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ON CHARACTERIZATIONS OF THE PARETO DISTRIBUTION BY THE INDEPENDENT PROPERTY OF UPPER RECORD VALUES

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Abstract. We present characterizations of the Pareto distribution by the independent property of upper record values in such a way that $F(x)$ has a Pareto distribution if and only if $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$ are independent for $1 \leq m < n$. Futhermore, the characterizations should find that $F(x)$ has a Pareto distribution if and only if $\frac{X_{U(n)}}{X_{U(n)} \pm X_{U(m)}}$ and $X_{U(m)}$ are independent for $1 \leq m < n$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = max(min)$ $\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of this sequence, if $Y_j > (<) Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

We call the random variable $X \in PAR(a, \alpha)$ if the corresponding probability cdf $F(x)$ of X is of the form

$$F(x) = \begin{cases} 1 - \left(\frac{a}{x}\right)^{\alpha}, \ x \geq a, \ \alpha > 0 \\ 0, \ otherwise. \end{cases} \tag{1.1}$$

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Lee and Chang(2008) obtained characterization that $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n+1)}}{X_{U(n+1)}+X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$. Also, Lee and Lim(2010) generalized that $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $1 \leq m < n$.

In this article, by the similar way of above papers, we obtain characterizations of the Pareto distribution by the independent property of upper record values.

2. Main results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for all $x > 1$. Then $F(x) = 1 - x^{-\alpha}$ for all $x > 1$, $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$ are independent for $1 \leq m < n$.*

Proof. The joint pdf $f_{m,n}(x,y)$ of $X_{U(m)}$ and $X_{U(n)}$ is found to be

$$f_{m,n}(x,y) = \frac{R(x)^{m-1}}{\Gamma(m)} r(x) \frac{\{R(y) - R(x)\}^{n-m-1}}{\Gamma(n-m)} f(y) \tag{2.1}$$

where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x))$

Consider the functions $U = X_{U(m)}$ and $W = \frac{X_{U(n)}}{X_{U(m)}}$. It follows that $x_{U(m)} = u$, $x_{U(m)} = uw$ and $\mid J \mid = u$. Thus we can write the joint pdf $f_{U,W}(u,w)$ of U and W as

$$f_{U,W}(u,w) = \frac{R(vw)^{m-1}}{(m-1)!} r(vw) \frac{\{R(w) - R(vw)\}^{n-m-1}}{(n-m-1)!} w f(w) \tag{2.2}$$

for $u > 1$, $w > 1$.

If $F(x) = 1 - x^{-\alpha}$ for all $x > 1$, $\alpha > 0$, then we get

$$\begin{aligned} &f_{U,W}(u,w) \\ &= \frac{\alpha^n}{\Gamma(m)\Gamma(n-m)} u^{-\alpha-1}\{\ln u\}^{m-1} w^{-\alpha-1}\{\ln w\}^{n-m-1} \end{aligned} \tag{2.3}$$

for all $u > 1$, $w > 1$ and $\alpha > 0$.

The marginal pdf of W is given by

$$
\begin{aligned}
f_W(w) &= \int_1^{\infty} f_{U,W}(u,w)du \\
&= \frac{\alpha^{n-m}}{\Gamma(n-m)} w^{-\alpha-1}\{\ln w\}^{n-m-1},
\end{aligned} \tag{2.4}
$$

for all $w > 1$, $\alpha > 0$.

Also, the pdf $f_U(u)$ of U is given by

$$
f_U(u) = \frac{R(u)^{m-1}}{\Gamma(m)} f(u) = \frac{\alpha^m}{\Gamma(m)} u^{-\alpha-1}\{\ln u\}^{m-1}. \tag{2.5}
$$

From (2.3), (2.4) and (2.5), we obtain $f_{U,W}(u,w) = f_U(u)f_W(w)$.

Hence U and W are independent for $1 \le m < n$.

Now we will prove the sufficient condition.

Let us use the transformation $U = X_{U(m)}$ and $W = \frac{X_{U(n)}}{X_{U(m)}}$. The Jacobian of the transformation is $\mid J \mid = u$. Thus we can write the joint pdf $f_{U,W}(u,w)$ of U and W as

$$
f_{U,W}(u,w) = \frac{R(u)^{m-1}}{\Gamma(m)} r(u) \frac{\{R(uw) - R(u)\}^{n-m-1}}{\Gamma(n-m)} f(uw)u \tag{2.6}
$$

for all $u > 1$, $w > 1$ and $\alpha > 0$.

The pdf $f_U(u)$ of U is given by

$$
f_U(u) = \frac{R(u)^{m-1}}{\Gamma(m)} f(u) \tag{2.7}
$$

for all $u > 1$, $m \ge 1$.

