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WEYL STRUCTURES ON COMPACT CONNECTED LIE GROUPS

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Abstract. Let G be a compact connected semisimple Lie group, B the Killing form of the algebra $\mathfrak{g}$ of G, and g the invariant metric induced by B. Then, we obtain a necessary and sufficient condition for a left invariant linear connection D with a Weyl structure (D, g, ω) on (G, g) to be projectively flat (resp. Einstein-Weyl). And, we also get that if a left invariant linear connection D with a Weyl structure (D, g, ω) on (G, g) which has symmetric Ricci tensor Ric^D is projectively flat, then the connection D is Einstein-Weyl; but the converse is not true. Moreover, we show that if a left invariant connection D with Weyl structure (D, g, ω) on (G, g) is projectively flat (resp. Einstein-Weyl), then D is a Yang-Mills connection.

1. Introduction

A Weyl structure on a Riemannian manifold (M, g) is a torsion free affine connection D preserving a conformal structure $[g]$. Namely, a torsion free affine connection D is called a *Weyl structure* if $Dg = \omega \otimes g$ for a 1-form ω.

A Weyl structure (D, g, ω) on a Riemannian manifold (M, g) is said to be *Einstein-Weyl* (cf. [16], [18], [21]) if the symmetrized Ricci tensor Ric^D is proportional to g, that is,

$$SymRic^D(X, Y) = \wedge\, g(x, y), \qquad (\wedge \in C^\infty(M)). \tag{1.1}$$

Thus, an Einstein-Weyl structure is a generalization of Einstein metric in terms of affine connection.

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Let G be an n-dimensional ($n \geq 3$) compact connected semisimple Lie group, B the Killing form of the Lie algebra $\mathfrak{g}$ of G, and g the left invariant Riemannian metric induced by B. Then we obtain a necessary and sufficient condition for a left invariant affine connection D with Weyl structure (D, g, ω) on the Riemannian manifold (G, g) to be projectively flat (resp. Einstein-Weyl), (cf. Theorem 2.6 (resp. Lemma 3.1)).

In general, it is well known that
(i) the Levi-Civita connection ∇ on a Riemannian manifold (M, g) is projectively flat if and only if (M, g) is a space of constant curvature.
(ii) if the Levi-Civita connection ∇ on (M, g) is projectively flat then the metric g is Einstein.

In this paper, we obtain the fact that if a connection D with Weyl structure (D, g, ω) on (G, g) which has symmetric Ricci tensor Ric^D is projectively flat, then the connection D is Einstein-Weyl, but the converse is not true (cf. Theorem 2.6, Lemma 3.1 and Remark 3.2).

And then, we get that if a left invariant affine connection D with Weyl structure (D, g, ω) on (G, g) is projectively flat (resp. Einstein-Weyl), then the connection D is a Yang-Mills connection (cf. Corollary 2.8 and Theorem 3.4).

2. Projectively flat connections

2.1. In this subsection, we introduce the notion of projectively flat connection in the tangent bundle TM over an n-dimensional manifold M. We say that two affine connections D and $\tilde{D}$ are *projectively equivalent* if there exists a 1-form τ on M such that

$$\tilde{D}_X Y = D_X Y + \tau(X)Y + \tau(Y)X, \quad (X, Y \in \mathfrak{X}(M)). \tag{2.1}$$

And, we say that an affine connection D is *projectively flat* if D is projectively equivalent to a flat affine connection around each point of M.

Suppose that a connection D is torsion free and its Ricci tensor Ric^D is symmetric. We define the *projective curvature tensor* W_p by

$$\begin{aligned} W_p(X, Y)Z = {} & R^D(X, Y)Z \\ & - \frac{1}{n-1}\{Ric^D(Y, Z)X - Ric^D(X, Z)Y\}. \end{aligned} \tag{2.2}$$

If two torsion free affine connections D and $\tilde{D}$ with symmetric Ricci tensors are projectively equivalent, then their projective curvature tensors coincide. For $n \geq 3$, a torsion free affine connection D with symmetric Ricci tensor is projectively flat if and only if its projective curvature tensor vanishes ([9] Theorem 3.3).

