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A FIFTH-ORDER IMPROVEMENT OF THE EULER-CHEBYSHEV METHOD FOR SOLVING NON-LINEAR EQUATIONS

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Abstract. In this paper we present a new variant of the Euler-Chebyshev method for solving nonlinear equations. Analysis of convergence is given to show that the presented methods are at least fifth-order convergent. Several numerical examples are given to illustrate that newly presented methods can be competitive to other known fifth-order methods and the Newton method in the efficiency and performance.

1. Introduction

We consider iterative methods that use f, f' and f'' but not the higher derivatives of f to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.

The most well-known and widely used iterative method for the calculation of α is Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.1}$$

where x_0 is an initial approximation sufficiently close to α. This method is quadratically convergent [6]. In [1], a family of third-order methods improving Newton's method is proposed:

$$x_{n+1} = x_n - H(t(x_n))\frac{f(x_n)}{f'(x_n)}, \tag{1.2}$$

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where

$$t(x) = \frac{f''(x)f(x)}{f'^2(x)} \tag{1.3}$$

and H is any function with $H(0) = 1, H'(0) = 1/2$ and $|H''(0)| < \infty$. This family includes many well-known classical third-order methods such as Halley's method ($H(t) = (1 - \frac{1}{2}t)^{-1}$), Euler-Chebyshev's method ($H(t) = 1 + \frac{1}{2}t$), Hansen-Patrick family ($H(t) = (\lambda + 1)(\lambda + \sqrt{1 - (\lambda + 1)t})^{-1}$) [4] etc, as particular cases. For further details, we refer to [1]. In this paper, we present a variant of the Euler-Chebyshev method, improving the order of convergence to five with an additional functional evaluation. By precise analysis of convergence, we show that the presented methods are of at least fifth-order, and their efficiency and performance are demonstrated by numerical results.

2. Main result

To construct fifth-order methods, we present the following main result.

THEOREM 2.1. *Assume that the function f is sufficiently smooth in a neighborhood of its root α, where $f'(\alpha) \neq 0$. Let H be any function with $H(0) = 1, H'(0) = 1/2$ and $|H''(0)| < \infty$. Then the iterative scheme defined by, for $n = 0, 1, 2, \cdots$,*

$$x_{n+1} = x_n - H(\zeta_n)\frac{f(x_n) + f(z_n)}{f'(x_n)}, \tag{2.1}$$

where

$$\zeta_n = \frac{f''(x_n)[f(x_n) + f(z_n)]}{f'^2(x_n)}, \tag{2.2}$$

$$z_n = x_n - H(t(x_n))\frac{f(x_n)}{f'(x_n)}, \tag{2.3}$$

is of fifth order, and it then satisfies the following error equation:

$$e_{n+1} = 3\Big(2(H''(0) - 1)c_2^2 + c_3\Big)^2 e_n^5 + O(e_n^6), \tag{2.4}$$

where $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k!f'(\alpha)$, for $k = 2, 3, 4, \cdots$.

Proof. Let $e_n = x_n - \alpha$. From the Taylor expansions of $f(x_n)$, $f'(x_n)$, and $f''(x_n)$, and taking into account $f(\alpha) = 0$, we obtain

$$
\begin{aligned}
(2.5)\ f(x_n) &= f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)],\\
(2.6)\ f'(x_n) &= f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)],\\
(2.7)\ f''(x_n) &= f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + O(e_n^4)].
\end{aligned}
$$

From (2.5)-(2.7), we get

$$
\begin{aligned}
\frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4\\
(2.8)\qquad &\quad +(8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + O(e_n^6),\\
\frac{f''(x_n)}{f'(x_n)} &= 2c_2 + (-4c_2^2 + 6c_3)e_n + (8c_2^3 - 18c_2c_3 + 12c_4)e_n^2\\
(2.9)\qquad &\quad +(-16c_2^4 + 48c_2^2c_3 - 32c_2c_4 - 18c_3^2 + 20c_5)e_n^3 + O(e_n^4),
\end{aligned}
$$

whence

$$
\begin{aligned}
t(x_n) &= \frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}\\
&= 2c_2e_n + 6(-c_2^2 + c_3)e_n^2 + 4(4c_2^3 - 7c_2c_3 + 3c_4)e_n^3\\
(2.10)\qquad &\quad +10(-4c_2^4 + 10c_2^2c_3 - 5c_2c_4 - 3c_3^2 + 2c_5)e_n^4 + O(e_n^5),
\end{aligned}
$$

$$
\begin{aligned}
t^2(x_n) &= 4c_2^2e_n^2 + 24(-c_2^3 + c_2c_3)e_n^3\\
(2.11)\qquad &\quad +4(25c_2^4 - 46c_2^2c_3 + 12c_2c_4 + 9c_3^2)e_n^4 + O(e_n^5),\\
(2.12)\quad t^3(x_n) &= 8c_2^3e_n^3 + 72(-c_2^4 + c_2^2c_3)e_n^4 + O(e_n^5),\\
(2.13)\quad t^4(x_n) &= 16c_2^4e_n^4 + O(e_n^5).
\end{aligned}
$$

