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THE BAUM-KATZ LAW OF LARGE NUMBERS FOR NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper the Baum-Katz law of large numbers for negatively orthant dependent random variables is studied. The complete convergence of negatively orthant dependent random variables under some conditions of uniform integrablity is also obtained.

1. Introduction

The history and literature on Baum-Katz law of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concept of negative dependence is much more limited but still very interesting. Lehmann(1966) provided an extensive introductory overview of various concepts of positive and negative dependence in the bivariate case. Negative dependence has been particularly useful in obtaining complete convergence. This section contains some background material on negative dependence which will be used in obtaining the major complete convergence in the next section.

Two random variables X and Y are said to be negatively quadrant dependent(NQD) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \text{ for all } x, y \in \mathbb{R}. \tag{1.1}$$

A collection of random variables is said to be pairwise NQD if every pair of random variables in the collection satisfies (1.1). It is important to note that (1.1) implies

$$P(X > x, Y > y) \le P(X > x)P(Y > y) \text{ for all } x, y \in \mathbb{R}. \tag{1.2}$$

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Moreover, it follows that (1.2) implies (1.1) and hence, they are equivalent for pairwise NQD. Ebrahimi and Ghosh(1981) showed that (1.1) and (1.2) are not equivalent for $n \geq 3$. Consequently, the following definition is needed to define sequences of negatively dependent random variables:

DEFINITION 1.1. (Joag-Dev and Proschan(1983)) The random variables $X_1, X_2, \cdots, X_n$ are said to be lower negatively orthant dependent(LNOD) if for each n

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \tag{1.3}$$

for all $x_1, \cdots x_n \in \mathbb{R}$, they are said to be upper negatively orthant dependent(UNOD) if for each n

$$P(X_1 > x_1, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \tag{1.4}$$

for all $x_1, \cdots x_n \in \mathbb{R}$.

Random variables $X_1, X_2, \cdots, X_n$ are said to be negatively orthant dependent(NOD) if they are both LNOD and UNOD. Obviously sequences of NOD random variables are a family of very wide scope, which contain sequences of independent random variables. Joag-Dev and Proschan(1983) once pointed out that negative association(NA) implies NOD, but neither LNOD nor UNOD implies NA. Here we listed the studies of convergence for NOD random variables established by many authors. For examples, Taylor, Patterson and Bozorgnia(2001) for weak laws of large numbers, Taylor, Patterson and Bozorgnia(2002) for strong law of large numbers, Amini and Bozorgnia(2003) for complete convergence, Ko and Kim(2005) for almost convergence of weighted sums, Ko, Han and Kim(2006) for strong law of large numbers of weighted sums, Volodin, Cabrera and Hu(2006) for convergence rate of the dependent bootstrapped means and Wang, Hu, Yang and Ling(2010) for exponential inequalities and inverse moment.

2. Preliminaries

Recently, Liu(2009) extended the NOD structure and introduced the concept of extended negative orthant dependent random variables.

The following results are listed for reference in obtaining the main results in Section 3. Detailed proofs can be founded in the previously cited literatures.

Lemma 2.1. *If $X_1, X_2, \cdots, X_n$ are pairwise NQD random variables, then*

$$Cov(X_i, X_j) \leq 0 \ i \neq j. \tag{2.1}$$

Lemma 2.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of LNOD(UNOD) random variables and $\{f_n,\ n \geq 1\}$ a sequence of monotone increasing functions, then $\{f_n(X_n),\ n \geq 1\}$ is still a sequence of LNOD(UNOD) random variables.*

Corollary 2.3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of NOD random variables and $\{f_n,\ n \geq 1\}$ a sequence of monotone increasing functions, then $\{f_n(X_n),\ n \geq 1\}$ is still a sequence of NOD random variables.*

Lemma 2.4. *Let $X_1, X_2, \cdots, X_n$ be nonnegative NOD random variables. Then*

$$E(\prod_{i=1}^{n} X_i) \leq \prod_{i=1}^{n} E(X_i). \tag{2.2}$$

3. Main results

The following lemma is a Fuk and Nagaev type inequality for NOD random variables.

