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# PULL-BACK MORPHISMS, CONVOLUTION PRODUCTS AND STEINBERG VARIETIES

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Abstract. In this paper, we first show that the pull-back morphism between two $K$-groups of the Steinberg varieties, obtained respectively from partial flag varieties and quiver varieties of type $A$, is a ring homomorphism with respect to the convolution product. Then, we prove that this ring homomorphism yields a property of compatibility between two certain convolution actions.

## 1. Introduction

Let $M$ be a complex smooth variety, let $N$ be any variety over $\mathbb{C}$, and let $p : M \longrightarrow N$ be a proper morphism. Put

$$Z = M \times_N M = \{(m_1, m_2) \in M \times M \mid p(m_1) = p(m_2)\},$$

and we set $M_x = p^{-1}(x)$ for $x \in N$. Then according to the general set-up of the convolution product in $K$-theory (see Section 2 for details), we obtain the following important convolution products in representation theory:

$$K(Z) \otimes K(Z) \longrightarrow K(Z), \tag{1.1}$$

$$K(Z) \otimes K(M_x) \longrightarrow K(M_x). \tag{1.2}$$

We refer to [2, 3, 5] for the applications of (1.1) and (1.2) in the geometric representation theory of Weyl groups, affine Hecke algebras and quantum affine algebras.

In particular, in this paper we are interested in the convolution products (1.1) and (1.2) appeared in [3, 5], where we have the cases when the variety $M$ is a partial flag variety or a quiver variety. Though the

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author treated the general types of quiver varieties in [5], in our current study we will restrict our attention to quiver varieties of type $A$ because certain relations have been known between quiver varieties of type $A$ and the geometry of partial flag varieties over the nilpotent cone [4].

We will use these relationships to construct a group homomorphism between the Grothendieck groups of related Steinberg type varieties. Here Steinberg type varieties mean the varieties $Z$ in the above general set-up for the cases when $M$ is a partial flag variety or a quiver variety of type $A$. In addition, we will show that this homomorphism is actually a ring homomorphism with respect to the convolution products obtained from (1.1). (See Theorem (5.2).) By using this ring homomorphism, we will also present a commutative diagram which shows a property of compatibility between two convolution actions arising from (1.2). (See Corollary (5.3).)

## 2. Preliminaries

**Pull-back in $K$-theory.** Let $X$ be a smooth algebraic variety over $\mathbb{C}$, and let $K(X)$ be the Grothendieck group of all coherent sheaves on $X$. Recall that the Grothendieck group $K(X)$ also can be understood as the Grothendieck group of algebraic vector bundles on $X$ if $X$ is nonsingular.

Assume that $Y$ is a closed subvariety of the smooth variety $X$, and let $K(X,Y)$ be the relative Grothendieck group of the derived category of complexes of algebraic vector bundles which are exact outside $Y$ (see [1]). Then there is a natural isomorphism between $K(X,Y)$ and $K(Y)$.

Suppose that $f : Y \longrightarrow X$ is a morphism between smooth varieties. Let $X^{'}$ and $Y^{'}$ be closed subvarieties of $X$ and $Y$ respectively satisfying $f^{-1}\left(X^{'}\right) \subset Y^{'}$. Then, we have a homomorphism $K\left(X, X^{'}\right) \longrightarrow K\left(Y, Y^{'}\right)$ induced by the pull-back $E^{\bullet} \longmapsto f^{*}E^{\bullet}$, where $E^{\bullet}$ is a complex of algebraic vector bundle over $X$. From isomorphisms $K\left(X^{'}\right) \simeq K\left(X, X^{'}\right)$ and $K\left(Y^{'}\right) \simeq K\left(Y, Y^{'}\right)$, we finally obtain a homomorphism $f^{*} : K\left(X^{'}\right) \longrightarrow K\left(Y^{'}\right)$. We call this morphism the *pull-back with support* (group) homomorphism. Notice that this depends in an essential way on the ambient spaces.

