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SOME RESULTS RELATED WITH POISSON-SZEGÖ KERNEL AND BEREZIN TRANSFORM

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Abstract. Let μ be a finite positive Borel measure on the unit ball $B \subset \mathbb{C}^n$ and ν be the Euclidean volume measure such that $\nu(B) = 1$. For the unit sphere $S = \{z : |z| = 1\}$, σ is the rotation-invariant measure on S such that $\sigma(S) = 1$. Let $\mathcal{P}[f]$ be the Poisson-Szegö integral of f and $\tilde{\mu}$ be the Berezin transform of μ. In this paper, we show that if there is a constant $M > 0$ such that $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$ for all $f \in L^p(\sigma)$, then $\| \tilde{\mu} \|_\infty \equiv \sup_{z\in B} |\tilde{\mu}(z)| < \infty$, and we show that if $\| \tilde{\mu} \|_\infty < \infty$, then $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq C \| \tilde{\mu} \|_\infty \int_S |f(\zeta)|^p d\sigma(\zeta)$ for some constant C.

1. Introduction

Throughout this paper, $\mathbb{C}^n (n \geq 1)$ will be the Cartesian product of n copies of $\mathbb{C}$(set of complex numbers). For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $|z|^2 = \langle z, z\rangle$. For $w \in \mathbb{C}^n$ and $r > 0$, let $B(w, r) = \{z \in \mathbb{C}^n : |z - w| < r\}$. For simplicity, the unit ball $B(0, 1)$ will be denoted by B. The boundary of B is the unit sphere $S = \{z : |z| = 1\}$.

Let σ be the rotation-invariant measure on S such that $\sigma(S) = 1$. For $1 \leq p \leq \infty$, $L^p(\sigma)$ denote the Lebesgue space of S induced by σ.

For μ a finite positive Borel measure on B and g measurable, we write

$$\| g \|_\mu^p = \int_B |g(z)|^p d\mu(z).$$

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Let ν be the usual Euclidean volume measure on $\mathbb{C}^n$ such that $\nu(B) = 1$. For $\mu = \nu$, we suppress the subscript($\| g \|_\nu = \| g \|$).

For $f \in L^1(\sigma)$, $\mathcal{P}[f]$ denotes the Poisson-Szegö integral defined for $z \in B$ by

$$\mathcal{P}[f](z) = \int_S P(z,\zeta) f(\zeta) d\sigma(\zeta)$$

where

$$P(z,\zeta) = \left(\frac{1 - |z|^2}{|1 - \langle z, \zeta \rangle|^2} \right)^n$$

is the Poisson-Szegö kernel for B(See [10], [12] and [13]).

In section 2, we will show that if $\sup_{\zeta \in S} \int_B P(z,\zeta) d\mu(z) < \infty$, then

$$\int_B |P[f](z)|^p d\mu(z) \leq \sup_{\zeta \in S} \int_B P(z,\zeta) d\mu(z) \int_S |f(\zeta)|^p d\sigma(\zeta).$$

And we will investigate the integrability of some functions using this property.

Let $L^2(B, d\nu)$ be the usual space of Lebesgue square-integrable complex valued functions on B. The Bergman space $L^2_a(B, d\nu)$ is defined to be the subspace of $L^2(B, d\nu)$ consisting of analytic functions.

Fix a point $z \in B$. Since the functional e_z given by $e_z(f) = f(z), f \in L^2_a(B, d\nu)$, is continuous, there exists a function $K(\cdot, z) \in L^2_a(B, d\nu)$ such that

$$f(z) = \int_B f(w) \overline{K(w,z)} d\nu(w)$$

by the Riesz representation theorem. $K(\cdot, z)$ is called the Bergman reproducing kernel in $L^2_a(B, d\nu)$. The function $K(\cdot, \cdot)$ is well understood on bounded symmetric domains and have many useful properties which is key roles in the theories for the complex analysis. The function $K(\cdot, \cdot)$ is actually defined and continuous on $B \times \overline{B}$(where $\overline{B}$ is the closure of B in $\mathbb{C}^n$)(See [3],[4],[5] and [8]).

The normalized(in $L^2_a(B, d\nu)$) reproducing kernel is denoted by $k_z(\cdot) = K(z,z)^{-1/2} K(\cdot, z)$. We define the Berezin transform of μ by

$$\tilde{\mu}(z) = \int_B |k_z(w)|^2 d\mu(w)$$

and consider the usual supremum $\| \tilde{\mu} \|_\infty \equiv \sup_{z \in B} |\tilde{\mu}(z)|$.

The Bergman metric $\beta(\cdot, \cdot)$ gives the usual topology on B(See [9, p52]). Moreover, the closed metric balls

$$E(z, r) = \{ w : \beta(z, w) \leq r \}$$

are compact(See[9, p56]).

