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# 멀티피직스 환경하의 이방성 구조물 해석

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# Analysis of Anisotropic Structures under Multiphysics Environment

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## ABSTRACT

An anisotropic beam model is proposed by employing an asymptotic expansion method for thermo-mechanical multiphysics environment. An asymptotic method based on virtual work is introduced first, and then the variables of mechanical displacement and temperature rise are asymptotically expanded by taking advantage of geometrical slenderness of elastic bodies. Subsequently substituting these expansions into the virtual work principle allows us to asymptotically expand the virtual work. This will yield a set of recursive virtual works from which two-dimensional microscopic and one-dimensional macroscopic equations are systematically derived at each order. In this way, homogenized stiffnesses and thermomechanical coupling coefficients are derived. To demonstrate the validity and efficiency of the proposed approach, composite beams are taken as a test-bed example. The results obtained herein are compared to those of three-dimensional finite element analysis.

**Key Words** : Multiphysics, Asymptotic Analysis, Thermoelasticity, Composite Beams

## 1. 서 론

Application of composite materials has been increased to the various fields where high-strength, high-stiffness and weight-reduced materials are typically demanded, for example, aerospace and automobile industries. Especially, the modeling of composite rotor blades used for helicopters, wind power generator, and tilt-rotor aircraft is one of the popular research fields[1]. Even though the slender beam is made of composite materials, most analysis is yet carried out by adopting classical Euler-Bernoulli beam theory and/or Rankine-Timoshenko beam theory. If a beam is sufficiently slender, the classical theory can offer an accurate solution. However, it is inappropriate for the analysis of composite beams having anisotropic characteristics. Thus beam theories have been developed to predict the accurate behavior of composite beams[1]. Among the refined beam theories, an asymptotic method can mathematically guarantee the accurate solutions as the order of solutions increases.

In general, there are two types of asymptotic methods; a formal asymptotic method, and a variational asymptotic

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method. First, a variational asymptotic method was applied by Hodges and his co-workers[2] to derive a Rankine-Timoshenko like beam theory. This has been referred to as variational-asymptotic beam sectional analysis(VABS) since the work of Cesnik and Hodges[3].

On the other hand, there have been many efforts using the formal asymptotic method to find the exact interior solution by employing either the decay analysis method[4] or the averaged boundary conditions[5] for the displacement prescribed boundaries. The generalized averaged boundary conditions were successively applied to generic anisotropic composite beams, which require much less efforts than the decay analysis method. Recently, Kim[6] studied the asymptotic characteristics of heterogeneous plates with the consideration of end effects, and Kim and Wang[7] also presented the vibration analysis of composite beams.

In this paper, the asymptotic approach developed by the previous works[5-7] is extended to multiphysics problems, anisotropic composite beams under thermal loadings are taken as a test-bed example.

## 2. Asymptotic Formulation

A three-dimensional slender composite beam is shown in Fig. 1, which has arbitrary cross-sectional geometry and material anisotropy.

### 2.1 Recursive virtual work

The constitutive equations of a three-dimensional thermoelasticity can be derived by assuming the existence of the Helmholtz free energy[8].

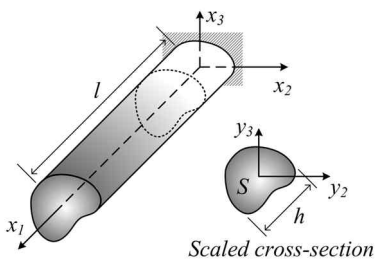



**Fig. 1 A composite beam structure with different length scales**

$$U(\varepsilon_{ij},T)=\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}-\beta_{ij}\varepsilon_{ij}\theta-\frac{1}{2}\alpha_T\theta^2 \quad (\theta=T-T_0\ ,\ \beta_{ij}=C_{ijkl}\alpha_{kl}) \tag{1}$$

where the quantities Cijkl and βij are the elastic and thermal expansion coefficients, respectively. From this, one can obtain three equations for a three-dimensional thermoelastostatic problem.

$$\begin{aligned}&\sigma_{ij,j}+\tilde{b}_i=0\\&\sigma_{ij}=\frac{\partial U}{\partial\varepsilon_{ij}}=C_{ijkl}\varepsilon_{kl}-\beta_{ij}\theta\\&\varepsilon_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k})\end{aligned} \tag{2}$$

The associated boundary conditions are given by

$$\begin{aligned}&\sigma_{ij}n_j=\overline{t}_i \quad \text{on} \quad \partial S_\sigma\\&u_i=\overline{u}_i \quad \text{on} \quad \partial S_u\end{aligned} \tag{3}$$

in which the overbar represents the prescribed quantity, and $S_{(\cdot)}$ denotes the boundary associated with (·).