2.2. In this subsection, we introduce some well known facts on a Weyl structure (D, g, ω) (cf. [8], [17], [19], [20]) on a smooth Riemannian manifold (M, g).

Let (D, g, ω) be a Weyl structure in the tangent bundle TM over a smooth Riemannian manifold (M, g), that is,

$$Dg = \omega \otimes g, \text{ and } T^D = 0 \text{ (torsion free)} \tag{2.3}$$

for some 1-form ω on M. Then, we have for $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(TM)$

$$\begin{cases} D^*{}_X Z = D_X Z + \omega(X) Z, \\ R^D(X,Y) - R^\nabla(X,Y) = [\nabla_X, \alpha_Y] + [\alpha_X, \nabla_Y] \\ \qquad\qquad\qquad\qquad\qquad + [\alpha_X, \alpha_Y] - \alpha_{[X,Y]}, \end{cases} \tag{2.4}$$

where $\alpha := D - \nabla \in \Gamma(TM^* \otimes \mathrm{End}(TM))$. Here, D^* is given by

$$g(D^*{}_X Y, Z) = X(g(Y,Z)) - g(Y, D_X Z)$$

for $X \in \mathfrak{X}(M)$ and $Y, Z \in \Gamma(TM)(= \mathfrak{X}(M))$. The connection D^* is said to be the *conjugate* connection (cf. [2], [3], [9], [15]) of D. Moreover, we have for $X, Y \in \mathfrak{X}(M)$

$$\alpha_X Y := D_X Y - \nabla_X Y = \alpha_Y X, \tag{2.5}$$

since the connections are torsion free. Let $\{X_i\}_{i=1}^n$ be an (locally defined) orthonormal frame on (M, g), where $n = \mathrm{dim} M$. Using (2.5) and fundamental properties of a connection, we get

$$\alpha_X Y = \frac{1}{2}\{g(X,Y)\omega^\sharp - \omega(X)Y - \omega(Y)X\}, \tag{2.6}$$

where $\omega^\sharp := \sum_{i=1}^n \omega(X_i) X_i$. For the (locally defined) orthonormal frame $\{X_i\}_{i=1}^n$, let $\{\theta^j\}_{j=1}^n$ be the local orthonormal coframe on (M, g). For the local frames $\{X_i\}_{i=1}^n$ and $\{\theta^j\}_{j=1}^n$, we introduce $\Gamma_{ij}{}^l := \theta^l(D_{X_i} X_j)$. Then we have

$$D_{X_i} X_j = \sum_{l=1}^n \Gamma_{ij}{}^l X_l, \text{ and } D_{X_i}\theta^j = -\sum_{l=1}^n \Gamma_{il}{}^j \theta^l. \tag{2.7}$$

By virtue of the fact $Dg = \omega \otimes g$, we have

$$\Gamma_{ij}{}^{j} = -\frac{1}{2}\omega(X_i) \text{ for each } j, \text{ and } \Gamma_{ij}{}^{k} = -\Gamma_{ik}{}^{j} \ (j \neq k). \tag{2.8}$$

Furthermore, a Weyl structure (D, g, ω) is said to be *Einstein-Weyl* with respect to $\wedge$, $(\wedge \in C^{\infty}(M))$, if $SymRic^{D} = \wedge\, g$.