**Convolutions in $K$-theory.** Let $X$ be a smooth algebraic variety, and let $Z_1$ and $Z_2$ be closed subsets of $X$.

For given $\mathcal{F}_1 \in K(Z_1)$ and $\mathcal{F}_2 \in K(Z_2)$, we write $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in K(Z_1 \times Z_2)$ for the external tensor product $p_{Z_1}^* \mathcal{F}_1 \otimes_{\mathcal{O}_{Z_1 \times Z_2}} p_{Z_2}^* \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$. Here $p_{Z_1}$ and $p_{Z_2}$ denote the projections of $Z_1 \times Z_2$ to corresponding factors. Next, let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding. Then we have $\Delta^{-1}(Z_1 \times Z_2) = Z_1 \cap Z_2$. So the restriction map $\Delta : Z_1 \cap Z_2 \longrightarrow Z_1 \times Z_2$ induces the map

$$\Delta^* : K(Z_1 \times Z_2) \longrightarrow K(Z_1 \cap Z_2)\,. \tag{2.1}$$

By combining the map (2.1) with the external tensor product, we obtain the following tensor product with support:
(2.2)

$$\otimes : K(Z_1) \otimes K(Z_2) \longrightarrow K(Z_1 \cap Z_2)\,, \quad (\mathcal{F}_1,\ \mathcal{F}_2) \mapsto \Delta^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)\,.$$

With the morphism (2.2), we define the convolution product in the algebraic $K$-theory as follows.

Let $M_1$, $M_2$ and $M_3$ be smooth, quisi-projective varieties with the $(i,j)$-projections $p_{ij} : M_1 \times M_2 \times M_3 \longrightarrow M_i \times M_j$. We suppose that $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ are closed subsets.

We also assume that

$$p_{13} : \left(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})\right) \longrightarrow M_1 \times M_3 \tag{2.3}$$

is a proper map.

Then we define the convolution product in $K$-theory

$$* : K(Z_{12}) \otimes K(Z_{23}) \longrightarrow K(Z_{12} \circ Z_{23}) \tag{2.4}$$

to be $\mathcal{F}_{12} * \mathcal{F}_{23} := (p_{13})_* \left((p_{12})^* \mathcal{F}_{12} \otimes (p_{23})^* \mathcal{F}_{23}\right)$. Here $Z_{12} \circ Z_{23}$ denotes the image of the map (2.3).

Concerned with the pull-back with support morphisms and the tensor products, the following lemma is known [1].

Lemma 2.1. *Let $f : Y \longrightarrow X$ be a morphism between smooth varieties. Assume that $X_1^{'},\ X_2^{'} \subset X$ and $Y_1^{'},\ Y_2^{'} \subset Y$ are closed subvarieties such that $f^{-1}\left(X_i^{'}\right) \subset Y_i^{'}$ for $i = 1,\ 2$. Then, we have*

$$f^*(\mathcal{F}_1 \otimes \mathcal{F}_2) = f^*(\mathcal{F}_1) \otimes f^*(\mathcal{F}_2) \quad \text{for } \mathcal{F}_i \in K\left(X_i^{'}\right).$$

## 3. Convolution actions on fibers, partial flag variety case

Given an integer $d \geq 1$, let $\mathbf{F}$ be the set of all $n$-step partial flags in $\mathbb{C}^d$, i.e., the set of sequences of vector spaces

$$\mathbf{F} = \left\{F = \left(0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d\right)\right\}.$$

Then $\mathbf{F}$ is a smooth compact manifold, and its connected components are parametrized by partitions $\mathbf{d} = (d_1, \cdots, d_n) \in (\mathbb{Z}_{\geq 0})^n$ of the integer $d$. In fact, the connected component of $\mathbf{F}$ corresponding to the partition $\mathbf{d}$ is given by

$$\mathbf{F_d} = \left\{ F = \left( 0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^d \right) \mid dim\,(F_i/F_{i-1}) = d_i \right\}.$$

Let $N = \left\{ x \in End\left(\mathbb{C}^d\right) \mid x^m = 0 \right\}$, and let

$$M = \{(x, F) \in N \times \mathbf{F} \mid x\,(F_i) \subset F_{i-1} \text{ for all } i = 1, 2, \cdots, n\}.$$

Then, it is known that $T^*\mathbf{F} \simeq M$ as vector bundles over $\mathbf{F}$. Thus the decomposition of $\mathbf{F}$ into connected components yields a decomposition $M = \sqcup M_{\mathbf{d}}$, where $M_{\mathbf{d}} \simeq T^* F_{\mathbf{d}}$.

