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ON ROBUSTLY POSITIVELY EXPANSIVE MAPS

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Abstract. In this paper, we show that every C^1-robustly positively expansive map is expanding.

1. Introduction

Let M be a compact smooth Riemannian manifold, and let $f : M \to M$ be a $C^r(r \geq 1)$ map. Let $d : M \times M \to \mathbb{R}$ be a metric on M and $e > 0$. We say that f is *e-positively expansive* if any $x \neq y$, there exists $n \geq 0$, such that $d(f^n(x, f^n(y)) \geq e$. In positively expansivity, an important class is the expanding ones defined as follows: we say that a map is *expanding* if there are constants $C > 0$ and $\gamma > 1$ such that

$$\|D_x f^n(v)\| \geq C\gamma^n \|v\|$$

for any $x \in M$. Recall that a C^1 map f on M is said to be *topologically transitive* if there is dense orbit; that is, $M = \overline{\{f^n(x) : n \geq 0\}}$ for some $x \in M$.

Since M is connected, it can be checked that the set of periodic points, $P(f)$, of f is dense(see, [7]).

Very recently, in [1] Arbieto proved that any C^1-persistently positively expansive map is expanding. Here, we introduce the definition of the C^1-persistently positively expansive of a C^1 map $f : M \to M$ if there exists a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, there exists $e(g) > 0$ such that g is $e(g)$-positively expansive, where $e(g)$ is the expansive constant for g. In this paper, we don't change to select the expansive constant in the initial condition. That is, the expansive

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constant doesn't change about the functions of the C^1-nearby f. It is more stronger condition than Arbieto's definition.

DEFINITION 1.1. *We say that a C^1 map $f : M \to M$ is C^1-robustly positively expansive map if there exist $e > 0$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g is e-positively expansive.*

In this paper, we shall prove the following result:

THEOREM 1.2. *Let M be a compact manifold. Any C^1-robustly positively expansive map $f : M \to M$ is expanding.*

2. Proof of Theorem 2.5

The following lemma is in [1, 4].

LEMMA 2.1. *Let f be a C^1-map and $\mathcal{U}(f)$ a C^1-neighborhood of f. Then there exists a neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f and a $\epsilon > 0$ such that, if for any $g \in \mathcal{U}_0(f)$, $S = \{p_1, \ldots, p_m\} \subset M$ is a finite set, and linear maps $L_i : T_{p_i}M \to T_{g(p_i)}M$ satisfying $\|L_i - D_{p_i}g\| \le \epsilon$ for $i = 1, \ldots, m$, there exists $g_1 \in \mathcal{U}_0(f)$ such that $g_(x) = g(x)$ if $x \in \{p_1, \ldots, p_m\}$ and $D_{p_i}g = L_i$. Moreover, if U is a neighborhood of S, then choose $\alpha > 0$ and $g_1 \in \mathcal{U}_0(f)$ such that*

(i) $g_1(x) = g(x)$, *for every* $x \in S \cup (M \setminus S)$,
(ii) $g_1(x) = \exp_{g(p_i)} \circ D_{p_i}g \circ \exp_{p_i}^{-1}(x)$, *for* $x \in B_\alpha(p_i)$,

where exp *is the exponential map of the Riemannian manifold M.*

Let f be a C^1 map on M and $e > 0$. By Lemma 2.1, if f is C^1-robustly expansive then there is a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}_0(f)$, g is e- positively expansive.

LEMMA 2.2. *Let $\mathcal{U}_0(f)$ be the C^1-neighborhood of f in the above. Suppose that f is C^1-robustly positively expansive. For any $g \in \mathcal{U}_0(f)$, and p as a periodic point for g with period $\pi(p)$ if λ is an eigenvalue of $Dg^{\pi(p)}(p)$ then $|\lambda| > 1$.*

Proof. Let M be a C^1-robustly positively expansive for a C^1-map f. Then there exist a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f and an expansive constant $e > 0$ such that for any $g \in \mathcal{U}_0(f)$, g is e-positive expansive. Let λ be an eigenvalue of $Dg^{\pi(p)}(p)$. We will derive a contradiction. Suppose that $|\lambda| \le 1$. To simplify, assume that $g^{\pi(p)}(p) = g(p) = p$.