Lemma 3.1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $0 < B_n = \sum_{i=1}^{n} EX_i^2$. Then, we have*

$$P(|S_n| \geq x) \leq \sum_{i=1}^{n} P(|X_i| \geq x) + 2\exp(\frac{x}{y} - \frac{x}{y}\log(1 + \frac{xy}{B_n})). \tag{3.1}$$

for all positive x and y.

Proof. The proof is similar to that of Theorem 2 in Fuk and Nagaev(1971). For completeness we repeat it here. Let $y > 0$, $Y_i = \min(X_i, y)$ and $T_n = \sum_{i=1}^{n} Y_i$. Then

$$EY_n \leq 0 \text{ and } E(Y_n^2) \leq E(X_n^2). \tag{3.2}$$

By Corollary 2.3 $\{e^{hY_i}, 1 \leq i \leq n\}$ is also a sequence of nonnegative NOD random variables for all $h > 0$. Thus, by Lemma 2.4

$$E(e^{hT_n}) = E(\prod_{i=1}^{n} e^{hY_i}) \leq \prod_{i=1}^{n} E(e^{hY_i}). \tag{3.3}$$

Denote $F_i(x) = P(X_i \leq x)$. Then

$$\begin{aligned}
E(e^{hY_i}) &= \int_{-\infty}^{\infty} e^{hy_i} dF(x) = \int_{-\infty}^{y} e^{hx} dF_i(x) + e^{hy} \int_{y}^{\infty} dF_i(x) \\
&= \int_{-\infty}^{y} e^{hx} dF_i(x) + e^{hy} P(X_i > y) \\
&= 1 + hEY_i + \int_{-\infty}^{y} (e^{hx} - 1 - hx) dF_i(x) \\
&\qquad + (e^{hy} - 1 - hy) P(X_i > y) \\
&\leq 1 + \int_{-\infty}^{y} (e^{hx} - 1 - hx) dF_i(x) + (e^{hy} - 1 - hy) P(X_i > y).
\end{aligned}$$

From the facts those for fixed $h > 0$, the function $f(x) = (e^{hx} - 1 - x)/x^2$ is increasing and $1 + u \leq e^u$ for all $u \in \mathbb{R}$ we have

$$\begin{aligned}
E(e^{hY_i}) &\leq 1 + \frac{e^{hy} - 1 - hy}{y^2} (\int_{-\infty}^{y} x^2 dF_i(x) + y^2 P(X_i > y)) \\
&\leq 1 + \frac{e^{hy} - hy}{y^2} EX_i^2 \\
&\leq \exp(\frac{e^{hy} - 1 - hy}{y^2} EX_i^2).
\end{aligned} \tag{3.4}$$

Hence, it follows from (3.3) and (3.4) that, for all $x > 0$ and all $h > 0$

$$e^{-hx} E e^{hT_n} \leq \exp(-hx + B_n \frac{e^{hy} - 1 - hy}{y^2}).$$

Putting $h = \log(1 + \frac{xy}{B_n})/y$, we have

$$\begin{aligned}
e^{-hx} E(e^{hT_n}) &\leq \exp(\frac{x}{y} - \frac{x}{y} \log(1 + \frac{xy}{B_n}) - \frac{B_n}{y^2} \log(1 + \frac{xy}{B_n})) \\
&\leq \exp(\frac{x}{y} - \frac{x}{y} \log(1 + \frac{xy}{B_n})).
\end{aligned} \tag{3.5}$$

Therefore

$$
\begin{aligned}
P(S_n \geq x) &\leq P(S_n \neq T_n) + P(T_n \geq x) \\
&\leq \sum_{k=1}^{n} P(X_k \geq y) + e^{-hx} E(e^{hT_n}) \qquad (3.6)\\
&\leq \sum_{k=1}^{n} P(X_k \geq y) + \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n}\right)\right),
\end{aligned}
$$

where $S_n = \sum_{i=1}^{n} X_i$. Similarly, since $\{-X_n, n \geq 1\}$ is a sequence of NOD random variables by Corollary 2.3 we also obtain

$$
P(-S_n \geq x) \leq \sum_{k=1}^{n} P(-X_k \geq y) + \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n}\right)\right). \qquad (3.7)
$$