Let us now consider a natural projection $\mu : M \longrightarrow N$. Define

$$Z = M \times_N M = \{(m_1, m_2) \in M \times M \mid \mu\,(m_1) = \mu\,(m_2)\}.$$

Then we have $Z \circ Z = Z$, and the convolution product

$$* : K(Z) \otimes K(Z) \longrightarrow K(Z) \tag{3.1}$$

is well-defined.

Let $\mathbf{F}_x$ be the fiber $\mu^{-1}(x)$. Then we also obtain the following convolution action on the fiber $\mathbf{F}_x$ due to the fact $Z \circ \mathbf{F}_x = \mathbf{F}_x$:

$$* : K(Z) \otimes K\left(\mathbf{F}_x\right) \longrightarrow K\left(\mathbf{F}_x\right). \tag{3.2}$$

## 4. Convolution actions on fibers, quiver variety case

Let $I = \{1, 2, \cdots, n-1\}$ be the set of vertices of the Dynkin diagram of the simple Lie algebra $\mathfrak{sl}_n\,(\mathbb{C})$. We also let $V = (V_i)_{i\in I}$ and $W = (W_i)_{i\in I}$ be collections of finite-dimensional vector spaces. We write $\mathbf{v}$ and $\mathbf{w}$ for $(dimV_1, \cdots, dimV_{n-1})$ and $(dimW_1, \cdots, dimW_{n-1})$ respectively.

Define

$$\begin{aligned} M\,(\mathbf{v}, \mathbf{w}) \;\; = \;\; & \left( \bigoplus_{k=1}^{n-2} Hom\,(V_k, V_{k+1}) \right) \oplus \left( \bigoplus_{k=1}^{n-2} Hom\,(V_{k+1}, V_k) \right) \\ & \oplus \left( \bigoplus_{k=1}^{n-1} Hom\,(W_k, V_k) \right) \oplus \left( \bigoplus_{k=1}^{n-1} Hom\,(V_k, W_k) \right). \end{aligned}$$

We denote by $(A = (A_k), B = (B_k), i = (i_k), j = (j_k))$ an element of $M\,(\mathbf{v}, \mathbf{w})$.

If $g = (g_k) \in G_{\mathbf{v}}$, then we define an action of $G_{\mathbf{v}} = \prod_{k=1}^{n-1} GL(V_k)$ on $M(\mathbf{v}, \mathbf{w})$ as follows:

$$g\cdot((A_k), (B_k), (i_k), (j_k)) = \left(\left(g_{k+1}A_k g_k^{-1}\right), \left(g_k B_k g_{k+1}^{-1}\right), (g_k i_k), \left(j_k g_k^{-1}\right)\right).$$

Next, we consider a subset $\mu(\mathbf{v}, \mathbf{w})$ of $M(\mathbf{v}, \mathbf{w})$ consisting of quadruples $\{(A, B, i, j)\}$ subject to the conditions $B_1 A_1 = i_1 j_1$, $B_k A_k = A_{k-1} B_{k-1} + i_k j_k$ for $2 \leq k \leq n-2$, and $A_{n-2}B_{n-2} + i_{n-1}j_{n-1} = 0$. An element $(A, B, i, j) \in \mu(\mathbf{v}, \mathbf{w})$ is called *stable* if each subspace $U = (U_1, \cdots, U_{n-1})$ of $V = (V_1, \cdots, V_{n-1})$, which contains $Im\, i$ (i.e., $Im\, i_k \subset U_k$) and invariant under $(A_k)$ and $(B_k)$ (i.e., $A_k(U_k) \subset U_{k+1}$ and $B_k(U_{k+1}) \subset U_k$), is actually equal to $V$. We denote by $\mu(\mathbf{v}, \mathbf{w})^s$ the set of stable elements in $\mu(\mathbf{v}, \mathbf{w})$. We notice that $\mu(\mathbf{v}, \mathbf{w})$ and $\mu(\mathbf{v}, \mathbf{w})^s$ are invariant under the action of $G_{\mathbf{v}}$. Thus we can consider the affine algebro-geometric quotient $\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \mu(\mathbf{v}, \mathbf{w}) // G_{\mathbf{v}}$ and the geometric invariant theory quotient $\mathfrak{M}(\mathbf{v}, \mathbf{w}) = \mu(\mathbf{v}, \mathbf{w}) //_{\chi} G_{\mathbf{v}}$. Here $\chi$ is the character on $G_{\mathbf{v}}$ given by $\chi((g_k)) = \prod_k det\left(g_k^{-1}\right)$. Recall that there is a natural projective morphism $\pi : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ which sends a $G_{\mathbf{v}}$-orbit $G_{\mathbf{v}}(A, B, i, j)$ to the unique closed orbit $G_{\mathbf{v}}(A_0, B_0, i_0, j_0)$ contained in $\overline{G_{\mathbf{v}}(A, B, i, j)}$.