Let $\epsilon > 0$ be as in Lemma 2.1. Take a linear map $L : T_pM \to T_pM$ such that $\|L - D_pg\| < \epsilon$. Then one can choose $0 < \alpha < e/4$ such that

$$g_1(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1}(x) & \text{if} \quad x \in B_\alpha(p), \\ g(x) & \text{if} \quad x \notin B_{4\alpha}(p), \end{cases}$$

where $B_\alpha(p) = \{x \in M : d(x,p) \leq \alpha\}$. Then $g_1(p) = g(p) = p$.

If $\lambda < 1$ then $T_pM = E_p^s$. Thus there exists $0 < \gamma < 1$ such that

$$\|D_pg^n(v)\| \leq \gamma^n \|v\|$$

for $v \in T_pM$. One can choose $0 < \epsilon_1 < \alpha/4$ such that $\exp_p(E_p^s(\epsilon_1)) \subset B_\alpha(p)$. Then for some $\gamma_1 > 1$,

$$g_1^n|_{\exp_p(E_p^s(\epsilon_1))} < g_1^{n-1}|_{\exp_p(E_p^s(\epsilon_1))} < \cdots < d(x,y) < \epsilon_1,$$

for $n \in \mathbb{N}$. Thus we can take distinct two points $x, y \in \exp_p(E_p^s(\epsilon_1)) \subset M$ such that

$$d(g_1^n(x), g_1^n(y)) < \epsilon_1 < \alpha < e,$$

for all $n \in \mathbb{N}$. This is a contradiction.

If $\lambda = 1$, then for some $k \in \mathbb{N}$, $g_1^k(x) = id(x) = x$ for $x \in B_\alpha(p)$, where id is the identity map. In this case, g_1 is not e-positively expansive.

Finally, if $\lambda \leq 1$ then $T_pM = E_p^c \oplus E_p^s$, where E_p^c associated to $\lambda = 1$ and E_p^s associated to $\lambda < 1$. Note that let $A \subset M$. If M is positively expansive of f, then A have to positive expansive of f. One can choose $0 < \epsilon_1 < \alpha/4$ such that $\exp_p(E_p^s(\epsilon_1)) \subset B_\alpha(p)$. Then choose $k > 0$ such that $g_1^k|_{\exp_p(E_p^c(\epsilon_1))} = id$, where id is the identity map. Thus we can take distinct two points $x, y \in \exp_p(E_p^s(\epsilon_1)) \subset M$ with $d(x,y) < \epsilon_1$ such that

$$d(g_1^n(x), g_1^n(y)) = d(x,y) < \epsilon_1 < \alpha < e,$$

for all $n \in \mathbb{N}$. But, $x \neq y$. This is a contradiction. □

It is known that f is positively expansive, f is open and local diffeomorphism since M is a manifold without boundary(see [3]).

Remark 2.3. *Let M be a compact Riemannian manifold and let $f : M \to M$ be a C^1 map. If f is a positively expansive open map then f is topological transitive.*

For $\delta > 0$, a sequence of points $\{x_i\}_{i=0}^n \subset M (0 < n \leq \infty)$ is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $0 \leq i \leq n-1$. We say that f has the *shadowing property* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=0}^n$ of f there is $y \in M$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $0 \leq i \leq n-1$.

In [7], Sakai showed that any positively expansive open map has the shadowing property.