Since

$$
\begin{aligned}
H(t(x_n)) &= 1 + \frac{1}{2}t(x_n) + \frac{1}{2}H''(0)t^2(x_n) + \frac{1}{6}H^{(3)}(0)t^3(x_n)\\
&\quad + \frac{1}{24}H^{(4)}(0)t^4(x_n) + O(t^5(x_n)),
\end{aligned}
$$

it follows from (2.10)-(2.13) that

$$
(2.14)\qquad H(t(x_n)) = 1 + c_2e_n + Ae_n^2 + Be_n^3 + Ce_n^4 + O(e_n^5),
$$

where

$$\begin{aligned} A &= (2H''(0)-3)c_2^2+3c_3, \\ B &= 4\left(\frac{1}{3}H^{(3)}(0)-3H''(0)+2\right)c_2^3+2(6H''(0)-7)c_2c_3+6c_4, \\ C &= 2\left(\frac{1}{3}H^{(4)}(0)-6H^{(3)}(0)+25H''(0)-10\right)c_2^4 \\ &\quad +2(6H^{(3)}(0)-46H''(0)+25)c_2^2c_3+(24H''(0)-25)c_2c_4 \\ &\quad +3(6H''(0)-5)c_3^2+10c_5. \end{aligned}$$

Hence, we easily obtain from (2.8) and (2.14) that the error equation of the method defined by (2.3) is given by

$$\begin{aligned} \tilde{e}_{n+1} &:= z_n-\alpha \\ &= \left(2(1-H''(0))c_2^2-c_3\right)e_n^3+K_1e_n^4+K_2e_n^5+O(e_n^6), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} K_1 &= \left(-\frac{4}{3}H^{(3)}(0)+14H''(0)-9\right)c_2^3+12(1-H''(0))c_2c_3-3c_4, \\ K_2 &= 2\left(-\frac{1}{3}H^{(4)}(0)+\frac{20}{3}H^{(3)}(0)-33H''(0)+15\right)c_2^4 \\ &\quad +3(-4H^{(3)}(0)+36H''(0)-21)c_2^2c_3+24(1-H''(0))c_2c_4 \\ &\quad +3(5-6H''(0))c_3^2-6c_5. \end{aligned}$$

On the other hand, since

$$f(z_n)=f'(\alpha)(z_n-\alpha)+O(e_n^6), \tag{2.16}$$

a simple calculation using (2.5), (2.6) and (2.15) shows that

$$\begin{aligned} \frac{f(x_n)+f(z_n)}{f'(x_n)} &= e_n-c_2e_n^2+\left(2(2-H''(0))c_2^2-3c_3\right)e_n^3 \\ &\quad +E_1e_n^4+E_2e_n^5+O(e_n^6), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} E_1 &= \left(-\frac{4}{3}H^{(3)}(0)+18H''(0)-17\right)c_2^3+3(7-4H''(0))c_2c_3-6c_4, \\ E_2 &= 2\left(-\frac{1}{3}H^{(4)}(0)+8H^{(3)}(0)-51H''(0)+32\right)c_2^4 \\ &\quad +3(-4H^{(3)}(0)+46H''(0)-39)c_2^2c_3+8(5-3H''(0))c_2c_4 \\ &\quad +6(4-3H''(0))c_3^2-10c_5, \end{aligned}$$

whence we get from (2.9)

$$\begin{aligned}\zeta_n &= \frac{f(x_n)+f(z_n)}{f'(x_n)} \cdot \frac{f''(x_n)}{f'(x_n)} \\ &= 2c_2e_n + 6(-c_2^2+c_3)e_n^2 + L_1e_n^3 + L_2e_n^4 + O(e_n^5), \quad (2.18)\end{aligned}$$

where

$$L_1 = 4(5-H''(0))c_2^3 - 30c_2c_3 + 12c_4, \quad (2.19)$$

$$\begin{aligned}L_2 &= 2\left(-\frac{4}{3}H^{(3)}(0) + 22H''(0) - 37\right)c_2^4 + 36(4-H''(0))c_2^2c_3 \\ &\quad -56c_2c_4 - 36c_3^2 + 20c_5.\end{aligned}$$