To apply the asymptotic expansion method, the small parameter is defined first, and the coordinates are scaled in the following manner.

$$y_1=x_1\ ,\ y_2=\frac{x_2}{\text{窒}},\ y_3=\frac{x_3}{} \tag{4}$$

where a small parameter is defined by

$$\grave{O}=\frac{h}{l_c} \tag{5}$$

in which h is the maximum dimension of the cross-section of the beam, and lc is the characteristic length of the beam.

The virtual work principle for a three-dimensional thermoelasticity can be written as:

$$\delta W=\int_\Omega\sigma_{ij}\delta\varepsilon_{ij}d\Omega-\int_{S_\sigma}\overline{t}_i\delta u_i dS \tag{6}$$

Substituting Eq. (2) into Eq. (5) yields the scaled strain as:

$$\varepsilon=\frac{1}{\grave{O}}\mathrm{L}_{23}\mathbf{u}+\mathrm{L}_1\mathbf{u}_{,1} \tag{7}$$

where the linear operator matrices are defined in the previous works[5-7].

In order to obtain the recursive virtual work at each order, the displacement vector and the temperature rise have to be asymptotically expanded as follows:

$$\begin{aligned} \mathbf{u}(x_i) &= \mathbf{u}^{(0)}(y_1) + \sum_{n=1}^{\infty} \delta^n \mathbf{u}^{(n)}(y_i) \\ \theta(x_i) &= \theta^{(0)}(y_1) + \sum_{n=1}^{\infty} \delta^n \theta^{(n)}(y_i) \end{aligned} \tag{8}$$

Substituting Eq. (8) into Eqs. (7) and (2) yields the asymptotically expanded strain and stress vectors.

$$\varepsilon(x_i) = \sum_{n=0}^{\infty} \delta^n \varepsilon^{(n)}(y_i), \ \sigma(x_i) = \sum_{n=0}^{\infty} \delta^n \sigma^{(n)}(y_i) \tag{9}$$

where

$$\begin{aligned} \varepsilon^{(n)} &= \mathrm{L}_{23}\mathbf{u}^{(n+1)} + \mathrm{L}_1 \mathbf{u}_{,1}^{(n)} \\ \sigma^{(n)} &= \mathbf{C}\varepsilon^{(n)} + \boldsymbol{\beta}\theta^{(n)} \end{aligned} \tag{10}$$

Finally the recursive virtual work can be obtained by substituting Eq. (9) into Eq. (6) as follows:

$$\delta W(x_i) = \delta W^{(0)}(y_i) + \sum_{n=1}^{\infty} \delta^n \delta W^{(n)}(y_i) \tag{11}$$

## 2.2 The fundamental solution

The fundamental solution of the problem can be obtained from the zeroth order virtual work that is summarized as follows:

$$\delta W^{(0)} = \int_{\Omega} \delta\varepsilon^{(0)t} \sigma^{(0)} d\Omega = 0 \tag{12}$$

where the superscript t denotes the transpose of matrices or vectors.

The problem is well posed and therefore one can state the followings:

$$\sigma^{(0)} = 0 \rightarrow \varepsilon^{(0)} = 0, \ \theta^{(0)} = 0 \tag{13}$$

From which, one can find the fundamental solution[4,5] that is given as:

$$\tilde{\mathbf{u}}^{(1)} = \Phi(y_2, y_3)\hat{\mathbf{u}}^{(1)}(y_1) + \hat{\mathbf{v}}^{(1)}(y_1), \ \tilde{\theta}^{(1)} = \hat{\theta}^{(1)}(y_1) \tag{14}$$

where

$$\begin{aligned} \Phi(y_2, y_3) &= \begin{bmatrix} 1 & -y_2 & -y_3 & 0 \\ 0 & 0 & 0 & -y_3 \\ 0 & 0 & 0 & y_2 \end{bmatrix} \\ \hat{\mathbf{u}}^{(1)} &= \begin{bmatrix} v_1^{(1)} & v_{2,1}^{(0)} & v_{3,1}^{(0)} & \phi^{(1)} \end{bmatrix}^t \\ \hat{\mathbf{v}}^{(1)} &= \begin{bmatrix} 0 & v_2^{(0)} & v_3^{(0)} \end{bmatrix}^t \end{aligned} \tag{15}$$

in which vi(k) and ϕ(k) are the kth order displacement and torsional angle, respectively, at the reference line.