Finally, from (3.6) and (3.7) the result (3.1) follows. □

THEOREM 3.2. *Let $1 \leq p < 2$ and $\{X_n,\ n \geq 1\}$ be a sequence of centered NOD random variables. If*

$$
\lim_{x\to\infty} \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} x^{1+\gamma} P(|X_k|^p \geq x) = 0, \qquad (3.8)
$$

for $\gamma > 2/p - 1$, $\alpha p > 1$, then, for all $\epsilon > 0$

$$
\sum_{n=1}^{\infty} n^{\alpha p - 2} P(|S_n| \geq \epsilon n^{\alpha}) < \infty. \qquad (3.9)
$$

Proof. Let

$$
\begin{aligned}
Y_k &= -xI[X_k \leq -x] + X_k I[|X_k| < x] + xI[X_k \geq x], \\
Z_k &= X_k - Y_k, \\
U_n &= \sum_{k=1}^{n} Y_k \text{ and } V_n = \sum_{k=1}^{n} Z_k.
\end{aligned}
$$

Taking $x = n^{\alpha(2-p)/4}$ we have, for any $\epsilon > 0$

(3.10)

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p - 2} P(|S_n| \geq \epsilon n^{\alpha}) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p - 2} P(|U_n - EU_n| \geq \frac{\epsilon}{2} n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p - 2} P(|V_n - EV_n| \geq \frac{\epsilon}{2} n^{\alpha}) \\
&=: I_1 + I_2.
\end{aligned}
$$

To prove (3.9) we need to prove $I_1 < \infty$ and $I_2 < \infty$. Note that Y_k and Z_k are NOD. Let $D_n = \sum_{k=1}^{n}(Y_k - EY_k)^2$, $x = y = \frac{\epsilon}{2}n^{\alpha}$. By Lemma 3.1 and Markov inequality we obtain

$$\begin{aligned} I_1 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2}\Big(\sum_{k=1}^{n} P(|Y_k - EY_k| \geq \frac{\epsilon}{2}n^{\alpha}) + C\frac{D_n}{D_n + n^{2\alpha}\epsilon^2/4}\Big) \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2}(n^{-2\alpha}D_n) \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{k=1}^{n} E(Y_k)^2 \qquad (3.11)\\ &\leq C\sum_{n=1}^{\infty} n^{-1-\alpha(2-p)/2} < \infty \end{aligned}$$

since $|Y_k| \leq n^{\alpha(2-p)/4}$.

By Lemma 3.1, Markov inequality and the fact that $|Z_k| \leq |X_k|I[|X_k| \geq x]$,

$$\begin{aligned} I_2 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{k=1}^{n} E|X_k|^2 I[|X_k| \geq x] \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{k=1}^{n}\int_0^{\infty} P(|X_k|^2 I[|X_k| \geq x] > t)dt \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{k=1}^{n}\Big\{\int_0^{x^2} P(|X_k|^2 I[|X_k| \geq x] > t)dt \qquad (3.12)\\ &\quad + \int_{x^2}^{\infty} P(|X_k|^2 I[|X_k| \geq x] > t)dt\Big\} \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{k=1}^{n}\Big\{\int_0^{x^2} P(|X_k| \geq x)dt + \int_{x^2}^{\infty} P(|X_k|^2 > t)dt\Big\} \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\Big\{\sum_{k=1}^{n} x^2 P(|X_k| \geq x) + \sum_{k=1}^{n}\int_{x^2}^{\infty} P(|X_k|^2 > t)dt\Big\} \\ &=: I_{21} + I_{22}. \end{aligned}$$

