS

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Abstract. Cameron and Storvick introduced change of scale formulas for Wiener integrals of bounded functions in the Banach algebra $\mathcal{S}$ of analytic Feynman integrable functions on classical Wiener space. Yoo and Skoug extended this result to an abstract Wiener space. Also Yoo, Song, Kim and Chang established a change of scale formula for Wiener integrals of functions on abstract Wiener space which need not be bounded or continuous. In this paper, we investigate a change of scale formula for generalized Wiener integrals of various functions on classical Wiener space.

1. Introduction

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [3] and under translations [4]. Cameron and Storvick [3] expressed the analytic Feynman integral for a rather large class of functionals as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals on classical Wiener space $(C_0[0,T], m_w)$. In [12, 13, 15], Yoo, Yoon and Skoug extended these results to classical Yeh-Wiener space and to an abstract Wiener space (H, B, ν). In particular, Yoo and Skoug [12] established a change of scale formula for Wiener integrals of functions in the Fresnel class $\mathcal{F}(B)$ on abstract Wiener space, and then they [13] developed this formula for a more generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$. But functions in $\mathcal{F}(B)$ and $\mathcal{F}_{A_1,A_2}$ are bounded. In [14], Yoo, Song, Kim and Chang investigated a change of scale formula for Wiener integrals of functions on abstract Wiener space which need not be bounded or continuous. In this paper,

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we investigate change of scale formulas for generalized Wiener integrals of various functions on classical Wiener space.

2. Definitions and preliminaries

Let $C_0[0,T]$ denote the Wiener space, that is, the space of $\mathbb{R}$-valued continuous functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m_w denote Wiener measure. $(C_0[0,T],\mathcal{M},m_w)$ is a complete measure space and we denote the Wiener integral of a functional F by $\int_{C_0[0,T]} F(x)\,dm_w(x)$.

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the set of complex numbers, complex numbers with positive real part, and nonzero complex numbers with nonnegative real part, respectively.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable provided αE is measurable for each $\alpha>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m_w(\alpha N)=0$ for each $\alpha>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., then we write $F\approx G$.

Let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(\lambda)=\int_{C_0[0,T]} F(\lambda^{-1/2}Z_h(x,\cdot))\,dm_w(x) \tag{2.1}$$

exists as a finite number for all real $\lambda>0$ where Z_h is the Gaussian process

$$Z_h(x,t)=\int_0^t h(s)\,\tilde{d}x(s) \tag{2.2}$$

where h is in $L_2[0,T]$ and $\int_0^t h(s)\,\tilde{d}x(s)$ denotes the Paley-Wiener- Zygmund(P.W.Z) integral [2]. If there exists an analytic function $J^*(\lambda)$ on $\mathbb{C}_+$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$, then $J^*(\lambda)$ is defined to be the generalized analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write

$$I_a^\lambda(F)=J^*(\lambda). \tag{2.3}$$

Let F be a functional on $C_0[0,T]$ such that $I_a^\lambda(F)$ exists for all $\lambda\in\mathbb{C}_+$. If the following limit exists for nonzero real q, then we call it the generalized analytic Feynman integral of F over $C_0[0,T]$ with parameter q and we write

$$I_a^q(F) = \lim_{\lambda \to -iq} I_a^\lambda(F) \tag{2.4}$$

where $\lambda \to -iq$ through $\mathbb{C}_+$. When $h \equiv 1$, the generalized analytic Wiener integral and generalized analytic Feynman integral reduced to the analytic Wiener integral and analytic Feynman integral, respectively [6,10].

The Banach algebra $\mathcal{S}$ consists of functionals on $C_0[0,T]$ expressible in the form

$$F(y) = \int_{L_2[0,T]} \exp\{i(u,y)\}\, d\mu(u) \tag{2.5}$$

for s-a.e. y in $C_0[0,T]$ where μ is an element of $M(L_2[0,T])$, the space of $\mathbb{C}$-valued countably additive Borel measures on $L_2[0,T]$, and (u,y) denotes the P.W.Z. integral $\int_0^T u(t)\, \tilde{d}y(t)$.

