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A NOTE ON WEAK QUASI GENERALIZED CONTINUITY ON BIGENERALIZED TOPOLOGICAL SPACES

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Abstract. We introduce the notion of weakly quasi generalized continuous functions, and investigate properties for the continuity.

1. Introduction

Császár [1] introduced the notions of generalized topologies and associated interior(closure) operators on generalized topological spaces. In [3], the concept of bitopology was introduced by Kelly. He called a set equipped with two topologies a bitopological space. Datta [2] has stated that a subset S of bitopological space (X, P, Q) is *quasiopen* if for every $x \in S$, there exists a P-open set U such that $x \in U \subseteq S$, or a Q-open set V such that $x \in V \subseteq S$. In [7], we introduced and investigated the notions of quasi generalized open sets and quasi generalized continuity. The purpose of this paper is to introduce and investigate the notion of weakly quasi generalized continuous functions, and investigate properties for the continuity.

2. Preliminaries

Let X be a nonempty set and ψ be a collection of subsets of X. Then ψ is called a *generalized topology* (briefly GT) on X iff $\emptyset \in \psi$ and $G_i \in \psi$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \psi$. We call the pair (X, ψ) a

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generalized topological space (briefly GTS) on X. The elements of ψ are called *ψ-open* sets and the complements are called *ψ-closed* sets. Denote $\mathcal{M}_\psi = \cup\{M \subseteq X : M \in \psi\}$.

If ψ is a generalized topology on X and $A \subseteq X$, the *interior* of A (denoted by $i_\psi(A)$) is the union of all $G \subseteq A$, $G \in \psi$, and the *closure* of A (denoted by $c_\psi(A)$) is the intersection of all ψ-closed sets containing A. Let ψ and μ be generalized topologies on X and Y, respectively. Then a function $f : (X, \psi) \to (Y, \mu)$ is said to be *(ψ, μ)-continuous* [1] if $G \in \mu$ implies that $f^{-1}(G) \in \psi$.

Let X be a nonempty set, and let ψ_1, ψ_2 be generalized topologies on X. A triple (X, ψ_1, ψ_2) is called a *bigeneralized topological space* (briefly *biGTS*) [7]. Let (X, ψ_1, ψ_2) be a biGTS. A subset A of X is said to be *quasi (ψ_1, ψ_2)-open* (briefly *quasi q_ψ-open*) [7] if for every $x \in A$, there exists a ψ_1-open set U such that $x \in U \subseteq A$, or a ψ_2-open set V such that $x \in V \subseteq A$. A subset A of X is said to be *quasi q_ψ-closed* if the complement of A is quasi q_ψ-open.

Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. We define the *quasi q_ψ-closure* (briefly $c_{q_\psi}(A)$) and the *quasi q_ψ-interior* (briefly $i_{q_\psi}(A)$) as the following:

$$c_{q_\psi}(A) = \cap\{F : A \subseteq F \text{ for a quasi } q_\psi\text{-closed set } F\};$$

$$i_{q_\psi}(A) = \cup\{G : G \subseteq A \text{ for a quasi } q_\psi\text{-open set } G\}.$$

LEMMA 2.1 ([7]). *Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. Then*

(1) A is quasi q_ψ-open if and only if A is a union of a ψ_1-open set and a ψ_2-open set.

(2) A is quasi q_ψ-closed if and only if A is a intersection of a ψ_1-closed set and a ψ_2-closed set.

(3) Any union of quasi q_ψ-open sets is quasi q_ψ-open.

Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. The ψ_i-closure and ψ_i-interior of A with respect to ψ_i are denoted by $c_{\psi_i}(A)$ and $i_{\psi_i}(A)$, respectively, for $i = 1, 2$.

THEOREM 2.2 ([7]). *Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. Then*

(1) $c_{q_\psi}(A) = c_{\psi_1}(A) \cap c_{\psi_2}(A)$.

(2) $i_{q_\psi}(A) = i_{\psi_1}(A) \cup i_{\psi_2}(A)$.

(3) A is quasi q_ψ-closed iff $c_{q_\psi}(A) = A$.

(4) A is quasi q_ψ-open iff $i_{q_\psi}(A) = A$.

(5) $x \in c_{q_\psi}(A)$ iff for every quasi q_ψ-open set U containing x, $A \cap U \neq \emptyset$.

(6) $x \in i_{q_\psi}(A)$ iff there exists a quasi q_ψ-open set U containing x such that $x \in U \subseteq A$.