2.3. Let G be an n-dimensional compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of G (the set of all left invariant vector fields on G), g the left invariant Riemannian metric on G which is induced by minus the Killing form (cf. [4], [10], [11], [13]) of $\mathfrak{g}$. The metric g is said to be the *canonical metric* on the Lie group G. Then the metric g is bi-invariant on G, and the Levi-Civita connection ∇ for the metric g is given by (cf. [5])

$$\nabla_X Y = \frac{1}{2}\,[X, Y], \quad (X, Y \in \mathfrak{g}). \tag{2.9}$$

The above Levi-Civita connection is bi-invariant. Let $\{X_i\}_{i=1}^{n}$ be an orthonormal basis of the semisimple Lie algebra $\mathfrak{g}$ with respect to the canonical metric g. Let $\{\theta^j\}_{j=1}^{n}$ be the dual basis of the basis $\{X_i\}_{i=1}^{n}$. Then each θ^j is left invariant, that is, $L_x^*(\theta^j) = \theta^j$ $(x \in G)$. From (2.9), we have

$$\theta^l(\nabla_{X_i} X_j) = \frac{1}{2}\, C_{ij}{}^{l}, \tag{2.10}$$

where $C_{ij}{}^{l} := \theta^l([X_i, X_j])$ for the orthonormal frame $\{X_i\}_{i=1}^{n}$. Let D be a left invariant connection in the tangent bundle over the Riemannian manifold (G, g) such that (D, g) is a Weyl structure with respect to a 1-form ω. Then we have the following

Lemma 2.1. *The 1-form ω is left invariant.*

Proof.

$$\begin{aligned} D_{X_i} g &= D_{X_i}\Big(\sum_{j=1}^{n} \theta^j \otimes \theta^j\Big) \\ &= -\sum_{j,l=1}^{n} \Gamma_{il}{}^{j}(\theta^l \otimes \theta^j + \theta^j \otimes \theta^l) = \omega(X_i) g, \end{aligned}$$

where $\Gamma_{ij}{}^{l} := \theta^l(D_{X_i} X_j)$. Thus, from this fact we have

$$\omega(X_i) = -2\,\Gamma_{ij}{}^{j} \text{ for each } j, \text{ and } \Gamma_{ik}{}^{j} = -\Gamma_{ij}{}^{k}, \ (j \neq k). \tag{2.11}$$

Since D is left invariant, each ${\Gamma_{ij}}^k$ is constant. So, $\omega(X_i)$ for each i is constant. Hence $L_x{}^*\omega = \omega$, $(x \in G)$. Thus the proof of this Lemma is completed. □

By virtue of (2.9) and properties of the Killing form on the semisimple Lie algebra $\mathfrak{g}$, we have for $X, Y, Z \in \mathfrak{g}$

$$g([X,Y],Z) + g(Y,[X,Z]) = 0, \quad R^{\nabla}(X,Y) = -\frac{1}{4}\, ad([X,Y]), \tag{2.12}$$

where ad is the adjoint representation of the semisimple Lie algebra $\mathfrak{g}$. From (2.12) and the definition of the Killing form B of the semisimple Lie algebra $\mathfrak{g}$, $(B(X,Y) := \text{Trace } (ad(X)ad(Y)),\ (X, Y \in \mathfrak{g}))$, we get for $Y, Z \in \mathfrak{g}$ (cf. [5])

$$\sum_{i=1}^{n} g(R^{\nabla}(X_i, Y)Z, X_i) = \frac{1}{4}\, g(Y,Z), \tag{2.13}$$

that is, the Riemannian manifold (G,g) is an Einstein manifold of Ricci curvature $\frac{1}{4}$. From the fact $g(\nabla_{X_i}X_j, X_k) + g(X_j, \nabla_{X_i}X_k) = 0$, (2.9) and (2.10), we have

$${C_{ij}}^k = -{C_{ik}}^j = -{C_{kj}}^i. \tag{2.14}$$

By virtue of the fact $g = -B$ and (2.14), we get

$$\sum_{i,l=1}^{n} {C_{il}}^k {C_{il}}^j = \delta_{kj}. \tag{2.15}$$

For later use, we have

Lemma 2.2.