Lemma 4.1. *Let $V' = \left(V_k'\right)$ be a collection of subspaces of $V = (V_k)$. Then there is a natural inclusion map $\mathfrak{M}_0\left(\mathbf{v}', \mathbf{w}\right) \hookrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ for a fixed collection of spaces $W = (W_k)$.*

*Proof.* See [5, Lemma (2.5.3)]. □

By Lemma (4.1), we can consider the projective morphisms

$$\pi : \mathfrak{M}\left(\mathbf{v}^k, \mathbf{w}\right) \longrightarrow \mathfrak{M}_0\left(\mathbf{v}^k, \mathbf{w}\right) \ \ (k = 1,\ 2)$$

as morphisms to $\mathfrak{M}_0\left(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w}\right)$. Thus we can define the following Steinberg-type variety:

$$Z\left(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}\right) = \left\{\left(x^1, x^2\right) \in \mathfrak{M}\left(\mathbf{v}^1, \mathbf{w}\right) \times \mathfrak{M}\left(\mathbf{v}^2, \mathbf{w}\right) \mid \pi\left(x^1\right) = \pi\left(x^2\right)\right\}.$$

Notice that

$$Z\left(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}\right) \circ Z\left(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w}\right) \subset Z\left(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w}\right).$$

Hence, if we let $Z(\mathbf{w}) = \sqcup_{\mathbf{v}^1, \mathbf{v}^2} Z\left(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}\right)$, then we obtain the convolution product

$$* : K(Z(\mathbf{w})) \otimes K(Z(\mathbf{w})) \longrightarrow K(Z(\mathbf{w})). \tag{4.1}$$

Let $\mathfrak{L}(\mathbf{v},\mathbf{w})$ be a fiber $\pi^{-1}(G_{\mathbf{v}}(0,0,0,0))$ for the projection $\pi : \mathfrak{M}(\mathbf{v},\mathbf{w}) \longrightarrow \mathfrak{M}_0(\mathbf{v},\mathbf{w})$, and let $\mathfrak{L}(\mathbf{w}) = \sqcup_{\mathbf{v}}\mathfrak{L}(\mathbf{v},\mathbf{w})$. Then we also have the following convolution action:

$$* : K(Z(\mathbf{w})) \otimes K(\mathfrak{L}(\mathbf{w})) \longrightarrow K(\mathfrak{L}(\mathbf{w})). \tag{4.2}$$

## 5. Compatibility between two convolution actions

We first review the relations between quiver varieties of type $A$ and the geometry of partial flag varieties over the nilpotent cone.

Let $x \in N = \{x \in End(\mathbb{C}^d) \mid x^m = 0\}$, and let $\{x, y, h\}$ be a $\mathfrak{sl}_2$−triple in $End(\mathbb{C}^d)$. Then we define $S_x = \{u \in N \mid [u - x, y] = 0\}$. Since the $GL(\mathbb{C}^d)$-orbits on $N$ are determined by partitions of $d$, we can denote by $\mathcal{O}_\lambda$ the orbit corresponding to the partition $\lambda$ of $d$.

