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CHARACTERIZATION OF STANDARD EXTREME VALUE DISTRIBUTIONS USING RECORDS

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Abstract. The paper deals with characterization of standard Gumbel distribution and standard Fréchet distribution and was motivated by [4], where the Weibull distribution is characterized. We present criterions using the independence of some suitable functions of lower records in a sequence of independent identically distributed random variables $\{X_n, n \geq 1\}$.

1. Introduction

Consider the sequence $\{X_n, n \geq 1\}$ of independent, identically distributed (iid) random variables with common absolutely continuous distribution function F and probability density function (pdf) f. Random variable X_n is a lower record if $X_n < min\{X_1, X_2, \ldots, X_{n-1}\}$. By convention X_1 is a record value. Let $\{T_n, n \geq 1\}$ be the lower record times at which record values occur. We consider discrete time and define $T_1 = 1$ and $T_n = min\{i, i > T_{n-1}, X_i < X_{T_{n-1}}, n > 1\}$. So the sequence $\{L_n, n \geq 1\} = \{X_{T_n}, n \geq 1\}$ is a sequence of lower record values. The joint pdf of L_m and L_n and the pdf of L_n are (see [2])

$$f_{m,n}(x, y) = \frac{(H(x))^{m-1}}{\Gamma(m)} h(x) \frac{(H(y) - H(x))^{n-m-1}}{\Gamma(n-m)} f(y), \tag{1.1}$$
$$\text{for } 1 \leq m < n \text{ and } -\infty < y < x < \infty,$$

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$$f_n(x) = \frac{(H(x))^{n-1}}{\Gamma(n)} f(x), \text{ for } n \geq 1, \quad -\infty < x < \infty, \tag{1.2}$$

where $H(x) = -\ln F(x)$ and $h(x) = -\frac{d}{dx}H(x)$ are the corresponding hazard function and hazard rate. Similar relations for upper records are proved in [5].

A continuous random variable X is distributed according to standard Gumbel law if its distribution function F is of the form

$$F(x) = exp(-e^{-x}), \ -\infty < x < \infty. \tag{1.3}$$

We say that a continuous random variable X is distributed according to standard Fréchet law if its distribution function F is of the form

$$F(x) = \begin{cases} exp(-x^{-\alpha}), & x > 0 \text{ and } \alpha > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Some characterizations by the independence of upper record values are known. For example Ahsanullah characterized the exponential distribution (see [2]), Chang in 2007 characterized Pareto distribution (see [3]) and Lee and Lim in 2010 Weibull distribution (see [4]).

In this paper, we will give some characterizations of the standard Gumbel and the standard Fréchet distribution by the independence of certain functions of lower record values. For proving our main theorems we need the following Lemma given by Ahsanullah ([2] p. 78).

Lemma 1.1. *Let $F_1(x)$ be an absolutely continuous function and $F_1(x) < 1$, for all $x > 0$. Suppose that $q(u,v) = H_1(u+v) - H_1(u)$ and $k(u,v) = (q(u,v))^r exp(-q(u,v))\frac{\partial}{\partial v}q(u,v)$, $r \geq 0$, where $k(u,v) \neq 0$, and $\frac{\partial}{\partial v}q(u,v) \neq 0$ for any positive u and v. If $k(u,v)$ is independent of u then $q(u,v)$ is a function of v only.*

2. Characterization of Gumbel distribution

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with absolutely continuous distribution function $F(x)$, pdf $f(x)$, where $0 < F(x) < 1$ for all x real and let $F(0) = \frac{1}{e}$. Then $F(x) = exp(-e^{-x})$, $-\infty < x < \infty$ if and only if $U = e^{-L_m}$ and $V = e^{-L_n} - e^{-L_m}$ are independent for $1 \leq m < n$.*