$$2 \sum_{i,s,t=1}^{n} {C_{ij}}^t {C_{tk}}^s {C_{si}}^l = {C_{jk}}^l.$$

Proof. By virtue of (2.12), (2.14) and (2.15),

$$\begin{aligned}
\sum_{i,s,t} {C_{ij}}^t {C_{tk}}^s {C_{si}}^l &= \sum_{i,s,t} g([[[X_i, X_j], X_k], X_i], X_l) \\
&= \sum_{i,s,t} g([[X_i, X_j], X_k], [X_i, X_l]) \\
&= -\sum_{i,s,t} g([[X_j, X_k], X_i] + [[X_k, X_i], X_j], [X_i, X_l])
\end{aligned}$$

$$= -\sum_{i,s,t}(C_{jk}{}^t C_{ti}{}^s C_{il}{}^s + C_{ki}{}^t C_{tj}{}^s C_{il}{}^s)$$

$$= C_{jk}{}^l - \sum_{i,s,t} C_{tj}{}^i C_{ks}{}^t C_{sl}{}^i = C_{jk}{}^l - \sum_{i,s,t} C_{ij}{}^t C_{tk}{}^s C_{si}{}^l.$$

Thus, the proof of this Lemma is completed. □

2.4. We retain the notations as in 2.3. By virtue of (2.4), (2.6), (2.9) and (2.12), we obtain on the Riemannian manifold (G, g) with Weyl structure (D, g, ω)

$$\begin{aligned} R^D(X,Y)Z =& \frac{1}{4}\ \{-[[X,Y],Z] + 2\ \omega([X,Y])\ Z \\ & + (\omega([X,Z]) - \omega(X)\cdot\omega(Z) + \omega(\omega^\sharp)\cdot g(X,Z))\ Y \\ & - (\omega([Y,Z]) - \omega(Y)\cdot\omega(Z) + \omega(\omega^\sharp)\cdot g(Y,Z))\ X \\ & + g(Y,Z)\ [X,\omega^\sharp] - g(X,Z)\ [Y,\omega^\sharp] \\ & + \omega(X)\cdot g(Y,Z)\ \omega^\sharp - \omega(Y)\cdot g(X,Z)\ \omega^\sharp\}, \end{aligned} \tag{2.16}$$

where $X, Y, Z \in \mathfrak{g}$. From (2.9), (2.10) and (2.16), we have for the orthonormal basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$

$$\begin{aligned} R^D(X_i,X_j)X_k =& \frac{1}{4}\ [\ 2\ \sum_t C_{ij}{}^t\omega_t X_k \\ & - (||\omega||_g{}^2\delta_{jk} - \omega_j\omega_k + \sum_t C_{jk}{}^t\omega_t)X_i \\ & + (||\omega||_g{}^2\delta_{ik} - \omega_i\omega_k + \sum_t C_{ik}{}^t\omega_t)X_j \\ & + \sum_l\{-\sum_t C_{ij}{}^t C_{tk}{}^l + \omega_i\omega_l\delta_{jk} - \omega_j\omega_l\delta_{ik} \\ & \quad + \sum_t(\omega_t\delta_{jk}C_{it}{}^l - \delta_{ik}\omega_t C_{jt}{}^l)\}X_l], \end{aligned} \tag{2.17}$$

where $\omega_i := \omega(X_i)$. The Ricci tensor Ric^D, of type (0,2), is defined by

$$Ric^D(Y,Z) := trace\{X \longmapsto R^D(X,Y)Z\}. \tag{2.18}$$

From (2.14), (2.15), (2.17) and (2.18), we get

$$\begin{aligned} Ric^D(X_j,X_k) =& \frac{1}{4}\ \{\delta_{jk} + (n-2)(\omega_j\omega_k - \delta_{jk}||\omega||_g{}^2) \\ & - n\sum_t \omega_t C_{jk}{}^t\}. \end{aligned} \tag{2.19}$$