The following existence theorem for generalized analytic Feynman integral of functions in the Banach algebra $\mathcal{S}$ is the result of Chung, Park and Skoug [6].

Theorem 2.1. ([6]) *Let $F \in \mathcal{S}$ be given by (2.5). Then the generalized analytic Feynman integral of F over $C_0[0,T]$ exists for all real $q \neq 0$ and*

$$I_a^q(F) = \int_{L_2[0,T]} \exp\{-\frac{i}{2q}\|uh\|_2^2\} d\mu(u). \tag{2.6}$$

In addition for each $\lambda \in \mathbb{C}_+$,

$$I_a^\lambda(F) = \int_{L_2[0,T]} \exp\{-\frac{1}{2\lambda}\|uh\|_2^2\} d\mu(u). \tag{2.7}$$

3. A change of scale formula for functionals in $\mathcal{S}$

In this section, we discuss a change of scale formula for generalized Wiener integrals of functions in $\mathcal{S}$ introduced by Cameron and Storvick.

We next introduce an integration formula which plays an important role in this section. This lemma is obtained by using a similar method as in the proof of Lemma 2 and 3 in [4] and hence we will state it without proof.

LEMMA 3.1. *Let λ be in $\mathbb{C}_+$, $h \in L_\infty[0,T]$ and $v \in L_2[0,T]$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a subset in $L_2[0,T]$ such that $\{\alpha_1 h, \alpha_2 h, \cdots, \alpha_n h\}$ are orthonormal on $L_2[0,T]$. Then*

$$\int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{k=1}^n (\alpha_k, Z_h(x,\cdot))^2 + i(v, Z_h(x,\cdot))\right\} dm_w(x)$$
$$= \lambda^{-n/2} \exp\left\{\frac{\lambda-1}{2\lambda}\sum_{k=1}^n \langle \alpha_k h, vh\rangle^2 - \frac{1}{2}\|vh\|_2^2\right\}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L_2[0,T]$.

Let h be in $L_\infty[0,T]$ with $1/h$ in $L_\infty[0,T]$ and let $Z_h(x,t)$ be given by (2.2). Let $\{\gamma_1, \cdots, \gamma_k, \cdots\}$ be a complete orthonormal set on $L_2[0,T]$. Now we set

$$\alpha_k = \gamma_k/h \quad \text{for } k = 1, 2, 3, \cdots, \tag{3.1}$$

and then the α_k's clearly belong to $L_2[0,T]$.

In the next theorem, we give a relationship between generalized Wiener integral and generalized analytic Wiener integral on Wiener space.

THEOREM 3.2. *Let $\{\alpha_k\}$ be given as in (3.1), let $F \in \mathcal{S}$, and let $\lambda \in \mathbb{C}^+$. Then*

$$I_a^\lambda(F) = \lim_{n\to\infty} \lambda^{n/2} \int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{k=1}^n (\alpha_k, Z_h(x,\cdot))^2\right\} F(Z_h(x,\cdot))dm_w(x).$$

Proof. Let $F \in \mathcal{S}$ be given by (2.5). By Fubini theorem and Lemma 3.1, we obtain that

$$\int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{k=1}^n (\alpha_k, Z_h(x,\cdot))^2\right\} F(Z_h(x,\cdot))dm_w(x)$$
$$= \int_{L_2[0,T]}\int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{k=1}^n (\alpha_k, Z_h(x,\cdot))^2 + i(v, Z_h(x,\cdot))\right\} dm_w(x)d\mu(v)$$
$$= \lambda^{-n/2}\int_{L_2[0,T]} \exp\left\{\frac{\lambda-1}{2\lambda}\sum_{k=1}^n \langle \alpha_k h, vh\rangle^2 - \frac{1}{2}\|vh\|_2^2\right\} d\mu(v).$$