It then easily follows that

$$\begin{aligned}\zeta_n^2 &= 4c_2^2e_n^2 + 24(-c_2^3+c_2c_3)e_n^3 \\ &\quad +\left(4(29-4H''(0))c_2^4 - 192c_2^2c_3 + 48c_2c_4 + 36c_3^2\right)e_n^4 + O(e_n^5), \quad (2.20)\end{aligned}$$

$$\zeta_n^3 = 8c_2^3e_n^3 + 72(-c_2^4+c_2^2c_3)e_n^4 + O(e_n^5), \quad (2.21)$$

$$\zeta_n^4 = 16c_2^4e_n^4 + O(e_n^5). \quad (2.22)$$

Since

$$H(\zeta_n) = 1 + \frac{1}{2}\zeta_n + \frac{1}{2}H''(0)\zeta_n^2 + \frac{1}{6}H^{(3)}(0)\zeta_n^3 + \frac{1}{24}H^{(4)}(0)\zeta_n^4 + O(\zeta_n^5),$$

it follows from (2.18)-(2.22) that

$$H(\zeta_n) = 1 + c_2e_n + M_1e_n^2 + M_2e_n^3 + M_3e_n^4 + O(e_n^5), \quad (2.23)$$

where

$$\begin{aligned}M_1 &= (2H''(0)-3)c_2^2 + 3c_3, \\ M_2 &= 2\left(\frac{2}{3}H^{(3)}(0) - 7H''(0) + 5\right)c_2^3 + 3(4H''(0)-5)c_2c_3 + 6c_4, \\ M_2 &= \left(\frac{2}{3}H^{(4)}(0) - \frac{40}{3}H^{(3)}(0) - 8H''^2(0) + 80H''(0) - 37\right)c_2^4 \\ &\quad +2(6H^{(3)}(0) - 57H''(0) + 36)c_2^2c_3 + 4(6H''(0)-7)c_2c_4 \\ &\quad +18(H''(0)-1)c_3^2 + 10c_5,\end{aligned}$$

whence from (2.17) we get

$$\begin{aligned}&\frac{f(x_n)+f(z_n)}{f'(x_n)}\cdot H(\zeta_n)\\&=e_n-3\Big(2(H''(0)-1)c_2^2+c_3\Big)^2e_n^5+O(e_n^6).\end{aligned} \tag{2.24}$$

Therefore, it is clear from (2.24) that the error equation of the method defined by (2.1) is given by

$$e_{n+1} = x_n-\alpha-\frac{f(x_n)+f(z_n)}{f'(x_n)}\cdot H(\zeta_n) \tag{2.25}$$

$$= 3\Big(2(H''(0)-1)c_2^2+c_3\Big)^2e_n^5+O(e_n^6). \tag{2.26}$$

This completes the proof. □

3. Some special cases of order five

In the case that $H(t)=\left(1-\dfrac{1}{2}t\right)^{-1}$, the scheme (2.1) yields a new fifth-order iterative method

$$x_{n+1}=x_n-\frac{2f'(x_n)(f(x_n)+f(z_n))}{2f'^2(x_n)-f''(x_n)(f(x_n)+f(z_n))}, \tag{3.1}$$

where

$$z_n=x_n-\frac{2f'(x_n)f(x_n)}{2f'^2(x_n)-f''(x_n)f(x_n)}. \tag{3.2}$$

Note that (3.2) is the well-known Halley method [1] and so, the method defined by (3.1) is a fifth-order variant of Halley's method.

In the case that $H(t)=\sqrt{1+t}$, the scheme (2.1) yields a new fifth-order iterative method

$$x_{n+1}=x_n-\sqrt{1+\frac{f''(x_n)(f(x_n)+f(z_n))}{f'^2(x_n)}}\,\frac{f(x_n)+f(z_n)}{f'(x_n)}, \tag{3.3}$$

where

$$z_n=x_n-\sqrt{1+\frac{f''(x_n)f(x_n)}{f'^2(x_n)}}\,\frac{f(x_n)}{f'(x_n)}. \tag{3.4}$$

In the case that $H(t) = \theta + (1-\theta)e^{\frac{1}{2(1-\theta)}t}$ where θ is any real number except 1, the scheme (2.1) yields the new one-parameter fifth-order family of methods