Furthermore, we have for the 1-form ω

$$d\omega(X_j, X_k) = -\sum_t \omega_t C_{jk}{}^t. \tag{2.20}$$

By virtue of (2.19) and (2.20), we get

PROPOSITION 2.3. *Let G be an n-dimensional $(n \geq 3)$ compact connected semisimple Lie group, and B the Killing form of the Lie algebra $\mathfrak{g}$ of G, and g the canonical metric induced from B. Let D be a left invariant connection in the tangent bundle over the manifold (G, g) such that (D, g) is a Weyl structure with respect to a 1-form ω. Then, a necessary and sufficient condition for the Ricci tensor Ric^D to be symmetric is $d\omega = 0$, that is, ω is closed.*

More concisely we get by virtue of (2.19) and (2.20) and Proposition 2.3

COROLLARY 2.4. *Under the same hypotheses as in Proposition 2.3, then the following conditions are equivalent :*
(i) *the Ricci tensor Ric^D is symmetric;*
(ii) *$d\omega = 0$, that is, ω is closed;*
(iii) $\sum_t C_{ij}{}^t \omega_t = 0$.

For Riemannian manifolds (M, g) with Weyl structure (D, g, ω), Proposition 2.3 holds good (cf. [1]).

Using (2.14), (2.17), (2.19) and Corollary 2.4, we obtain under the same hypotheses as in Proposition 2.3

LEMMA 2.5. *Assume Ric^D is symmetric. Then*

$$\begin{aligned} R^D(X_i, X_j)X_k = &\frac{1}{4}\ \{(\omega_j\omega_k - \delta_{jk}\|\omega\|_g{}^2)X_i - (\omega_i\omega_k - \delta_{ik}\|\omega\|_g{}^2)X_j \\ &+ \sum_l (\sum_t C_{ji}{}^t C_{tk}{}^l + \delta_{jk}\omega_i\omega_l - \delta_{jk}\omega_j\omega_l)X_l\}, \end{aligned} \tag{2.21}$$

$$Ric^D(X_i, X_k) = \frac{1}{4}\ \{\delta_{jk} + (n-2)(\omega_j\omega_k - \delta_{jk}\|\omega\|_g{}^2)\}. \tag{2.22}$$

2.5. In this subsection, we retain the same notations as in 2.3 and 2.4. We always assume Ric^D is symmetric, and $\dim G \geq 3$. And then, we get a necessary and sufficient condition for the connection D with Weyl structure (D, g, ω) to be projectively flat.

Assume a connection D with Weyl structure (D, g, ω) on (G, g) which has symmetric Ricci tensor Ric^D is projectively flat, and dim $G \geq 3$. Then, from the statements of the subsection 2.1, we get

$$R^D(X_i, X_j)X_k = \frac{1}{n-1} \{Ric^D(X_j, X_k)X_i - Ric^D(X_i, X_k)X_j\}. \tag{2.23}$$

By the help of (2.22) and (2.23), we have

$$\begin{aligned} R^D(X_i, X_j)X_k =& \frac{1}{4(n-1)} [\{\delta_{jk} + (n-2)(\omega_j\omega_k - \delta_{jk}\|\omega\|_g{}^2)\}X_i \\ & - \{\delta_{ik} + (n-2)(\omega_i\omega_k - \delta_{ik}\|\omega\|_g{}^2)\}X_j]. \end{aligned} \tag{2.24}$$

Taking mutually different indices i, j, k in (2.21) and (2.24), we have

$$\omega_j\omega_k X_i - \omega_i\omega_k X_j + (n-1)\sum_{l,t} C_{ji}{}^t C_{tk}{}^l X_l = 0. \tag{2.25}$$

From (2.25) with mutually different indices i, j, k, we get

$$\omega_j\omega_k = (1-n)\sum_t C_{ji}{}^t C_{tk}{}^i. \tag{2.26}$$

By the help of (2.14), (2.15) and (2.26), we have for mutually different indices j, k

$$\begin{aligned} (n-2)\omega_j\omega_k &= (1-n)\sum_{i,t,(i\neq j,k)} C_{ji}{}^t C_{tk}{}^i \\ &= (1-n)\sum_{i,t} C_{it}{}^j C_{it}{}^k = 0. \end{aligned} \tag{2.27}$$