(7) $c_{q_\psi}(A) = X - i_{q_\psi}(X - A)$; $i_{q_\psi}(A) = X - c_{q_\psi}(X - A)$

3. Weakly quasi (q_ψ, q_μ)-continuous functions

DEFINITION 3.1 ([7]). Let (X, ψ_1, ψ_2) and (Y, μ_1, μ_2) be two biGTS's. Then a function $f : X \to Y$ is said to be *quasi* (q_ψ, q_μ)*-continuous* (or *quasi generalized continuous*) if for every quasi q_μ-open set U in Y, $f^{-1}(U)$ is quasi q_ψ-open in X.

DEFINITION 3.2. Let (X, ψ_1, ψ_2) and (Y, μ_1, μ_2) be two biGTS's. Then a function $f : X \to Y$ is said to be *weakly quasi* (q_ψ, q_μ)*-continuous* if for each $x \in X$ and each quasi q_μ-open set V containing $f(x)$, there exists a quasi q_ψ-open set U containing x such that $f(U) \subseteq c_{q_\mu}(V)$.

Let ψ and ψ' be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is said to be *weakly* (ψ, ψ')*-continuous* [5] if for each $x \in X$ and each ψ'-open set V containing $f(x)$, there exists a ψ-open set U containing x such that $f(U) \subseteq c_{\psi'}(V)$.

REMARK 3.3. Let (X, ψ_1, ψ_2) and (Y, μ_1, μ_2) be biGTS's. Then

(1) if $\psi_1 = \psi_2$ and $\mu_1 = \mu_2$ and if $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ is a weakly quasi (q_ψ, q_μ)-continuous (resp., quasi (q_ψ, q_μ)-continuous) function, then it is weakly (ψ_1, μ_1)-continuous (resp., (ψ_1, μ_1)-continuous);

(2) if f is a quasi (q_ψ, q_μ)-continuous function, then it is weakly quasi (q_ψ, q_μ)-continuous. But the converse may not be true in general as shown in the following example.

EXAMPLE 3.4. Let $X = \{a, b, c\}$ and $Y = \{a, b, c, d\}$. Consider
$\psi_1 = \{\emptyset, \{a\}\}$ and $\psi_2 = \{\emptyset, \{b, c\}\}$ on X;
$\mu_1 = \{\emptyset, \{a\}\}$ and $\mu_2 = \{\emptyset, \{a, b\}\}$ on Y.

Let us define a function $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ as follows $f(a) = a$, $f(b) = c$, $f(c) = b$. Then it is obvious that f is weakly quasi (q_ψ, q_μ)-continuous but it is not quasi (q_ψ, q_μ)-continuous.

THEOREM 3.5. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on two biGTS's (X, ψ_1, ψ_2) and (Y, μ_1, μ_2). Then the following are equivalent:*

(1) f is weakly quasi (q_ψ, q_μ)-continuous.

(2) $f^{-1}(V) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$ for every quasi q_μ-open subset V of Y.

(3) $c_{q_\psi}(f^{-1}(i_{q_\mu}(F))) \subseteq f^{-1}(F)$ for every quasi q_μ-closed set F of Y.

(4) $c_{q_\psi}(f^{-1}(i_{q_\mu}(c_{q_\mu}(B)))) \subseteq f^{-1}(c_{q_\mu}(B))$ *for every set* B *of* Y.

(5) $f^{-1}(i_{q_\mu}(B)) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(i_{q_\mu}(B)))$ *for every set* B *of* Y.

(6) $c_{q_\psi}(f^{-1}(V)) \subseteq f^{-1}(c_{q_\mu}(V))$ *for every quasi* q_μ*-open subset* V *of* Y.

Proof. (1) $\Rightarrow$ (2) Let V be a quasi q_μ-open subset of Y and $x \in f^{-1}(V)$. Then there exists a quasi q_ψ-open subset U of X containing x such that $f(U) \subseteq c_{q_\mu}(V)$. Since $x \in U \subseteq f^{-1}(c_{q_\mu}(V))$, from Theorem 2.2, $x \in i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$. Hence $f^{-1}(V) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$.

(2) $\Rightarrow$ (3) Let F be a quasi q_μ-closed subset in Y. Then by (2), $f^{-1}(Y - F) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(Y - F))) = i_{q_\psi}(f^{-1}(Y - i_{q_\mu}(F))) \subseteq X - c_{q_\psi}(f^{-1}(i_{q_\mu}(F)))$.

Thus $c_{q_\psi}(f^{-1}(i_{q_\mu}(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) For a subset B of Y, since $c_{q_\mu}(B)$ is quasi q_μ-closed in Y, from (3), it follows $c_{q_\psi}(f^{-1}(i_{q_\mu}(c_{q_\mu}(B)))) \subseteq f^{-1}(c_{q_\mu}(B))$.