Since U and W are independent, we get the pdf $f_W(w)$ of W from (2.6), (2.7) as

$$
\begin{aligned}
f_W(w) &= \frac{1}{\Gamma(n-m)} (R(uw) - R(u))^{n-m-1} \frac{f(uw)u}{\bar{F}(u)} \\
&= \frac{1}{\Gamma(n-m)} \left(-\ln \frac{\bar{F}(uw)}{\bar{F}(u)} \right)^{n-m-1} \frac{\bar{F}(uw)}{\bar{F}(u)} \left\{ \frac{\partial}{\partial w} \left(-\ln \frac{\bar{F}(uw)}{\bar{F}(u)} \right) \right\}
\end{aligned}
$$

where $\bar{F}(x) = 1 - F(x)$.

By the Lemma of Ahsanullah[see Ahsanullah(1995), p. 48], the pdf $f_W(w)$ of W is a function of w only. Thus we have

$$
\bar{F}(uw) = \bar{F}(u)G(w) \tag{2.8}
$$

where $G(w)$ is a function of w only. By functional equations[see Aczel (1966)], the only continuous solution of (2.8) with the boundary conditions $\bar{F}(1) = 1$ and $\bar{F}(\infty) = 0$ is

$$\bar{F}(x) = x^{-\alpha}$$

for all $x > 1$ and $\alpha > 0$. Thus we have $F(x) = 1 - x^{-\alpha}$.

This completes the proof. □

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for all $x > 1$. Then $F(x) = 1 - x^{-\alpha}$ for all $x > 1$, $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n)}+X_{U(m)}}$ and $X_{U(m)}$ are independent for $1 \leq m < n$.*

Proof. The necessary condition is easy to establish. Now we prove the sufficient condition.

Let us use the transformation $U = X_{U(m)}$ and $V = \frac{X_{U(n)}}{X_{U(n)}+X_{U(m)}}$. The Jacobian of the transformation is $\mid J \mid = \frac{u}{(1-v)^2}$. Thus we can write the joint pdf $f_{U,V}(u, v)$ of U and V as

$$\begin{aligned} &f_{U,V}(u, v) \\ &= \frac{R(u)^{m-1}}{\Gamma(m)} r(u) \frac{\{R(\frac{uv}{1-v}) - R(u)\}^{n-m-1}}{\Gamma(n-m)} f(\frac{uv}{1-v}) \frac{u}{(1-v)^2} \end{aligned} \tag{2.9}$$

for all $u > 1$, $\frac{1}{2} < v < 1$ and $\alpha > 0$.

The pdf $f_U(u)$ of U is given by

$$f_U(u) = \frac{R(u)^{m-1}}{\Gamma(m)} f(u) \tag{2.10}$$

for $u > 1$.

We get the pdf $f_V(v)$ of V from (2.9), (2.10) as

$$\begin{aligned} f_V(v) &= \frac{1}{\Gamma(n-m)} (R(\frac{uv}{1-v}) - R(u))^{n-m-1} \frac{f(\frac{uv}{1-v}) \frac{u}{(1-v)^2}}{\bar{F}(u)} \\ &= \frac{1}{\Gamma(n-m)} \Big(- \ln \frac{\bar{F}(\frac{uv}{1-v})}{\bar{F}(u)} \Big)^{n-m-1} \frac{\bar{F}(\frac{uv}{1-v})}{\bar{F}(u)} \Big\{ \frac{\partial}{\partial w} \Big(- \ln \frac{\bar{F}(\frac{uv}{1-v})}{\bar{F}(u)} \Big) \Big\}. \end{aligned}$$

By the independent property of U and V, the pdf $f_V(v)$ of V is a function of v only [see Ahsanullah(1995), p. 48] . Thus we must have

$$\bar{F}(\frac{uv}{1-v}) = \bar{F}(u) G(\frac{v}{1-v}) \tag{2.11}$$

where $G(\frac{v}{1-v})$ is a function of v only. By functional equations[see Aczel (1966)], the only continuous solution of (2.11) with the boundary conditions $\bar{F}(1) = 1$ and $\bar{F}(\infty) = 0$ is

$$\bar{F}(x) = x^{-\alpha}$$

for all $x > 1$ and $\alpha > 0$. Thus we have $F(x) = 1 - x^{-\alpha}$.

This completes the proof. □

THEOREM 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $\bar{F}(1) = 1$ and $F(x) < 1$ for all $x > 1$. Then $F(x) = 1 - x^{-\alpha}$ for all $x > 1$, $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n)} - X_{U(m)}}$ and $X_{U(m)}$ are independent for $1 \leq m < n$.*

Proof. The proof can be done in exactly the same way as that of Theorem 2.2. □

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