From the fact dim$G \geq 3$ and (2.27), we get

$$\omega = 0. \tag{2.28}$$

From (2.21), (2.24) and (2.28), we get

$$\sum_t C_{ji}{}^t C_{tk}{}^l = \frac{1}{n-1} (\delta_{jk}\delta_i^l - \delta_{ik}\delta_j{}^l). \tag{2.29}$$

Conversely, assume that

$$\omega = 0, \text{ and } \sum_t C_{ji}{}^t C_{tk}{}^l = \frac{1}{n-1}(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j^l). \tag{2.30}$$

Then, by virtue of (2.21) and (2.22), we obtain

$$R^D(X_i, X_j)X_k = \frac{1}{n-1} \{Ric^D(X_j, X_k)X_i - Ric^D(X_i, X_k)X_j\}. \tag{2.31}$$

Thus, we obtain

THEOREM 2.6. *Let G be an n-dimensional $(n \geq 3)$ compact connected semisimple Lie group, and B the Killing form of the Lie algebra of G, and g the canonical metric induced from B. Let D be a left invariant linear connection on (G, g) such that (D, g) is a Weyl structure with respect 1-form ω and Ric^D is symmetric. Then, a necessary and sufficient condition for D to be projectively flat is*

$$\omega = 0, \quad \textit{and}$$

$$\sum_t C_{ji}{}^t C_{tk}{}^l = \frac{1}{(n-1)} \left(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j{}^l\right) \tag{2.32}$$

$$(= 4 \cdot g(R^D(X_i, X_j)X_k, X_l)).$$

REMARK 2.7. By virtue of Theorem 2.6, we find the fact that if a left invariant connection D with Weyl structure (D, g, ω) on (G, g) which has symmetric Ric^D is projectively flat, then

(i) $\omega = 0$, that is, D coincides with the Levi-Civita connection ∇ on (G, g),

(ii) (G, g) is a space of constant curvature, and

(iii)
$$\begin{aligned}(\delta_D R^D)(X_j)X_k &= -\sum_i (D^*_{X_i} R^D)(X_i, X_j)X_k \\ &= -\sum_i (\nabla_{X_i} R^{\nabla})(X_i, X_j)X_k = 0,\end{aligned}$$

where δ_D is the formal adjoint operator of the covariant exterior differentiation d_D (cf. [3], [6], [7], [12], [14], [15]), and D^* is the conjugate connection of D.

The fact (iii) above is easily obtained by simple computation, since (G, g) is a space of constant curvature. Thus, we have from the fact (iii) above

COROLLARY 2.8. *Under the same hypotheses as in Theorem 2.6, we assume the connection D is projectively flat. Then, D is a Yang-Mills connection.*

3. Einstein-Weyl structures

In this section, we retain the notations as in 2.3 and 2.4. We get a necessary and sufficient condition for an invariant linear connection D with Weyl structure (D, g, ω) on the Riemannian manifold (G, g) to be

Einstein-Weyl. And then, we show that the Einstein-Weyl connection D on (G, g) is a Yang-Mills connection.

Let D be a left invariant connection on the manifold (G, g) such that (D, g, ω) is a Weyl structure.