## 2.3 The classical solution

The classical solution can be obtained from the second order virtual work, since the first order virtual work is found to be zero. This solution is comparable to the Euler-Bernoulli theory for beams. At this point, one needs to decompose the displacement and temperature rise into two parts: fundamental and warping solutions. Let these two variables be

$$\begin{aligned} \mathbf{u}^{(2)}(y_i) &= \tilde{\mathbf{u}}^{(2)}(y_i) + \mathbf{u}_w^{(2)}(y_i) \\ \theta^{(2)}(y_i) &= \tilde{\theta}^{(2)}(y_1) + \theta_w^{(2)}(y_i) \end{aligned} \tag{16}$$

where the terms with tilde and subscript *w* represent the fundamental and warping solutions, respectively.

The second order virtual work can be obtained by substituting Eq. (16) into Eq. (11) and collecting the fundamental and warping variables as follows:

$$\delta W^{(2)} = \delta\hat{W}^{(2)}(\delta\tilde{\mathbf{u}}^{(2)}, \delta\tilde{\theta}^{(2)}) + \delta W_w^{(2)}(\delta\mathbf{u}_w^{(2)}, \delta\theta_w^{(2)}) \tag{17}$$

where the first term in RHS forms the one -dimensional macroscopic equation, while the second order solution forms the two-dimensional microscopic equation.

Let us consider the two-dimensional microscopic equation in order to find the warping solution by neglecting the applied traction. This is written as:

$$\delta W_w^{(2)} = \int_{S_c} \delta(\mathrm{L}_{23}\mathbf{u}_w^{(2)})^t \sigma^{(1)} \mathrm{d}S = 0 \tag{18}$$

which yields the equation to be solved for the mechanical warping displacement. We assume that the temperature rise is prescribed over the cross-section. The mechanical displacement is discretized by employing a standard

isoparametric two-dimensional finite element. After discretization, Eq. (18) becomes

$$\mathbf{K}\bar{\mathbf{u}}_w^{(2)} + \mathbf{F}_{23E}\mathbf{e}^{(1)} - \mathbf{F}_{23\beta}\theta^{(1)} = 0 \tag{19}$$

in which

$$\begin{aligned} &\mathbf{e}^{(1)} = \left[ v_{1,1}^{(1)} \quad v_{2,11}^{(0)} \quad v_{3,11}^{(0)} \quad \phi_{,1}^{(1)} \right]^t, \\ &\mathbf{K} = \left\langle (\mathrm{L}_{23}\mathbf{N}_u)^t \mathbf{C} (\mathrm{L}_{23}\mathbf{N}_u) \right\rangle, \\ &\mathbf{F}_{23E} = \left\langle (\mathrm{L}_{23}\mathbf{N}_u)^t \mathbf{C} (\mathrm{L}_1 \Phi) \right\rangle, \\ &\mathbf{F}_{23\beta} = \left\langle (\mathrm{L}_{23}\mathbf{N}_u)^t \boldsymbol{\beta} \right\rangle, \ \langle \square \rangle = \int_{S_c} \square \, \mathrm{d}S \end{aligned} \tag{20}$$

wehre $\mathbf{N}_u$ is the shape function matrix.

Equation (19) can be solved under the orthogonal constraint of the warping displacement to the rigid body displacement. This orthogonality is realized by introducing the Lagrange multiplier in such a way that

$$\int_{S_c} (\delta \mathbf{u}_w^{(2)})^t \cdot \mathbf{u}_R \mathrm{d}S = 0 \tag{21}$$

which yields

$$\begin{bmatrix} \mathbf{K} & \mathbf{H} \\ \mathbf{H}^t & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}_w^{(2)} \\ \Lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{23\beta}\theta^{(1)} - \mathbf{F}_{23E}\mathbf{e}^{(1)} \\ \mathbf{0} \end{Bmatrix} \tag{22}$$

where $\mathbf{H}$ and $\Lambda$ denote the constraint matrix and associated Lagrange multiplier vector[9].