Since $\{\alpha_k\}$ is given by (3.1), $\{\alpha_k h\}$ is a complete orthonormal set in $L_2[0,T]$. By the bounded convergence theorem and Parseval's relation, we obtain

$$\lim_{n\to\infty} \lambda^{n/2} \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2} \sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2 \right\} F(Z_h(x,\cdot)) dm_w(x)$$
$$= \int_{L_2[0,T]} \exp\{-\frac{1}{2\lambda}\|vh\|_2^2\} d\mu(v) = I_a^\lambda(F)$$

which completes the proof. □

The following theorem is a relationship between generalized Wiener integral and generalized analytic Feynman integral on Wiener space which follows from Theorem 3.2 and using (2.6) instead of (2.7).

THEOREM 3.3. *Let α_k and F be given as in Theorem 3.2 and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of complex numbers from $\mathbb{C}_+$ such that $\lambda_n \to -iq$. Then*

$$I_a^q(F) = \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda_n}{2} \sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2 \right\}$$
$$F(Z_h(x,\cdot)) dm_w(x).$$

The next theorem shows our change of scale formula for generalized Wiener integrals on Wiener space which follows from Theorem 3.2 above.

THEOREM 3.4. *Let $\rho > 0$ be given and let $\{\alpha_k\}$ be given as in (3.1). Then for $F \in \mathcal{S}$,*

$$\int_{C_0[0,T]} F(\rho Z_h(x,\cdot)) dm_w(x) \tag{3.2}$$
$$= \lim_{n\to\infty} \rho^{-n} \int_{C_0[0,T]} \exp\left\{ \frac{\rho^2-1}{2\rho^2} \sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2 \right\}$$
$$F(Z_h(x,\cdot)) dm_w(x).$$

Proof. By Theorem 3.2 and letting ρ^{-2} instead of λ, we obtain the change of scale formula for generalized Wiener integrals of functions having the form (2.5). □

COROLLARY 3.5. ([3]) *When $h \equiv 1$ in Theorem 3.4, we obtain Cameron and Storvick's change of scale formula for Wiener integrals* *of functions having the form (2.5),*

$$\int_{C_0[0,T]} F(\rho x)\, dm_w(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_0[0,T]} \exp\Big\{ \frac{\rho^2-1}{2\rho^2} \sum_{k=1}^{n} (\alpha_k, x)^2 \Big\} F(x)\, dm_w(x). \tag{3.3}$$

4. A change of scale formula for unbounded functions

Cameron and Storvick [5] introduced the class of functions of the form

$$F(x) = G(x)\Psi((\alpha_1, x), (\alpha_2, x), \cdots, (\alpha_r, x)) \tag{4.1}$$

for $G \in \mathcal{S}$ and $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^r)$, $1 \le p < \infty$, α_k's given as in (3.1) in Section 3, and $\phi \in \hat{M}(\mathbb{R}^r)$, the set of functions ϕ defined on $\mathbb{R}^r$ by

$$\phi(\vec{s}) = \int_{\mathbb{R}^r} \exp\Big\{ i \sum_{k=1}^{r} s_k t_k \Big\}\, d\rho(\vec{t}) \tag{4.2}$$

where ρ is a complex Borel measure of bounded variation on $\mathbb{R}^r$, $\vec{s} = (s_1, \cdots, s_r)$ and $\vec{t} = (t_1, \cdots, t_r)$. And they showed that the above functions (4.1) which need not be bounded or continuous are analytic Feynman integrable.

In this section, we establish a change of scale formula for generalized Wiener integrals of functions of the form (4.1).