(4) $\Rightarrow$ (5) Let B be a subset of Y. From (4) and Theorem 2.2, it follows $f^{-1}(i_{q_\mu}(B)) = X - f^{-1}(c_{q_\mu}(Y - B)) \subseteq X - c_{q_\psi}(f^{-1}(i_{q_\mu}(c_{q_\mu}(Y - B)))) = i_{q_\psi}(f^{-1}(c_{q_\mu}(i_{q_\mu}(B))))$. Thus we get the result.

(5) $\Rightarrow$ (6) Let V be a quasi q_μ-open subset of Y. Suppose $x \notin f^{-1}(c_{q_\mu}(V))$. Then by Theorem 2.2, there exists a quasi q_μ-open set U containing $f(x)$ such that $U \cap V = \emptyset$, and it implies $c_{q_\mu}(U) \cap V = \emptyset$. For the quasi q_μ-open set U, by (5), $x \in f^{-1}(U) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(U)))$. Thus there exists a quasi q_ψ-open set G containing x such that $x \in G \subseteq f^{-1}(c_{q_\mu}(U))$. Since $c_{q_\mu}(U) \cap V = \emptyset$ and $f(G) \subseteq c_{q_\mu}(U)$, we have $G \cap f^{-1}(V) = \emptyset$. By Theorem 2.2, $x \notin c_{q_\psi}(f^{-1}(V))$ and consequently, $c_{q_\psi}(f^{-1}(V)) \subseteq f^{-1}(c_{q_\mu}(V))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and V a quasi q_μ-open set in Y containing $f(x)$. Then $V = i_{q_\mu}(V) \subseteq i_{q_\mu}(c_{q_\mu}(V))$, from (6), it follows

$x \in f^{-1}(V) \subseteq f^{-1}(i_{q_\mu}(c_{q_\mu}(V))) = X - f^{-1}(c_{q_\mu}(Y - c_{q_\mu}(V))) \subseteq X - c_{q_\psi}(f^{-1}(Y - c_{q_\mu}(V))) = i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$. So there exists a quasi q_ψ-open subset U containing x in X such that $U \subseteq f^{-1}(c_{q_\mu}(V))$. Hence f is weakly quasi (q_ψ, q_μ)-continuous. □

Definition 3.6. Let (X, ψ_1, ψ_2) be a biGTS. X is said to be *G-quasi* q_ψ*-regular* if for each $x \in \mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}$ and a quasi q_ψ-closed set F with $x \notin F$, there exist disjoint quasi q_ψ-open sets U and V such that $x \in U$ and $F \cap (\mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}) \subseteq V$.

REMARK 3.7. Let (X,μ) be a generalized topological space. Then X is said to be *relative G-regular* (simply, *G-regular*) [6] on $\mathcal{M}_\mu$ if for $x \in \mathcal{M}_\mu$ and a μ-closed set F with $x \notin F$, there exist $U, V \in \mu$ such that $x \in U$, $F \cap \mathcal{M}_\mu \subseteq V$ and $U \cap V = \emptyset$. For a biGTS (X,ψ_1,ψ_2), if $\psi_1 = \psi_2$, then X is the G-regular space.

We recall: Let (X,ψ_1,ψ_2) be a biGTS. Then X is said to be *bi-strong* [7] if $X = \mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}$.

LEMMA 3.8. *Let (X,ψ_1,ψ_2) be a biGTS. If X is bi-strong and G-quasi q_ψ-regular, then for each $x \in X$ and a quasi q_ψ-closed set F with $x \notin F$, there exist disjoint quasi q_ψ-open sets U and V such that $x \in U$ and $F \subseteq V$.*

THEOREM 3.9. *Let (X,ψ_1,ψ_2) be a biGTS. Then X is G-quasi q_ψ-regular if and only if for $x \in \mathcal{M}_\psi$ and each quasi q_ψ-open set U containing x, there is a quasi q_ψ-open set V containing x such that $x \in V \subseteq c_{q_\psi}(V) \cap \mathcal{M}_\psi \subseteq U$ where $\mathcal{M}_\psi = \mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}$.*

Proof. Let X be G-quasi q_ψ-regular. Then for $x \in \mathcal{M}_\psi$ and a quasi q_ψ-open set U containing x, since the quasi q_ψ-closed set U^c does not contain the point x, there are disjoint quasi q_ψ-open sets V, W such that $x \in V$ and $U^c \cap \mathcal{M}_\psi \subseteq W$. Since $V \subseteq W^c$ and W^c is quasi q_ψ-closed, $c_\psi V \subseteq W^c$ and $c_\psi(V) \cap (U^c \cap \mathcal{M}_\psi) \subseteq c_\psi V \cap W = \emptyset$. It implies $c_\psi V \cap \mathcal{M}_\psi \subseteq U$.