In section 3, we will show that if there is a constant $M > 0$ such that $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$ for all $f \in L^p(\sigma)$, then $\| \tilde{\mu} \|_\infty \equiv \sup_{z\in B} |\tilde{\mu}(z)| < \infty$. We will also show that if $\| \tilde{\mu} \|_\infty < \infty$, then $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq C \| \tilde{\mu} \|_\infty \int_S |f(\zeta)|^p d\sigma(\zeta)$ for some constant C.

2. Results related with Poisson-Szegö kernel

Let $A(B)$ be the class of all $f : B \to \mathbb{C}$ that are continuous on the closed ball $\overline{B}$ and that are holomorphic in its interior B. Equipped with the supremum norm $\| f \|_\infty$, $A(B)$ is a Banach Algebra.

THEOREM 2.1. *(a) $f \in C(S)$ and F is defined on $\overline{B}$ so that $F = f$ on S and $F = \mathcal{P}[f]$ in B, then $F \in C(\overline{B})$ and $\| F \|_\infty = \| f \|_\infty$.*

(b) If $1 \leq p \leq \infty$, $f \in L^p(\sigma)$, $F = \mathcal{P}[f]$ and $F_r(\zeta) = F(r\zeta)$ $(0 \leq r < 1, \zeta \in S)$, then $\| F_r \|_p \leq \| f \|_p$. If also $1 \leq p < \infty$, then

$$\lim_{r\to 1} \| F_r - f \|_p = 0.$$

Proof. See [13, Theorem 5.5 and Proposition 5.6]. □

THEOREM 2.2. *If $f \in A(B)$, then $f(z) = \mathcal{P}[f](z)$ for all $z \in B$.*

Proof. See [13, Corollary 5.2]. □

THEOREM 2.3. *If $f \in A(B)$ and $z \in B$, Then*

$$f(z) = \int_S \frac{f(\zeta)}{(1 - \langle z, \zeta\rangle)^n} d\sigma(\zeta).$$

Proof. See [12, Theorem 3.2.4]. □

THEOREM 2.4. *If $0 \leq r < 1$, $\zeta \in S$ and $\eta \in S$, then*

$$\mathcal{P}(r\eta, \zeta) = \mathcal{P}(r\zeta, \eta).$$

Also,

$$\int_S \mathcal{P}(r\eta, \zeta) d\sigma(\zeta) = 1 = \int_S \mathcal{P}(r\zeta, \eta) d\sigma(\eta).$$

Proof. See [13, Lemma 5.3]. □

THEOREM 2.5. *The measures ν and σ are related by the formula*

$$\int_{C^n} f d\nu = 2n \int_0^\infty r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr.$$

In particular,

$$\int_B f d\nu = 2n \int_0^1 r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr.$$

Proof. See [12, Proposition 1.4.3]. □

THEOREM 2.6. *If μ is a measure such that $\sup_{\zeta\in S} \int_B P(z,\zeta) d\mu(z) < \infty$, then*

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \le \left(\sup_{\zeta\in S} \int_B P(z,\zeta) d\mu(z) \right) \int_S |f(\zeta)|^p d\sigma(\zeta)$$

for all $f \in L^p(\sigma)$ $(1 \le p < \infty)$.

Proof. By Hölder inequlity,

$$\begin{aligned} |\mathcal{P}[f](z)| &= \left| \int_S P(z,\zeta) f(\zeta) d\sigma(\zeta) \right| \\ &\le \left\{ \int_S |f(\zeta)|^p P(z,\zeta) d\sigma(\zeta) \right\}^{\frac{1}{p}}. \end{aligned}$$

This implies that

$$\begin{aligned} \int_B |\mathcal{P}[f](z)|^p d\mu(z) &= \int_B \int_S |f(\zeta)|^p P(z,\zeta) d\sigma(\zeta) d\mu(z) \\ &= \int_S |f(\zeta)|^p \int_B P(z,\zeta) d\mu(z) d\sigma(\zeta) \\ &\le \left(\sup_{\zeta\in S} \int_B P(z,\zeta) d\mu(z) \right) \int_S |f(\zeta)|^p d\sigma(\zeta). \end{aligned}$$