To simplify the expressions, we use the following notations:

$$(\vec{\alpha}, x) = ((\alpha_1, x), (\alpha_2, x), \cdots, (\alpha_r, x))$$

and

$$(\vec{\alpha}h, x) = ((\alpha_1 h, x), (\alpha_2 h, x), \cdots, (\alpha_r h, x)).$$

The following theorem is the existence theorem for generalized analytic Feynman intgral of above functions (4.1) which corresponds to Cameron and Storvick's theorem [5] for analytic Feynman integral of these functions. Using the similar methods as in the proof of Cameron and Storvick's theorem, we obtain the following existence theorem and so we will state it without proof.

THEOREM 4.1. *Let* $F(x) = G(x)\Psi((\vec{\alpha}, x))$ *where* $G \in \mathcal{S}$, $\Psi = \psi + \phi \in L_p(\mathbb{R}^r) + \hat{M}(\mathbb{R}^r)$, $1 \le p < \infty$. *Then for each* $\lambda \in \mathbb{C}_+$, F *is generalized* *analytic Wiener integrable. Moreover if G and ϕ are given by (2.5) and (4.2), respectively and $\psi \in L_p(\mathbb{R}^r)$, then*

$$(4.3)\quad I_a^\lambda(F) = \Big(\frac{\lambda}{2\pi}\Big)^{r/2} \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\{S_1(\lambda, v, \vec{t})\}\psi(\vec{t})\, d\vec{t}\, d\mu(v) + \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\{S_2(\lambda, v, \vec{t})\}\, d\rho(\vec{t})\, d\mu(v)$$

where

$$S_1(\lambda, v, \vec{t}) = \frac{1}{2\lambda}\Big[\sum_{k=1}^{r} (i\lambda t_k + \langle \alpha_k h, vh\rangle)^2 - \|vh\|_2^2\Big]$$

and

$$S_2(\lambda, v, \vec{t}) = -\frac{1}{2\lambda}\Big[\|vh\|_2^2 + \sum_{k=1}^{r} 2t_k\langle \alpha_k h, vh\rangle + \sum_{k=1}^{r} t_k^2\Big].$$

In case $p = 1$, for each real $q \neq 0$, F is generalized analytic Feynman integrable. Moreover if $\{\alpha_k\}$ is given as in (3.1), then

$$(4.4)\quad I_a^q(F) = \Big(-\frac{iq}{2\pi}\Big)^{r/2} \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\{S_1(-iq, v, \vec{t})\}\psi(\vec{t})\, d\vec{t}\, d\mu(v) + \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\{S_2(-iq, v, \vec{t})\}\, d\rho(\vec{t})\, d\mu(v).$$

Now we give a relationship between generalized Wiener integral and generalized analytic Wiener integral on Wiener space.

Theorem 4.2. *Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots\}$ be given as in (3.1). Let $F(x) = G(x)\ \psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\psi \in L_p(\mathbb{R}^r), 1 \leq p < \infty$. Then for each $\lambda \in \mathbb{C}_+$, we have*

$$I_a^\lambda(F) = \lim_{n\to\infty} \lambda^{n/2} \int_{C_0[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2\right\} F(Z_h(x,\cdot))dm_w(x).$$

Proof. Let n be a natural number with $n > r$ and let

$$\Gamma(n) = \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2\Big\} F(Z_h(x,\cdot))\, dm_w(x).$$

By the Fubini theorem, we have

$$\Gamma(n) = \int_{L_2[0,T]} \int_{C_0[0,T]} \exp\Big\{ \frac{1-\lambda}{2} \sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2 + i(v, Z_h(x,\cdot)) \Big\}$$
$$\psi\left((\vec{\alpha}, Z_h(x,\cdot))\right)\, dm_w(x)\, d\mu(v)$$
$$= \Big(\frac{\lambda}{2\pi}\Big)^{r/2} \Big(\frac{1}{\lambda}\Big)^{n/2} \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\Big\{ \frac{\lambda-1}{2\lambda} \sum_{k=1}^{n} \langle \alpha_k h, vh\rangle^2 - \frac{1}{2}\|vh\|_2^2 \Big\}$$
$$\exp\Big\{ \frac{1}{2\lambda} \sum_{k=1}^{r} \left(i\lambda t_k + \langle \alpha_k h, vh\rangle\right)^2 \Big\} \psi(\vec{t})\, d\vec{t}\, d\mu(v).$$