Now, assume that the connection D with Weyl structure (D, g, ω) is Einstein-Weyl, that is,

$$(3.1) \qquad Dg = \omega \otimes g, \quad T^D = 0, \quad \text{and} \quad SymRic^D = \wedge\, g,$$

where $\wedge \in C^\infty(G)$. In this case, $\wedge$ is constant. From (2.14) and (2.19), we have

$$(3.2) \qquad SymRic^D(X_j, X_k) = \frac{1}{4}\, \{\delta^{jk} + (n-2)(\omega_j\omega_k - \delta_{jk}\|\omega\|_g{}^2)\}.$$

Taking different indices j, k in (3.2), by the help of (3.1) and the fact dim $G \geq 3$ we get

$$\omega = 0, \quad \text{from which} \quad \wedge = \frac{1}{4}.$$

Conversely, from (3.2) we find that if $\omega = 0$ in the Weyl structure (D, g, ω), then D is Einstein Weyl. Thus, we have

LEMMA 3.1. *Let D be a left invariant linear connection on (G, g) such that (D, g, ω) is a Weyl structure. Then, a necessary and sufficient condition for the connection D to be Einstein-Weyl is $\omega = 0$.*

REMARK 3.2. By virtue of Theorem 2.6 and Lemma 3.1, we find out the fact that if a left invariant connection D with Weyl structure (D, g, ω) on (G, g) is projectively flat, then D is Einstein-Weyl, but the converse is not true.

A Yang-Mills connection is a critical point of the Yang-Mills functional (cf. [3], [12], [22]) which is defined on the space $\mathfrak{C}_E$ of all connections in a smooth vector bundle E over a compact Riemannian manifold (M, g). A connection D in the tangent bundle TM over (M, g) is a Yang-Mills connection if and only if it satisfies the Yang-Mills equation $\delta_D R^D = 0$.

Finally, we assume that the connection D with Weyl structure (D, g, ω) on (G, g) is Einstein-Weyl. Then, $\omega = 0$ by Lemma 3.1, and so D coincides with the Levi-Civita connection ∇ on (G, g). We get from these facts, (2.14) and (2.17)

$$(3.4) \qquad R^D(X_i, X_j)X_k = R^\nabla(X_i, X_j)X_k = \frac{1}{4}\sum_{l,t} C_{ji}{}^t C_{tk}{}^l X_l.$$

Then, we have

$$(\delta_D R^D)(X_j)X_k = (\delta_\nabla R^\nabla)(X_j)X_k$$
$$= -\sum_i (\nabla_{X_i} R^\nabla)(X_i, X_j)X_k \tag{3.5}$$
$$= -\sum_i \{\nabla_{X_i}(R^\nabla(X_i, X_j)X_k) - R^\nabla(X_i, \nabla_{X_i}X_j)X_k - R^\nabla(X_i, X_j)\nabla_{X_i}X_k\}.$$

In order to analyze (3.5), we obtain from (2.14), (2.15) and (3.4)

LEMMA 3.3. *The following equations are hold.*

(i) $\sum_i \nabla_{X_i}(R^\nabla(X_i, X_j)X_k) = \frac{1}{8} \sum_{i,l,s,t} C_{ij}{}^t C_{tk}{}^s C_{si}{}^l X_l,$

(ii) $\sum_i R^\nabla(X_i, \nabla_{X_i}X_j)X_k = \frac{1}{8} \sum_l C_{jk}{}^l X_l,$

(iii) $\sum_i R^\nabla(X_i, X_j)\nabla_{X_i}X_k = -\frac{1}{8} \sum_{i,l,s,t} C_{ij}{}^t C_{tk}{}^s C_{si}{}^l X_l.$

By virtue of Lemma 2.2, (3.5) and Lemma 3.3, we have $\delta_D R^D = 0$. Thus, we have

THEOREM 3.4. *Let G be an n-dimensional ($n \geq 3$) compact connected semisimple Lie group, B the Killing form of the Lie algebra of G, and g the canonical metric induced by B. Let D be a left invariant linear connection with Weyl structure (D, g, ω) on (G, g). Then, if the connection D is Einstein-Weyl, D is a Yang-Mills connection.*

REMARK 3.5. A compact connected semi-simple Lie group with respect to the killing metric g is a Riemannian symmetric space. Hence the Levi-Civita connection ∇ for g is a Yang-Mills connection. Theorem 3.4 also follows from this fact, since ω is vanishing under the assumptions of Theorem 3.4.

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