□

THEOREM 2.7. *For $f \in L^p(\sigma)$ $(1 \le p < \infty)$,*

$$\int_B |\mathcal{P}[f](z)|^p d\nu(z) \le \int_S |f(\zeta)|^p d\sigma(\zeta).$$

Proof. Since $\sup_{\zeta\in S} \int_B P(z,\zeta) d\nu(z) = 1$ by Theorem 2.4 and Theorem 2.5,

$$\begin{aligned} \int_B |\mathcal{P}[f](z)|^p d\nu(z) &\le \left(\sup_{\zeta\in S} \int_B P(z,\zeta) d\nu(z) \right) \int_S |f(\zeta)|^p d\sigma(\zeta) \\ &\le \int_S |f(\zeta)|^p d\sigma(\zeta) \end{aligned}$$

where the first inequality follows from Theorem 2.6. □

COROLLARY 2.8. *If μ is a measure such that $\sup_{\zeta\in S}\int_B P(z,\zeta)d\mu(z) < \infty$, then*

$$\sup_{w\in B}\int_B \frac{(1-|w|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(z) \le \sup_{\zeta\in S}\int_B P(z,\zeta)d\mu(z).$$

Proof. For the following function g such that

$$g(\zeta) = \left(\frac{(1-|w|^2)^n}{(1-\langle\zeta,w\rangle)^{2n}}\right)^{1/p},$$

$$\int_S |g(\zeta)|^p d\sigma(\zeta) = 1$$

by Theorem 2.3 and

$$\mathcal{P}[g](z) = \left(\frac{(1-|w|^2)^n}{(1-\langle z,w\rangle)^{2n}}\right)^{1/p}$$

by Theorem 2.2.

$$\begin{aligned}
&\sup_{w\in B}\int_B \frac{(1-|w|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\mu(z)\\
&= \sup_{w\in B}\int_B |\mathcal{P}[g](z)|^p d\mu(z)\\
&\le \sup_{w\in B}\left(\sup_{\zeta\in S}\int_B P(z,\zeta)d\mu(z)\right)\int_S |g(\zeta)|^p d\sigma(\zeta)\\
&\le \sup_{\zeta\in S}\int_B P(z,\zeta)d\mu(z)
\end{aligned}$$

where the second inequality follows from Theorem 2.6. □

COROLLARY 2.9.

$$\sup_{w\in B}\int_B \frac{(1-|w|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\nu(z) \le 1.$$

Proof. By Corollary 2.8,

$$\sup_{w\in B}\int_B \frac{(1-|w|^2)^n}{|1-\langle w,z\rangle|^{2n}}d\nu(z) \le \sup_{\zeta\in S}\int_B P(z,\zeta)d\nu(z) = 1.$$

□

3. Berezin transform of μ

In this section, we will show that if there is a constant $M > 0$ such that $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \le M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$ for all $f \in L^p(\sigma)$, then $\| \tilde{\mu} \|_\infty \equiv \sup_{z\in B} |\tilde{\mu}(z)| < \infty$. We will also show that if $\| \tilde{\mu} \|_\infty < \infty$, then $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \le C \| \tilde{\mu} \|_\infty \int_S |f(\zeta)|^p d\sigma(\zeta)$ for some constant C.

Lemma 3.1. *For $r > 0$, there are constants $M(r)$ and $n(r)$ so that*

$$0 < n(r) \le |k_z(w)|^2 \nu(E(z,r)) < M(r) < \infty$$

for all $z, w \in B$ with $\beta(z,w) \le r$.

Proof. See [2, Lemma 8]. □

Theorem 3.2. *For a, b in B with $\beta(a,b) \le R$ and $r, s > 0$, we have*

$$0 < m(R,r,s) \le \frac{\nu(E(a,r))}{\nu(E(b,s))} \le M(R,r,s) < \infty.$$

Proof. See [2, Lemma 6]. □

Theorem 3.3. *For fixed $r > 0$, there is a sequence $\{w_j\}$ in B such that*

(1) $\cup_{j=1}^{\infty} E(w_j, r) = B$,

(2) there is a positive integer N_0 such that, for any z in B, z is contained in at most N_0 of the sets $E(w_k, 2r)$.

For the above sequence $\{w_j\}$ and any positive Borel measure m, we have

$$\sum_{k=1}^{\infty} m(E(w_k, 2r)) \le N_0 m(B).$$

Proof. See [3, Lemma 5 and Lemma 6]. □

Theorem 3.4. *For $f \in L^p(\sigma), 1 \le p \le \infty$,*

$$|\mathcal{P}[f](w)|^p \le \frac{C_r}{\nu(E(w,r))} \int_{E(w,r)} |\mathcal{P}[f](z)|^p d\nu(z).$$

Proof. See [14, Theorem 3.6]. □

Theorem 3.5. *If $z \in E(w_n, r)$, then*

$$|\mathcal{P}[f](z)|^p \le \frac{C_r M}{\nu(E(w_n,r))} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$

where $1 \le p \le \infty$.