Proof. Let $F(x) = exp(-e^{-x})$, $-\infty < x < \infty$. Then $H(x) = e^{-x}$ and according to (1.1) the joint pdf of L_m and L_n is for $-\infty < y < x < \infty$ and $1 \leq m < n$

$$f_{m,n}(x, y) = \frac{(e^{-x})^{m-1}}{\Gamma(m)} e^{-x} \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} e^{-y} e^{-e^{-y}}. \tag{2.1}$$

Consider the transformation

$$\begin{aligned} t &: \begin{pmatrix} L_m \\ L_n \end{pmatrix} \rightarrow \begin{pmatrix} e^{-L_m} \\ e^{-L_n} - e^{-L_m} \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} \\ \tau &: \begin{pmatrix} U \\ V \end{pmatrix} \rightarrow \begin{pmatrix} -\ln U \\ -\ln(U+V) \end{pmatrix}. \end{aligned} \tag{2.2}$$

The Jacobian of the transformation τ is

$$|J_\tau| = \frac{1}{U(U+V)}, \tag{2.3}$$

thus for the joint pdf of transformed U and V we get

$$f_{U,V}(u, v) = \frac{u^{m-1} v^{n-m-1}}{\Gamma(m)\Gamma(n-m)} e^{-u} e^{-v}, \text{ for } u > 0, v > 0. \tag{2.4}$$

The marginal pdf of U is given by

$$f_U(u) = \int_0^\infty f_{U,V}(u, v) dv = \frac{u^{m-1}}{\Gamma(m)} e^{-u}, \quad u > 0, \tag{2.5}$$

and the marginal pdf of V is

$$f_V(v) = \int_0^\infty f_{U,V}(u, v) du = \frac{v^{n-m-1}}{\Gamma(n-m)} e^{-v}, \quad v > 0. \tag{2.6}$$

From (2.4), (2.5) and (2.6) we obtain $f_U(u) f_V(v) = f_{U,V}(u, v)$. Hence U and V are independent.

Now we will prove the sufficient condition. So we suppose that U, V are independent for $1 \leq m < n$. The joint pdf $f_{m,n}(x, y)$ of variables L_m, L_n is given by (1.1). Consider the transformation (2.2) with Jacobian (2.3). Then for the joint pdf of transformed U, V we get

$$\begin{aligned} f_{U,V}(u, v) = & \frac{[H(-\ln u)]^{m-1} [H(-\ln(u+v)) - H(-\ln u)]^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \times \\ & \times \frac{h(-\ln u) f(-\ln(u+v))}{u(u+v)}, u > 0, v > 0 \end{aligned}$$

and the pdf of transformed U by (1.2) is

$$f_U(u) = \frac{[H(-\ln u)]^{m-1}}{\Gamma(m)} \cdot \frac{f(-\ln u)}{u}, u > 0.$$

Since U and V are independent, for the pdf of V holds

$$\begin{aligned} f_V(v) &= \frac{[H(-\ln(u+v)) - H(-\ln(u))]^{n-m-1}}{\Gamma(n-m)F(-\ln u)} \cdot \frac{f(-\ln(u+v))}{u+v} \\ &= \frac{[H(-\ln(u+v)) - H(-\ln(u))]^{n-m-1}}{\Gamma(n-m)F(-\ln u)} \cdot \frac{-\partial}{\partial v}F(-\ln(u+v)), \\ & u > 0, v > 0. \end{aligned}$$

If we denote $F_1(x) = F(-\ln x)$ and $H_1(x) = H(-\ln x))$, for all $x > 0$, then

$$f_V(v) = \frac{[H_1(u+v) - H_1(u)]^{n-m-1}}{\Gamma(n-m)} \cdot \frac{-\partial}{\partial v}\left(\frac{F_1(u+v)}{F_1(u)}\right). \tag{2.7}$$

Now let

$$H_1(u+v) - H_1(u) = -\ln\frac{F_1(u+v)}{F_1(u)} = q(u,v).$$

Writing (2.7) in terms of $q(u,v)$ as in Lemma 1.1 we get

$$f_V(v) = \frac{1}{\Gamma(n-m)}(q(u,v))^{n-m-1}exp(-q(u,v))\frac{\partial}{\partial v}q(u,v). \tag{2.8}$$