(3.5)
$$x_{n+1} = x_n - \left[\theta + (1-\theta)e^{\dfrac{1}{2(1-\theta)}\dfrac{f''(x_n)(f(x_n)+f(z_n))}{f'^2(x_n)}}\right]\frac{f(x_n)+f(z_n)}{f'(x_n)},$$

where

$$z_n = x_n - \left[\theta + (1-\theta)e^{\dfrac{1}{2(1-\theta)}\dfrac{f''(x_n)f(x_n)}{f'^2(x_n)}}\right]\frac{f(x_n)}{f'(x_n)}. \tag{3.6}$$

In the case that $H(t) = (\lambda+1)(\lambda+\sqrt{1-(\lambda+1)t})^{-1}$ where λ is any real number, the scheme (2.1) yields another new one-parameter fifth-order family of methods

$$\begin{aligned} x_{n+1} &= x_n - (\lambda+1)\left(\lambda + \sqrt{1-(\lambda+1)\frac{f''(x_n)(f(x_n)+f(z_n))}{f'^2(x_n)}}\right)^{-1} \\ &\quad \cdot \frac{f(x_n)+f(z_n)}{f'(x_n)}, \end{aligned} \tag{3.7}$$

where

$$z_n = x_n - (\lambda+1)\left(\lambda + \sqrt{1-(\lambda+1)\frac{f''(x_n)f(x_n)}{f'^2(x_n)}}\right)^{-1} \cdot \frac{f(x_n)}{f'(x_n)}. \tag{3.8}$$

In a similar fashion as in the above, with any other functions H satisfying $H(0) = 1, H'(0) = 1/2$ and $|H''(0)| < \infty$, we can continuously apply formula (2.1) to obtain the fifth-order methods; per iteration each of them requires two evaluations of the given function, one of its first derivative and one of its second derivative. If we consider the definition of efficiency index [2] as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method, we have that all of the methods obtained from formula (2.1) have the efficiency index equal to $5^{\frac{1}{4}} \approx 1.495$, which is better than the ones of the third-order methods $3^{\frac{1}{3}} \approx 1.442$ obtained from (1.2) and Newton's method $\sqrt{2} \approx 1.414$.

4. Numerical examples and conclusions

All computations were done using MAPLE using 64 digit floating point arithmetics (Digits:=64). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-15}$.

We present some numerical test results for various fifth-order iterative schemes in Table 1. Compared were the Newton method (NM), the method of Grau (GM) given by

$$x_{n+1} = x_n - \left(1 + \frac{f''(x_n)(f(x_n) + f(z_n))}{2f'^2(x_n)}\right) \frac{f(x_n) + f(z_n)}{f'(x_n)},$$
$$z_n = x_n - \left(1 + \frac{1}{2}t(x_n)\right) \frac{f(x_n)}{f'(x_n)},$$

the method of Kou et al [5] (KM) defined by

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n) + f''(x_n)(z_n - x_n)},$$
$$z_n = x_n - \left(1 + \frac{1}{2}\frac{f''(x_n)f(x_n)}{f'^2(x_n)}\right) \frac{f(x_n)}{f'(x_n)},$$

and the methods (3.1) with (3.2) (CM1), (3.5) with $\theta = 1/2$ and (3.6) (CM2), and (3.7) with $\lambda = 1$ and (3.8) (CM3), respectively, introduced in the present contribution. We used the following test functions:

$$\begin{aligned}
f_1(x) &= x^3 + 4x^2 - 10, \\
f_2(x) &= \sin^2 x - x^2 + 1, \\
f_3(x) &= x^2 - e^x - 3x + 2, \\
f_4(x) &= \cos x - x, \\
f_5(x) &= (x-1)^3 - 1, \\
f_6(x) &= e^{-x} + \cos x, \\
f_7(x) &= e^x - 4x^2, \\
f_8(x) &= x^2 + \sin(x/5) - 1/4.
\end{aligned}$$

As convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the

Table 1. Comparison of various fifth-order iterative methods and the Newton method