For the converse, let F be a quasi q_ψ-closed set and $x \notin F$ for $x \in \mathcal{M}_\psi$. Then by hypothesis, there is a quasi q_ψ-open set V containing x such that $x \in V \subseteq c_\psi V \cap \mathcal{M}_\psi \subseteq F^c$, so $c_\psi V \cap \mathcal{M}_\psi \cap F = \emptyset$ and $\mathcal{M}_\psi \cap F \subseteq (c_\psi V)^c$. Hence X is G-quasi q_ψ-regular. □

COROLLARY 3.10. Let (X,ψ_1,ψ_2) be a biGTS, and let X be bi-strong. Then X is G-quasi q_ψ-regular if and only if for $x \in X$ and each quasi q_ψ-open set U containing x, there is a quasi q_ψ-open set V containing x such that $x \in V \subseteq c_{q_\psi}(V) \subseteq U$.

Let (X,ψ_1,ψ_2) and (Y,μ_1,μ_2) be two biGTS's. In Theorem 4.3 (6) of [7], we showed that f is quasi (q_ψ,q_μ)-continuous if and only if for each $x \in X$ and each quasi q_μ-open set V containing $f(x)$, there exists a quasi q_ψ-open set U containing x such that $f(U) \subseteq V$.

From the above fact, we have the following theorem.

THEOREM 3.11. *Let $f : (X,\psi_1,\psi_2) \to (Y,\mu_1,\mu_2)$ be a function on biGTS's. Let Y be G-quasi q_μ-regular and $f(\mathcal{M}_\psi) \subseteq \mathcal{M}_\mu$. If f is weakly quasi (q_ψ,q_μ)-continuous, then f is quasi (q_ψ,q_μ)-continuous.*

Proof. Let $x \in X$ and V any quasi q_μ-open set containing $f(x)$. Then from G-quasi q_μ-regularity, there exists a quasi q_μ-open set W such that $f(x) \in W \subseteq c_\mu(W) \cap \mathcal{M}_\mu \subseteq V$. For a quasi q_μ-open set W containing $f(x)$, since f is weakly quasi (q_ψ, q_μ)-continuous, there exists a quasi q_ψ-open set U such that $f(U) \subseteq c_\mu(W)$. From $f(U) \subseteq \mathcal{M}_\mu$, it follows

$$f(U) \subseteq f(U) \cap \mathcal{M}_\mu \subseteq c_\mu(W) \cap \mathcal{M}_\mu \subseteq V.$$

Hence from Theorem 4.3 (6) of [7], f is quasi (q_ψ, q_μ)-continuous. □

Corollary 3.12. Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. If Y is G-quasi q_μ-regular and bi-strong, and if f is weakly quasi (q_ψ, q_μ)-continuous, then f is quasi (q_ψ, q_μ)-continuous.

Proof. Since $f(\mathcal{M}_\psi) \subseteq Y = \mathcal{M}_\mu$, by Theorem 3.11, it is obvious. □

Definition 3.13. Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. Then a function $f : X \to Y$ is said to be *quasi* (q_ψ, q_μ)*-open* if for every quasi q_ψ-open set G in X, $f(G)$ is quasi q_μ-open in Y.

Obviously we get the following lemmas.

Lemma 3.14. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be quasi (q_ψ, q_μ)-open on two biGTS's $(X, \psi_1, \psi_2), (Y, \mu_1, \mu_2)$. If f is surjective and X is bi-strong, then Y is also bi-strong.*

Lemma 3.15. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be quasi (q_ψ, q_μ)-countious on two biGTS's $(X, \psi_1, \psi_2), (Y, \mu_1, \mu_2)$. If Y is bi-strong, then X is also bi-strong.*

Theorem 3.16. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. If Y is G-quasi q_μ-regular and if f is quasi (q_ψ, q_μ)-open, then every weakly quasi (q_ψ, q_μ)-continuous is quasi (q_ψ, q_μ)-continuous.*

Proof. It follows from Theorem 3.11. □

Let (X, ψ) and (Y, μ) be generalized topological spaces. Then $f : (X, \psi) \to (Y, \mu)$ is said to be (ψ, μ)*-open* [4] if for any ψ-open set U in X, $f(U)$ is μ-open in Y.

Corollary 3.17. Let $f : (X, \psi_1) \to (Y, \mu_1)$ be a function on GTS's. If Y is G-regular and if f is (ψ_1, μ_1)-open, then every weakly (ψ_1, μ_1)-continuous is (ψ_1, μ_1)-continuous.

Proof. Consider two biGTS's (X, ψ_1, ψ_1) and (Y, μ_1, μ_1) on X and Y, respectively. Then from Remark 3.3, Remark 3.7 and Theorem 3.16, the corollary is obtained □

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