Equation (22) is now solvable, and the warping solution is consequently expressed as follows:

$$\bar{\mathbf{u}}_w^{(2)} = \Gamma_e^{(1)}\mathbf{e}^{(1)} + J_\theta^{(1)}\theta^{(1)} \tag{23}$$

where $\Gamma_e^{(1)}$ represents the cross-sectional deformation due to the macroscopic mechanical strain, and $J_\Theta^{(1)}$ represents the one due to the prescribed temperature rise over the cross-section.

The one-dimensional macroscopic equation is derived by the second term in RHS of Eq. (17), which is associated with the virtual fundamental displacement. This is summarized as:

$$\delta \hat{W}^{(2)} = \int_{y_1} (\delta \mathbf{e}^{(1)})^t \tilde{\mathrm{N}}^{(1)} \mathrm{d}y_1 - \mathrm{F}^{(1)} = 0 \tag{24}$$

where

$$\begin{aligned} \tilde{\mathrm{N}}^{(1)} &= \int_{S_C} (\mathrm{L}_1 \Phi)^t \sigma^{(1)} \mathrm{d}S \\ \mathrm{F}^{(1)} &= \int_{S_T} \left( \bar{\mathbf{t}}^{(1)t} \delta \mathbf{u}^{(1)} + \bar{\mathbf{t}}^{(2)t} \delta \mathbf{u}^{(0)} \right) \mathrm{d}S \end{aligned} \tag{25}$$

The stress resultant vector in Eq. (25) can be explicitly expressed by

$$\tilde{\mathrm{N}}^{(1)} = \mathrm{A}^{(1)}\mathbf{e}^{(1)} + \mathrm{B}^{(1)}\theta^{(1)} \tag{26}$$

where

$$\begin{aligned} \mathrm{A}^{(1)} &= \left\langle (\mathrm{L}_1 \Phi)^t \mathbf{C} (\mathrm{L}_1 \Phi) \right\rangle + \left\langle (\mathrm{L}_1 \Phi)^t \mathbf{C} (\mathrm{L}_{23}\mathbf{N}_u) \right\rangle \Gamma_e^{(1)} \\ \mathrm{B}^{(1)} &= -\left\langle (\mathrm{L}_1 \Phi)^t \boldsymbol{\beta} \right\rangle + \left\langle (\mathrm{L}_1 \Phi)^t \mathbf{C} (\mathrm{L}_{23}\mathbf{N}_u) \right\rangle J_\theta^{(1)} \end{aligned} \tag{27}$$

The first-order classical strain measure $\mathbf{e}^{(1)}$ is discretized by using a standard one-dimensional finite element. The discretized beam equation is then given as follows:

$$\mathbf{K}_b^{(1)}\mathbf{V}^{(1)} = -\mathbf{P}_\theta^{(1)} + \mathbf{P}_F^{(1)} \tag{28}$$

where

$$\begin{aligned} &\mathbf{K}_b^{(1)} = \int_{y_1} \left( \mathrm{L}_b^{(1)}\mathbf{N}_b \right)^t \mathrm{A}^{(1)} \left( \mathrm{L}_b^{(1)}\mathbf{N}_b \right) \mathrm{d}y_1 \\ &\mathbf{P}_\theta^{(1)} = \int_{y_1} \left( \mathrm{L}_b^{(1)}\mathbf{N}_b \right)^t \mathrm{B}^{(1)}\theta^{(1)} \mathrm{d}y_1, \ \mathbf{P}_F^{(1)} = (\mathbf{N}_b)^t \mathrm{F}^{(1)} \end{aligned} \tag{29}$$

in which $\mathbf{N}_b$ is the shape function matrix for the one-dimensional beam equation.

The displacement boundary condition at the clamped end can be interpreted as a constraint equation. In the present study, the generalized boundary condition proposed by Kim et al.[5,9] is adopted, which kinematically modifies the boundary condition for $\mathbf{V}^{(1)}$.