From assumption (3.8) there exists a positive constant $M < x$ such that

$$\sup_{n\geq 1} n^{-1}\sum_{k=1}^{n} P(|X_k|^p \geq x) \leq x^{-(1+\gamma)}.$$

From $x = n^{\alpha(2-p)/4}$, there exists $m \leq n$ such that $x > M$. Hence we have

$$\begin{aligned}
I_{21} &\leq C\{1 + \sum_{n=m}^{\infty} n^{\alpha p-2-2\alpha} \sum_{k=1}^{m} x^2 P(|X_k| \geq x)\} \\
&\leq C \sum_{n=m}^{\infty} n^{\alpha p-1-2\alpha} u^{2/p} \sum_{k=1}^{n} n^{-1} P(|X_k|^p \geq u) \\
&\leq C \sum_{n=m}^{\infty} n^{-1-\alpha(2-p)} u^{2/p-1-\gamma} \qquad (3.13) \\
&\quad \text{by taking } u = x^p \\
&= C \sum_{n=m}^{\infty} n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\gamma-2/p)/4} \\
&\leq C \sum_{n=1}^{\infty} n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\gamma-2/p)/4} \\
&< \infty \\
&\quad \text{by assumption } 1+\gamma-2/p > 0.
\end{aligned}$$

$$\begin{aligned}
I_{22} &\leq C\{1 + \sum_{n=m}^{\infty} n^{\alpha p-1-2\alpha} \int_{x^2}^{\infty} n^{-1} \sum_{k=1}^{n} P(|X_k|^2 \geq t)dt\} \\
&\leq C \sum_{n=m}^{\infty} n^{-1-\alpha(2-p)} \int_{x^p}^{\infty} y^{2/p-1} n^{-1} \sum_{k=1}^{n} P(|X_k|^p \geq y)dy \\
&\quad \text{by } t = y^{2/p} \\
&\leq C \sum_{n=m}^{\infty} n^{-1-\alpha(2-p)} \int_{x^p}^{\infty} y^{2/p-\gamma-2} dy \qquad (3.14) \\
&\leq C \sum_{n=m}^{\infty} n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\gamma-2/p)/4} \\
&\leq C \sum_{n=1}^{\infty} n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\gamma-2/p)/4} \\
&< \infty \\
&\quad \text{since } 1+\gamma-2/p > 0.
\end{aligned}$$

From (3.13) and (3.14) $I_2 < \infty$ and then the proof is complete. $\square$

Definition 3.3. (Landers and Rogge(1997)) A sequence $\{X_n, n \geq 1\}$ of random variables is said to be Cesàro uniformly integrable if

$$\lim_{x\to\infty} \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} E|X|^p I[|X_k| \geq x] = 0. \tag{3.15}$$

Now we prove the complete convergence for an NOD random variables under the condition of Cesàro uniformly integrability.

Theorem 3.4. *Let $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ be a sequence of centered NOD random variables which are Cesàro uniformly integrable. Then, for $\alpha p \geq 1$ and all $\epsilon > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P(|S_n| \geq \epsilon n^{\alpha}) < \infty. \tag{3.16}$$

Proof. From the fact that for a nonnegative random variable Y, $EY = \int_0^{\infty} P(Y > t)dt$ we obtain

(3.17)

$$\begin{aligned} E|X_k|^p I[|X_k| \geq x] &= \int_0^{\infty} P(|X_k|^p I[|X_k| \geq x] > t)dt \\ &= \int_0^{x^p} P(|X_k|^p I[|X_k| \geq x] > t)dt + \int_{x^p}^{\infty} P(|X_k|^p I[|X_k| \geq x] > t)dt \\ &\leq x^p P(|X_k| \geq x) + \int_{x^p}^{\infty} P(|X_k|^p > t)dt. \end{aligned}$$

From (3.15) and (3.17) we have

$$\lim_{x\to\infty} \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} x^p P(|X_k| \geq x) = 0, \tag{3.18}$$

which yields

$$\lim_{u\to\infty} \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} u P(|X_k|^p \geq u) = 0, \tag{3.19}$$

by taking $u = x^p$. Hence by the similar method as in the proof of Theorem 3.2 we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P(|S_n| \geq \epsilon n^{\alpha}) < \infty$$

for all $\epsilon > 0$ and thus the proof is complete. □

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