Proof. By Theorem 3.2,

$$\frac{1}{\nu(E(z,r))} \leq \frac{M}{\nu(E(w_n,r))}$$

for some constant M. By Theorem 3.4, if $z \in E(w_n, r)$,

$$\begin{aligned}|\mathcal{P}[f](z)|^p &\leq \frac{C_r}{\nu(E(z,r))}\int_{E(z,r)} |\mathcal{P}[f](w)|^p d\nu(w)\\ &\leq \frac{C_r}{\nu(E(z,r))}\int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)\\ &\leq \frac{C_r M}{\nu(E(w_n,r))}\int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w).\end{aligned}$$

□

THEOREM 3.6. *If there is a constant M such that*

$$\int_B |P[f](z)|^p d\mu(z) \leq M \int_B |P[f](z)|^p d\nu(z)$$

for all $f \in L^p(\sigma)(1 \leq p < \infty)$, then $\| \tilde{\mu} \|_\infty < \infty$.

Proof. For the following function f such that

$$f(\zeta) = \left(\frac{(1-|w|^2)^{n+1}}{(1-\langle \zeta, w\rangle)^{2(n+1)}}\right)^{1/p},$$

$$\begin{aligned}\mathcal{P}[f](z) &= \int_S \mathcal{P}(z,\zeta)\left(\frac{(1-|w|^2)^{n+1}}{(1-\langle \zeta, w\rangle)^{2(n+1)}}\right)^{1/p} d\sigma(\zeta)\\ &= \left(\frac{(1-|w|^2)^{n+1}}{(1-\langle z, w\rangle)^{2(n+1)}}\right)^{1/p}\end{aligned}$$

by Theorem 2.2. This implies that

$$\begin{aligned}&\int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w\rangle|^{2(n+1)}} d\mu(z)\\ &= \int_B |\mathcal{P}[f](z)|^p d\mu(z)\\ &\leq M\int_B |\mathcal{P}[f](z)|^p d\nu(z)\\ &= M\int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w\rangle|^{2(n+1)}} d\nu(z).\end{aligned}$$

Since $k_w(z) = \frac{(1-|w|^2)^{\frac{n+1}{2}}}{(1-\langle z,w\rangle)^{(n+1)}}$ is unit vector in $L^2_a(B, d\nu)$,

$$\begin{aligned}&\int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z,w\rangle|^{2(n+1)}} d\mu(z)\\ &\leq M \int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z,w\rangle|^{2(n+1)}} d\nu(z)\\ &= M.\end{aligned}$$

This implies that

$$\begin{aligned}\| \tilde{\mu} \|_\infty &= \sup_{w\in B} \int_B |k_w(z)|^2 d\mu(z)\\ &= \int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z,w\rangle|^{2(n+1)}} d\mu(z)\\ &\leq M.\end{aligned}$$

□

THEOREM 3.7. *If $f \in L^p(\sigma)$ $(1 \leq p < \infty)$ and $\| \tilde{\mu} \|_\infty < \infty$, then*

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq C \| \tilde{\mu} \|_\infty \int_S |f(\zeta)|^p d\sigma(z)$$

for some constant C.

Proof. By Theorem 3.3 and Theorem 3.5,

$$\begin{aligned}&\int_B |\mathcal{P}[f](z)|^p d\mu(z)\\ &\leq \sum_{n=1}^{\infty} \int_{E(w_n,r)} |\mathcal{P}[f](z)|^p d\mu(z)\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \mu(E(w_n,r)) \frac{C_r M}{\nu(E(w_n,r))} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w).$$

So, we obtain the following inequalities

$$\begin{aligned}&\int_B |\mathcal{P}[f](z)|^p d\mu(z)\\ &\leq MC_r \sum_{n=1}^{\infty} \frac{\mu(E(w_n,r))}{\nu(E(w_n,r))} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)\end{aligned}$$

$$\leq MC_r \sum_{n=1}^{\infty} \left(\int_{E(w_n,r)} \frac{1}{\nu(E(w_n,r))} d\mu(w) \right) \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\leq MC_r \sum_{n=1}^{\infty} \left(\int_{E(w_n,r)} |k_w(z)|^2 d\mu(z) \right) \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\leq MC_r \sum_{n=1}^{\infty} \tilde{\mu}(w_n) \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\leq C_r M \parallel \tilde{\mu} \parallel_\infty \sum_{n=1}^{\infty} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\leq C_r M \parallel \tilde{\mu} \parallel_\infty N_0 \int_B |\mathcal{P}[f](w)|^p d\nu(w)$$

where the third inequality follows from Lemma 3.1 and the last inequality follows from Theorem 3.3. Since

$$\int_B |\mathcal{P}[f](w)|^p d\nu(w) \leq \int_S |f(\zeta)|^p d\sigma(\zeta),$$

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq C_r M N_0 \parallel \tilde{\mu} \parallel_\infty \int_S |f(\zeta)|^p d\sigma(\zeta).$$

□

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