Let $M = \sqcup_{\mathbf{d}\in P(d)} M_{\mathbf{d}}$ be the decomposition introduced in Section 3, and let $\mu_{\mathbf{d}} : M_{\mathbf{d}} \longrightarrow N$ be the restriction of the map $\mu : M \longrightarrow N$. Let $\rho = (\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n)$ be a permutation of $\mathbf{d}$ and define the partition

$$\lambda_{\mathbf{d}} = 1^{\rho_1-\rho_2} 2^{\rho_2-\rho_3} \cdots n^{\rho_n}.$$

Then $\lambda_{\mathbf{d}}$ is a partition of $d$, and it is known that $\mu_{\mathbf{d}}(M_{\mathbf{d}}) \subset \overline{\mathcal{O}}_{\lambda_{\mathbf{d}}}$. For $x \in \overline{\mathcal{O}}_{\lambda_{\mathbf{d}}}$, we now define $S_{\mathbf{d},x} = S_x \cap \overline{\mathcal{O}}_{\lambda_{\mathbf{d}}}$ and $\tilde{S}_{\mathbf{d},x} = \mu_{\mathbf{d}}^{-1}(S_{\mathbf{d},x}) = \mu_{\mathbf{d}}^{-1}(S_x)$.

Next, for $\mathbf{v}$, $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^{n-1}$ we define an $n$-tuple $\mathbf{a}(\mathbf{v},\mathbf{w}) = (a_1, \cdots, a_n)$ as follows:

$$\begin{gathered} a_1 = w_1 + \cdots + w_{n-1} - v_1, \\ a_k = w_k + \cdots + w_{n-1} - v_k + v_{k-1} \quad \text{for} \quad 2 \leq k \leq n-1, \\ a_n = v_{n-1}. \end{gathered} \tag{5.1}$$

If we fix the dimension vector $\mathbf{w}$, then Equation (5.1) yields a bijection between $(n-1)$-tuples $\mathbf{v}$ and partitions $\mathbf{a}(\mathbf{v},\mathbf{w})$ of $\sum_{k=1}^{n-1} kw_k$. If no confusion is likely to arise, then we will simply write $\mathbf{a}$ for $\mathbf{a}(\mathbf{v},\mathbf{w})$. We also assume that $S_{\mathbf{a},x} = \tilde{S}_{\mathbf{a},x} = \emptyset$ if $a_i < 0$ for some $i$.

We now present the result of Maffei [4].

THEOREM 5.1. *Let* $\mathbf{v}$, $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^{n-1}$, $d = \sum_{k=1}^{n-1} kw_k$ *and* $\mathbf{a}(\mathbf{v},\mathbf{w})$ *be as (5.1). Let* $x \in N$ *be a nilpotent element of type* $1^{w_1}2^{w_2}\cdots(n-1)^{w_{n-1}}$. *Then there exists an isomorphism* $\varphi : \mathfrak{M}(\mathbf{v},\mathbf{w}) \longrightarrow \tilde{S}_{\mathbf{a},x}$ *and*

*$\psi : \pi\left(\mathfrak{M}\left(\mathbf{v},\mathbf{w}\right)\right) \longrightarrow S_{\mathbf{a},x}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{M}\left(\mathbf{v},\mathbf{w}\right) & \xrightarrow{\varphi} & \tilde{S}_{\mathbf{a},x} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\mu_{\mathbf{a}}} \\ \pi\left(\mathfrak{M}\left(\mathbf{v},\mathbf{w}\right)\right) & \xrightarrow{\psi} & S_{\mathbf{a},x}. \end{array}$$

*Moreover, $\psi$ maps $G_{\mathbf{v}}(0,0,0,0)$ to $x \in S_{\mathbf{a},x}$.*

*Proof.* See [4]. □

From now on, we fix $d = \sum_{k=1}^{n-1} kw_k$, a collection $W = (W_1, \cdots, W_{n-1})$ of vector spaces with a dimension vector $\mathbf{w} = (w_1, \cdots, w_{n-1})$ and a nilpotent element $x \in N$ of type $1^{w_1}2^{w_2}\cdots(n-1)^{w_{n-1}}$.

In the following theorem, we construct a ring homomorphism which yields a property of compatibility between two convolution actions on $K\left(\mathbf{F}_x\right)$ and $K\left(\mathcal{L}\left(\mathbf{w}\right)\right)$.

Theorem 5.2. *Let $K\left(Z\right)$ and $K\left(Z\left(\mathbf{w}\right)\right)$ be equipped with ring structures via the convolution products (3.1) and (4.1), respectively. Then there exists a ring homomorphism from $K\left(Z\right)$ to $K\left(Z\left(\mathbf{w}\right)\right)$ with respect to the convolution products.*

*Proof.* Let $M^{'} = \mu^{-1}\left(S_x\right)$ for the projection $\mu : M \longrightarrow N$. Then, we have $M^{'} = \sqcup_{\mathbf{d}}\tilde{S}_{\mathbf{d},x}$.