	IT	COC	x_*	$f(x_*)$	δ
$f_1, x_0 = 1.27$					
NM	5	2	1.3652300134140968457608068290	2.70e-41	1.83e-21
GM	3	5.039	1.3652300134140968457608068290	0.0e-01	4.35e-27
KM	3	5.024	1.3652300134140968457608068290	0.0e-01	2.36e-32
CM1	3	5.018	1.3652300134140968457608068290	0.0e-01	6.94e-32
CM2	3	5.013	1.3652300134140968457608068290	0.0e-01	5.14e-32
CM3	3	5	1.3652300134140968457608068290	0.0e-01	5.33e-38
$f_2, x_0 = 1$					
NM	7	2.0	1.4044916482153412260350868178	-1.04e-50	7.33e-26
GM			divergent		
KM	4	4.841	1.4044916482153412260350868178	-2.0e-63	5.79e-31
CM1	4	5.004	1.4044916482153412260350868178	1.3e-63	6.01e-43
CM2	3	4.568	1.4044916482153412260350868178	1.3e-63	2.50e-17
CM3	3	5.359	1.4044916482153412260350868178	1.3e-63	4.12e-16
$f_3, x_0 = 0$					
NM	5	2	0.25753028543986076045536730494	1.56e-49	6.64e-25
GM	3	4.868	0.25753028543986076045536730494	-1.0e-63	5.47e-33
KM	3	4.910	0.25753028543986076045536730494	1.0e-63	1.05e-30
CM1	3	4.926	0.25753028543986076045536730494	0.0e-01	3.49e-30
CM2	3	4.925	0.25753028543986076045536730494	0.0e-01	3.29e-30
CM3	3	4.967	0.25753028543986076045536730494	0.0e-01	2.97e-28
$f_4, x_0 = 1.2$					
NM	5	2	0.73908513321516064165531208767	-1.90e-35	7.16e-18
GM	3	4.842	0.73908513321516064165531208767	0.0e-01	5.81e-18
KM	3	4.954	0.73908513321516064165531208767	0.0e-01	1.06e-18
CM1	3	4.926	0.73908513321516064165531208767	1.0e-64	1.32e-18
CM2	3	4.926	0.73908513321516064165531208767	0.0e-01	1.34e-18
CM3	3	5.055	0.73908513321516064165531208767	0.0e-01	1.65e-19
$f_5, x_0 = 1.8$					
NM	6	2	2	2.87e-41	3.09e-21
GM	4	5.007	2	0.0e-01	1.99e-41
KM	4	5	2	0.0e-01	1.23e-17
CM1	3	5.119	2	0.0e-01	5.07e-16
CM2	3	5.006	2	0.0e-01	2.28e-17
CM3	3	5.013	2	0.0e-01	7.34e-21
$f_6, x_0 = 0.1$					
NM	6	2	1.7461395304080124176507030890	6.80e-43	1.97e-21
GM	4	4.723	1.7461395304080124176507030890	-9.0e-64	1.0e-63
KM	3	4.122	1.7461395304080124176507030890	3.0e-64	1.78e-16
CM1	4	4.332	1.7461395304080124176507030890	-9.0e-64	1.0e-63
CM2	4	4.317	1.7461395304080124176507030890	-9.0e-64	1.0e-63
CM3	3	4.177	1.7461395304080124176507030890	-9.0e-64	6.94e-16
$f_7, x_0 = 2$					
NM	7	2	0.71480591236277780613762220811	-2.06e-47	2.63e-24
GM	4	4.888	0.71480591236277780613762220811	0.0e-01	1.07e-21
KM	4	4.989	0.71480591236277780613762220811	0.0e-01	3.69e-27
CM1	4	4.934	0.71480591236277780613762220811	0.0e-01	1.90e-24
CM2	4	4.938	0.71480591236277780613762220811	0.0e-01	2.28e-24
CM3	4	5.012	0.71480591236277780613762220811	0.0e-01	7.29e-34
$f_8, x_0 = 2$					
NM	8	2	0.4099920179891371316212583765	6.06e-56	2.46e-28
GM	4	4.745	0.4099920179891371316212583765	0.0e-01	5.50e-16
KM	4	5.251	0.4099920179891371316212583765	0.0e-01	1.39e-34
CM1	4	4.901	0.4099920179891371316212583765	-1.0e-64	1.17e-21
CM2	4	4.902	0.4099920179891371316212583765	0.0e-01	3.86e-21
CM3	3	4.988	0.4099920179891371316212583765	0.0e-01	1.04e-27

computational order of convergence (COC), the approximate zero x_*, and the value $f(x_*)$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so it making all looking the same though they may in fact differ.

The test results in Table 1 show that the computed order of convergence of the presented iterative methods is all five, which agree with the theoretical result developed in this paper. It can be observed that for most of the functions we tested, the methods introduced in this presentation show at least equal performance compared to the other fifth-order methods. Moreover, the presented methods can compete with Newton's method.

5. Conclusion

In this work we improved a third-order family of iterative methods, which includes the well-known methods such as Halley's method, Euler-Chebyshev's method, etc as particular cases, to five at the expense of an additional functional evaluation. Numerical results confirmed that the methods obtained in this paper demonstrate at least equal performance compared to other well-known methods in the literature.

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