Note that, by the Bessel inequality, we have

$$\Big| \exp\Big\{ \frac{\lambda-1}{2\lambda} \sum_{k=1}^{n} \langle \alpha_k h, vh\rangle^2 - \frac{1}{2}\|vh\|_2^2$$
$$+ \frac{1}{2\lambda} \sum_{k=1}^{r} \left(i\lambda t_k + \langle \alpha_k h, vh\rangle\right)^2 \Big\} \psi(\vec{t}) \Big| \le \exp\Big\{ -\frac{Re\lambda}{2} \sum_{k=1}^{r} t_k^2 \Big\} |\psi(\vec{t})|$$

and the right hand side of the inequality above is integrable on $L_2[0,T] \times \mathbb{R}^r$, since $\psi \in L_p(\mathbb{R}^r)$ and $\mu \in M(L_2[0,T])$. Hence by the dominated convergence theorem and Parseval's relation, we obtain

$$\lim_{n\to\infty} \lambda^{n/2} \Gamma(n) = \Big(\frac{\lambda}{2\pi}\Big)^{r/2} \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\Big\{ S_1(\lambda, v, \vec{t}) \Big\} \psi(\vec{t})\, d\vec{t}\, d\mu(v).$$

By Theorem 4.1, the proof is completed. □

Moreover if $p = 1$, we obtain a relationship between generalized Wiener integral and generalized analytic Feynman integral on Wiener space.

THEOREM 4.3. *Let $\{\alpha_k\}$ be given as in (3.1). Let $F(x) = G(x)\psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\psi \in L_1(\mathbb{R}^r)$ and let $\{\lambda_n\}$ be a sequence of complex numbers in $\mathbb{C}_+$ such that $\lambda_n \longrightarrow -iq$. Then*

$$I_a^q(F) = \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\Big\{ \frac{1-\lambda_n}{2} \sum_{k=1}^{n} (\alpha_k, Z_h(x,\cdot))^2 \Big\}$$
$$F(Z_h(x,\cdot))\, dm_w(x).$$

Proof. Let n be a natural number with $n > r$ and let

$$\begin{aligned}\Gamma(n,\lambda_n) &= \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda_n}{2}\sum_{k=1}^{n}(\alpha_k, Z_h(x,\cdot))^2\Big\} \\ &\qquad F(Z_h(x,\cdot))\,dm_w(x).\end{aligned}$$

By the same method as in the proof of Theorem 4.2, we have

$$\begin{aligned}&\Gamma(n,\lambda_n) \\ &= \Big(\frac{\lambda_n}{2\pi}\Big)^{r/2}\Big(\frac{1}{\lambda_n}\Big)^{n/2}\int_{L_2[0,T]}\int_{\mathbb{R}^r}\exp\Big\{\frac{\lambda_n-1}{2\lambda_n}\sum_{k=1}^{n}\langle\alpha_k h, vh\rangle^2 - \frac{1}{2}\|vh\|_2^2\Big\} \\ &\quad \exp\Big\{\frac{1}{2\lambda_n}\sum_{k=1}^{r}(i\lambda_n t_k + \langle\alpha_k h, vh\rangle)^2\Big\}\psi(\vec{t})\,d\vec{t}\,d\mu(v).\end{aligned}$$

Using the Bessel inequality in the first exponent above, we have that the absolute value of the exponentials above is bounded by unity. And also $|\psi(\vec{t})|$ is integrable on $L_2[0,T]\times\mathbb{R}^r$ since $\psi\in L_1(\mathbb{R}^r)$ and $\mu\in M(L_2[0,T])$. Hence by the dominated convergence theorem and Parseval's relation, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\lambda_n^{n/2}\Gamma(n,\lambda_n) \\ &= \Big(-\frac{iq}{2\pi}\Big)^{r/2}\int_{L_2[0,T]}\int_{\mathbb{R}^r}\exp\{S_1(-iq,v,\vec{t})\}\psi(\vec{t})\,d\vec{t}\,d\mu(v).\end{aligned}$$