By the independence of U and V, we have for all $v > 0$ that $f_V(v)$ defined by (2.8) is independent of u. So if we use the Lemma 1.1 we get that $q(u,v)$ is a function of v only. Hence

$$G(v) = q(u,v) = -\ln\frac{F_1(u+v)}{F_1(u)}.$$

Now we have to solve a functional equation of the form

$$G_1(v)F_1(u) = F_1(u+v), \tag{2.9}$$

where $G_1(v) = exp(-G(v))$. The only continuous solution of (2.9) is (see [1]) $F_1(x) = ae^{cx}$, $x > 0$, where a and c are arbitrary constants. Then

$$ae^{cx} = F_1(x) = F(-\ln x).$$

If we use substitution $x = e^{-y}, -\infty < y < \infty$ and the boundary conditions $F(-\infty) = 0$, $F(\infty) = 1$, $F(0) = \frac{1}{e}$, then we get

$$F(x) = exp(-e^{-x}), -\infty < x < \infty,$$

what is the Gumbel distribution (1.3). Thus the proof is complete. □

3. Characterization of Fréchet distribution

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with absolutely continuous distribution function $F(x)$, pdf $f(x)$, $F(x) < 1$, for all $x > 0$, $F(0) = 0$ and $F(1) = \frac{1}{e}$. Then $F(x) = exp(-x^{-\alpha})$, $x > 0$ and $\alpha > 0$ if and only if $U = L_m^{-\alpha}$ and $V = L_n^{-\alpha} - L_m^{-\alpha}$ are independent for $1 \leq m < n$.*

Proof. If $F(x) = exp(-x^{-\alpha})$, $x > 0$, then $H(x) = x^{-\alpha}$ and according to (1.1) the joint pdf of L_m and L_n is for $1 \leq m < n$ and $-\infty < y < x < \infty$

$$f_{m,n}(x,y) = \frac{(x^{-\alpha})^{m-1}}{\Gamma(m)} \alpha x^{-\alpha-1} \frac{(y^{-\alpha} - x^{-\alpha})^{n-m-1}}{\Gamma(n-m)} \alpha y^{-\alpha-1} e^{-y^{-\alpha}}. \tag{3.1}$$

Consider the transformation

$$\begin{aligned} t &: \begin{pmatrix} L_m \\ L_n \end{pmatrix} \to \begin{pmatrix} L_m^{-\alpha} \\ L_n^{-\alpha} - L_m^{-\alpha} \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} \\ \tau &: \begin{pmatrix} U \\ V \end{pmatrix} \to \begin{pmatrix} U^{-\frac{1}{\alpha}} \\ (U+V)^{-\frac{1}{\alpha}} \end{pmatrix}. \end{aligned} \tag{3.2}$$

The Jacobian of the transformation τ is

$$|J_\tau| = \frac{1}{\alpha^2} \left[U(U+V)\right]^{-\frac{1+\alpha}{\alpha}}, \tag{3.3}$$

thus the joint pdf of transformed U and V we get in form

$$f_{U,V}(u,v) = \frac{u^{m-1} v^{n-m-1}}{\Gamma(m)\Gamma(n-m)} e^{-u} e^{-v}, \text{ for } u > 0, v > 0. \tag{3.4}$$

The marginal pdf of U is given by

$$f_U(u) = \int_0^\infty f_{U,V}(u,v) dv = \frac{u^{m-1}}{\Gamma(m)} e^{-u}, \quad u > 0, \tag{3.5}$$

and the marginal pdf of V we can write as

$$f_V(v) = \int_0^\infty f_{U,V}(u,v) du = \frac{v^{n-m-1}}{\Gamma(n-m)} e^{-v}, \quad v > 0. \tag{3.6}$$

From (3.4), (3.5) and (3.6) we obtain $f_U(u) f_V(v) = f_{U,V}(u,v)$, what means that U and V are independent.