Next, we set $Z^{'} = M^{'} \times_{S_x} M^{'}$. Then, by Theorem (5.1), the isomorphism $\varphi^{-1}$ gives an isomorphism

$$Z^{'} \simeq Z\left(\mathbf{w}\right) = \sqcup_{\mathbf{v}^1,\mathbf{v}^2} Z\left(\mathbf{v}^1,\mathbf{v}^2;\mathbf{w}\right). \tag{5.2}$$

Let us now consider the following commutative diagram:

$$\begin{array}{ccc} M^{'} \times M^{'} & \overset{i}{\hookrightarrow} & M \times M \\ \big\uparrow & & \big\uparrow \\ Z^{'} & \hookrightarrow & Z\ , \end{array} \tag{5.3}$$

where $Z = M \times_N M$.

In the commutative diagram (5.3), we notice that the embedding $i$ is a smooth embedding (see [2, Corollary 3.5.9]), and $i^{-1}\left(Z\right) = Z^{'}$. Furthermore, $Z^{'}$ and $Z$ are closed subvarieties of $M^{'} \times M^{'}$ and $M \times M$, respectively.

Thus, the diagram (5.3) yields the pull-back with support homomorphism $i^* : K\left(Z\right) \longrightarrow K\left(Z^{'}\right) \simeq K\left(Z\left(\mathbf{w}\right)\right)$. Notice from Theorem 5.1

that the isomorphism $K\left(Z^{'}\right) \simeq K\left(Z\left(\mathbf{w}\right)\right)$ is compatible with respect the convolution products.

Now, we prove that $i^*$ is a ring homomorphism with respect to the convolution products. In other words, we show that the following diagram is commutative:

$$\begin{array}{ccc} K(Z) \otimes K\left(Z\right) & \xrightarrow{\;*\;} & K\left(Z\right) \\ {\scriptstyle i^*\otimes i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ K\left(Z^{'}\right) \otimes K\left(Z^{'}\right) & \xrightarrow[*^{'}]{} & K\left(Z^{'}\right). \end{array}$$

Let $p_{ij} : M \times M \times M \longrightarrow M \times M$ and $p_{ij}^{'} : M^{'} \times M^{'} \times M^{'} \longrightarrow M^{'} \times M^{'}$ be the projections to the $(i, j)$-factor. For $\mathcal{F}_1, \mathcal{F}_2 \in K(Z) = K(M \times M, Z)$, recall that $p_{12}^*\mathcal{F}_1 \otimes p_{23}^*\mathcal{F}_2 = \triangle^*\left(p_{12}^*\mathcal{F}_1 \boxtimes p_{23}^*\mathcal{F}_2\right)$, where $\triangle^* : K\left(p_{12}^{-1}(Z) \times p_{23}^{-1}(Z)\right) \longrightarrow K\left(p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z)\right)$ is the pull-back induced by the diagonal embedding $\triangle : M \times M \times M \hookrightarrow (M \times M \times M) \times (M \times M \times M)$. Thus, we may consider $p_{12}^*\mathcal{F}_1 \otimes p_{23}^*\mathcal{F}_2 \in K(M \times M \times M, p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z))$.

Let us now consider the following commutative diagram:

$$\begin{array}{ccc} M^{'} \times M^{'} \times M^{'} & \stackrel{\tilde{i}}{\hookrightarrow} & M \times M \times M \\ {\scriptstyle p_{ij}^{'}}\downarrow & & \downarrow{\scriptstyle p_{ij}} \\ M^{'} \times M^{'} & \stackrel{i}{\hookrightarrow} & M \times M. \end{array} \tag{5.4}$$