By Theorem 4.1, the proof is completed. □

THEOREM 4.4. *Let $\{\alpha_k\}$ be given as in (3.1). Let $F(x) = G(x)\times\phi((\vec{\alpha},x))$ where $G\in\mathcal{S}$ and $\phi\in\hat{M}(\mathbb{R}^r)$. Then the formulas in Theorem 4.2 and 4.3 hold.*

Proof. Let n be a natural number with $n > r$ and let

$$\Gamma(n) = \int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n}(\alpha_k, Z_h(x,\cdot))^2\Big\}F(Z_h(x,\cdot))\,dm(x).$$

By the Fubini theorem, we have

$$\begin{aligned}\Gamma(n) &= \int_{L_2[0,T]}\int_{\mathbb{R}^r}\int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n}(\alpha_k, Z_h(x,\cdot))^2 \\ &\qquad + i(v, Z_h(x,\cdot)) + i\sum_{k=1}^{r} t_k(\alpha_k, Z_h(x,\cdot))\Big\}\, dm_w(x)\, d\rho(\vec{t})\, d\mu(v) \\ &= \lambda^{-n/2}\int_{L_2[0,T]}\int_{\mathbb{R}^r}\exp\big\{S_3(\lambda, v, \vec{t})\big\}\, d\rho(\vec{t})\, d\mu(v)\end{aligned}$$

where

$$S_3(\lambda, v, \vec{t}) = \frac{\lambda-1}{2\lambda}\sum_{k=1}^{n}(\alpha_k h, vh) - \frac{1}{\lambda}\sum_{k=1}^{r} t_k(\alpha_k h, vh) - \frac{1}{2\lambda}\sum_{k=1}^{r} t_k^2 - \frac{1}{2}\|vh\|_2^2.$$

Using the Bessel inequality, we have that the exponential of the last expression above is bounded in absolute value by unity. Hence by the dominated convergence theorem and Parseval's relation, we obtain

$$\lim_{n\to\infty}\lambda^{n/2}\Gamma(n) = \int_{L_2[0,T]}\int_{\mathbb{R}^r}\exp\big\{S_2(\lambda, v, \vec{t})\big\} d\rho(\vec{t})\, d\mu(v).$$

By Theorem 4.1, the proof is completed. □

From Theorem 4.2, Theorem 4.3, Theorem 4.4 and the linearity of the generalized analytic Wiener integral on Wiener space, we have the following corollaries.

Corollary 4.5. *Let $\{\alpha_k\}$ be given as in (3.1). Let F be given as in Theorem 4.1. Then the formula in Theorem 4.2 holds.*

Corollary 4.6. *Let$\{\alpha_k\}$ and $\{\lambda_k\}$ be given as in Theorem 4.3. Let $F(x) = G(x)\Psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\Psi = \psi + \phi \in L_1(\mathbb{R}^r) + \hat{M}(\mathbb{R}^r)$. Then the formula in Theorem 4.3 holds.*

Our main result, namely a change of scale formula for generalized Wiener integrals on Wiener space now follows from Corollary 4.5.

Theorem 4.7. *Let $\{\alpha_k\}$ be given as in (3.1). Let F be given as in Theorem 4.1. Then the change of scale formula (3.2) holds.*

Proof. By letting $\lambda = \rho^{-2}$ in Corollary 4.5, we obtain the change of scale formula for generalized Wiener integrals of functions having the form (4.1). □

Corollary 4.8. ([14]) *When $h \equiv 1$ in Theorem 4.7, we have the change of scale formula (3.3) for Wiener integrals of functions of the form (4.1).*

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