Now we will prove the sufficient condition. Suppose that U, V are independent for $1 \leq m < n$. The joint pdf $f_{m,n}(x,y)$ of variables L_m, L_n

is given by (1.1). Consider the transformation (3.2) with Jacobian (3.3). Then for the joint pdf of transformed U, V we obtain

$$f_{U,V}(u,v) = \frac{[H(u^{-\frac{1}{\alpha}})]^{m-1}[H((u+v)^{-\frac{1}{\alpha}}) - H(u^{-\frac{1}{\alpha}})]^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \times$$
$$\times \frac{h(u^{-\frac{1}{\alpha}})f((u+v)^{-\frac{1}{\alpha}})[u(u+v)]^{-\frac{\alpha+1}{\alpha}}}{\alpha^2}, u > 0, v > 0$$

and the marginal pdf of transformed U we get by (1.2) in form

$$f_U(u) = \frac{[H(u^{-\frac{1}{\alpha}})]^{m-1}}{\Gamma(m)} \cdot \frac{f(u^{-\frac{1}{\alpha}})u^{-\frac{\alpha+1}{\alpha}}}{\alpha}, u > 0.$$

Since U and V are independent so the pdf of V is given by

$$f_V(v) = \frac{[H((u+v)^{-\frac{1}{\alpha}}) - H(u^{-\frac{1}{\alpha}})]^{n-m-1}}{\Gamma(n-m)} \cdot \frac{f((u+v)^{-\frac{1}{\alpha}})(u+v)^{-\frac{\alpha+1}{\alpha}}}{\alpha F(u^{-\frac{1}{\alpha}})}$$
$$= \frac{[H((u+v)^{-\frac{1}{\alpha}}) - H(u^{-\frac{1}{\alpha}})]^{n-m-1}}{\Gamma(n-m)F(u^{-\frac{1}{\alpha}})} \cdot \frac{-\partial}{\partial v} F((u+v)^{-\frac{1}{\alpha}}),$$
$$u > 0, v > 0.$$

If we denote $F_1(x) = F(x^{-\frac{1}{\alpha}})$ and $H_1(x) = H(x^{-\frac{1}{\alpha}}))$ for all $x > 0$, then the remainder part of the proof is the same as the proof of Theorem 2.1.

For the marginal pdf $f_V(v)$ the term (2.7) holds. The condition of the independence of U and V leads (using terms as in Lemma 1.1) to the functional equation (2.9). The only continuous solution of this equation is again $F_1(x) = ae^{cx}$, $x > 0$, where a and c are arbitrary constants. So

$$ae^{cx} = F_1(x) = F(x^{-\frac{1}{\alpha}}).$$

Now we use a substitution $x = y^{-\alpha}$, for $y > 0$. Then according to the boundary conditions $F(0) = 0$, $F(\infty) = 1$ and $F(1) = \frac{1}{e}$ we get

$$F(x) = exp(-x^{-\alpha}), x > 0, \alpha > 0,$$

what is the Fréchet distribution (1.4). This completes the proof. □

References

[1] J. ACZEL, *Lectures on Functional Equations and Their Applications*, Academic Press, NY, 1966.

[2] M. AHSANULLAH, *Record values - theory and application*, University Press of America, Inc., Lanham, Maryland, USA, 2005, ISBN 0-7618-2794-3.

[3] S. K. CHANG, *Characterizations of the Pareto distribution by the independence of record values*, J. Chungcheong Math. Soc. **20** (2007), no. 1 , 51-57.

[4] M. Y. LEE and E. H. LIM, *On characterizations of the Weibull distribution by the independent property of record value*, J. Chungcheong Math. Soc. **23** (2010), no. 2 , 245-250.

[5] V. SKŘIVÁNKOVÁ and M. JUHÁS, *Records in Non-life Insurance*, IM Preprint, series A, No. 9/2010, June 2010.

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