Then,

$$\begin{aligned}
& i^*\left(\mathcal{F}_1 * \mathcal{F}_2\right) \\
= \;& i^*\left(p_{13}\right)_*\left(p_{12}^*\mathcal{F}_1 \otimes p_{23}^*\mathcal{F}_2\right) \text{ (we consider } \mathcal{F}_1 * \mathcal{F}_1 \in K(M \times M, Z)) \\
= \;& \left(p_{13}^{'}\right)_* \tilde{i}^*\left(p_{12}^*\mathcal{F}_1 \otimes p_{23}^*\mathcal{F}_2\right) \text{ (see the proof of [2, Theorem 5.3.9])} \\
= \;& \left(p_{13}^{'}\right)_*\left(\tilde{i}^*p_{12}^*\mathcal{F}_1 \otimes \tilde{i}^*p_{23}^*\mathcal{F}_2\right) \text{ (by Lemma 2.1)} \\
= \;& \left(p_{13}^{'}\right)_*\left(\left(p_{12} \circ \tilde{i}\right)^*\mathcal{F}_1 \otimes \left(p_{23} \circ \tilde{i}\right)^*\mathcal{F}_2\right) \\
= \;& \left(p_{13}^{'}\right)_*\left(\left(i \circ p_{12}^{'}\right)^*\mathcal{F}_1 \otimes \left(i \circ p_{23}^{'}\right)^*\mathcal{F}_2\right) \\
= \;& \left(p_{13}^{'}\right)_*\left(\left(p_{12}^{'}\right)^* i^*\mathcal{F}_1 \otimes \left(p_{23}^{'}\right)^* i^*\mathcal{F}_2\right) \\
= \;& i^*\mathcal{F}_1 *^{'} i^*\mathcal{F}_2.
\end{aligned}$$

Hence, $i^* : K(Z) \longrightarrow K\left(Z'\right)$ is a ring homomorphism with respect to the convolution product. Furthermore, we remind that the induced $K$-group homomorphisms from homotopy equivalences commute with the convolution products. The theorem is now immediate from Equation (5.2). □

We now present the main result of this paper.

COROLLARY 5.3. *The following diagram commutes for a nilpotent element $x \in N$ of type $1^{w_1}2^{w_2}\cdots(n-1)^{w_{n-1}}$:*

$$\begin{array}{ccccc} K(Z)\otimes K(\mathbf{F}_x) & \xrightarrow{i^*\otimes i^*} & K\left(Z'\right)\otimes K(\mathbf{F}_x) & \xrightarrow{\simeq} & K(Z(\mathbf{w}))\otimes K(\mathcal{L}(\mathbf{w})) \\ \downarrow{\scriptstyle convolution} & & \downarrow{\scriptstyle convolution} & & \downarrow{\scriptstyle convolution} \\ K(\mathbf{F}_x) & \xrightarrow[i^*]{} & K(\mathbf{F}_x) & \xrightarrow[\simeq]{} & K(\mathcal{L}(\mathbf{w})). \end{array}$$

*Proof.* We first notice that the fiber $\mathbf{F}_x = \mu^{-1}(x)$ can be considered as a subvariety of $M'$ or $M$, respectively. So, $\mathbf{F}_x$ can be viewed as a subvariety of $M'\times M'$ or $M\times M$ through the diagonal embeddings. Thus, we obtain the pull-back with support homomorphism $i^* : K(\mathbf{F}_x) \longrightarrow K(\mathbf{F}_x)$ from $i : M' \times M' \hookrightarrow M \times M$. Hence, $i^* : K(\mathbf{F}_x) \longrightarrow K(\mathbf{F}_x)$ is just the restriction of $i^* : K(Z) \longrightarrow K\left(Z'\right)$ because they are the pull-backs obtained from the same embedding $i : M' \times M' \hookrightarrow M \times M$.

Now, the following diagram is commutative by the same argument as the proof of Theorem (5.2):

$$\begin{array}{ccc} K(Z)\otimes K(\mathbf{F}_x) & \xrightarrow{\text{convolution}} & K(\mathbf{F}_x) \\ \downarrow{\scriptstyle i^*\otimes i^*} & & \downarrow{\scriptstyle i^*} \\ K\left(Z'\right)\otimes K(\mathbf{F}_x) & \xrightarrow{\text{convolution}} & K(\mathbf{F}_x). \end{array}$$

On the other hand, we note that the isomorphism $\varphi^{-1}$ yields an isomorphism $\mathbf{F}_x \simeq \mathcal{L}(\mathbf{w})$ because $\psi(G_{\mathbf{v}}(0,0,0,0)) = x$. The theorem is now immediate because the isomorphism $Z' \simeq Z(\mathbf{w})$ is also obtained from the same isomorphism $\varphi^{-1}$. □

## References

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