The non-classical solutions can be found in the higher order virtual works. These calculations are lengthy but straightforward, and therefore they are omitted here for a brevity.

## 4. Results and Discussion

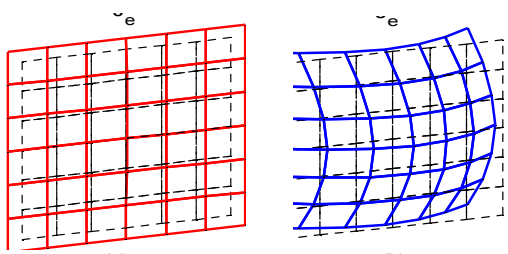
In this study, a simple two-layer angle-ply [45/−45] composite beam is considered as a test-bed example in order to demonstrate the capability of the proposed approach. The width of the beam is 0.08m and the total

thickness is 0.08m. The length of the beam is 0.8m. The beam is subjected to a 50°C uniform temperature rise. The ply material properties(see Table 1) are given by Wang and Yu[10].

The first-order cross-sectional mechanical deformation modes of an isotropic beam are investigated first. The Poisson's effects can be seen for extension and two bending modes, and the Saint-Venant's warping for a torsion mode. The thermoelastic mode is also shown in Fig. 2(a), $J_e^{(1)}$, where a simple thermal expansion of the cross-section is observed as expected. The second-order cross-sectional deformation modes of an isotropic beam are also investigated.

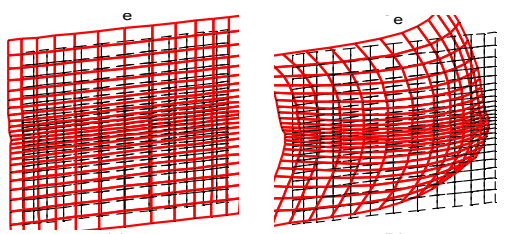
**Table 1 The ply material properties**

| Properties | Quantities / Units |
|---|---|
| $E_1$ | 133.4MPa |
| $E_2=E_3$ | 8.29MPa |
| $G_{12}=G_{23}=G_{12}$ | 3.81MPa |
| $\nu_{12}=\nu_{13}=\nu_{23}$ | 0.26 |
| $\alpha_{11}$ | $2.0\times10^{-6}$ /℃ |
| $\alpha_{22}=\alpha_{33}$ | $27.34\times10^{-6}$ /℃ |



(a) (b)

**Fig. 2 The cross-sectional thermal deformation mode of an isotropic beam: (a) the first-order mode, (b) the second-order mode**



(a) (b)

**Fig. 3 The cross-sectional thermal deformation mode of a [45/-45] composite beam: (a) the first-order mode, (b) the second-order mode**

The non-classical cross-sectional deformations appear at the second order warping solution, $\Gamma_e^{(2)}$. The shear deformation effects can be seen in two bending modes and the cross-sectional distorsion is seen in a torsion mode. The second-order thermoelastic mode is identified as the thermal higher-order extension mode, as shown in Fig. 2(b). These results enable us to determine the dominant deformation modes in the thermo-mechanical beam problem.

The first-order mechanical deformation modes and the first and second thermoelastic deformation modes of a [45/−45] composite beam are calculated to investigate the composite coupling effects. Because of the anisotropic behavior of the angle-ply composite beam(e.g., the extension-shear coupling in this case), the mechanical deformation modes are more complicated than the isotropic beam. Therefore there are the out-of plane deformations even for the first-order extension and bending modes. The thermal deformation modes are shown in Fig. 3, where the complicated thermal deformation modes are seen.

## 5. Conclusion

A multiphysics beam analysis using an asymptotic expansion method is presented for a thermo-elasto- static problem. The mechanical displacement and temperature rise are asymptotically expanded, so that the recursive virtual work is obtained at each level. This virtual work renders the warping solution and the macroscopic beam formulation. A finite element discretization is employed to handle composite beams with arbitrary cross-sections and material distributions. When the temperature rise is prescribed over the beam cross-section, the thermo-elasto-static warping solutions are obtained. Unlike the isotropic beam, it is found that the thermal warping mode shows the complicated distribution over the beam cross-section. The proposed approach can be extended further to deal with fully coupled thermo-electro-mechanical problems for the optimization of cross-sectional geometry and material distributions.

## Acknowledgement

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