Denote by $\mathcal{M}(M)$ the set of all probability on the Borel σ-algebra of M endowed with the usual topology such that for $\mu_n, \mu \in \mathcal{M}(M)$,

$$\mu_n \to \mu \Leftrightarrow \int \phi d\mu_n \to \int \phi d\mu,$$

for every continuous functions $\phi : M \to \mathbb{R}$. For a continuous map f, we denote by $\mathcal{M}_f(M)$, the set of all f-invariant elements of $\mathcal{M}(M)$. Take $x \in M$ and define a probability $\mu_n \in \mathcal{M}(M)(n > 0)$ by

$$\mu_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f_j(x)}.$$

Here δ_x is the so-called Dirac measure supported at point x. Then it is known that $\mu_n \to \mu \in \mathcal{M}_f(M)(n \to \infty)$.

For any $x \in M$ and $v \in T_xM$, let

$$\lambda(x, v) = \lim_{k\to\infty} \frac{1}{k} \log \|D_x f^k(v)\|$$

whenever this limit exists. It is well known that given an invariant measure μ, the limit exists for μ-almost all x by the Oselecdec theorem([5]) and it is called *Lyapunov exponent*. Lyapunov exponents are often used to characterize the non-uniform rates of expansion or contraction of tangent vectors.

Note that by the Oseledets theorem([5]), there exists a set $B \subset M$ such that $\mu(B) = 1$, for any $\mu \in \mathcal{M}(M)$ with the following properties.

- There is a measurable function $s : B \to \mathbb{Z}^+$ with $s \circ f = s$.
- If $x \in B$ there are real numbers $\lambda_1(x) < \cdots < \lambda_{s(x)}(x)$.
- If $x \in B$ there are linear subspaces

$$\{0\} = V_{(0)}(x) \subset \cdots \subset V_{s(x)}(x) = T_xM.$$

- If $x \in B$ and $0 < i \le s(x)$ then

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda(x),$$

  for any $v \in V_i(x) \setminus V_{i-1}(x)$.

Lemma 2.4. *Let μ be a finite invariant measure of f. Suppose that f is C^1-robustly positively expansive. Then for μ-almost all x, the Lyapunov exponents $\lambda_i(x)$ are positive.*

Proof. See, [1] Lemma 2.3. □

THEOREM 2.5. [2] *Let $f : M \to M$ be a C^1-local diffeomorphism. If the Lyapunov exponents of every f-invariant probability measure are positive, then f is uniformly expanding.*

Proof of Theorem 1.2 : Suppose that f is C^1-robustly positively expansive. Since f is positively expansive, f is a C^1-local diffeomorphism and it is open. Also, f has the shadowing property. Since M is connected, $M = \overline{P(f)}$. By Remark 2.3, f is transitive. Thus $M = \overline{P(f)}$ is transitive set and $\mu(\overline{P(f)}) = 1$ for $\mu \in \mathcal{M}_f(M)$. By the assumption and Lemma 2.1, there is $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}_0(f)$, g is positively expansive. For $x \in P(g)$, let λ be an eigenvalue of $D_x g$. By Lemma 2.2,

$$\begin{aligned}\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \log \|D_x g^n\| \\ &\geq \lim_{n\to\infty} \frac{1}{n} \log |\lambda|^n \geq \lim_{n\to\infty} \log |\lambda| \\ &= \log |\lambda| > 0.\end{aligned}$$

Thus $\lambda(x)$ is positive for μ-almost all x. Therefore, by Theorem 2.5, f is uniformly expanding. □

References

[1] A. Arbieto, *On persistently positively expansive maps*, Annal. Brazil. Acad. Sci. **82** (2010), 263-266.

[2] Y. Cao, *Non-zero lyapunov exponents and unform hyperbolicity*, Nonlinearity, **16** (2003), 1473-1479.

[3] E. M. Coven and W. L. Reddy, *Positively expansive maps of compact manifolds*, Lecture Notes in Math. **819**, Springer-Verlag, Berlin, 1980, 96-110.

[4] J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301-308.

[5] V. Oseledets, *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc.**19**(1968), 197-231.

[6] M. J. Pacifico, E. Pujals and J. L. Vieitez, *Robustly expansive homoclinic classes*, Ergodi. Th. & Dynm. Syst **25** (2005), 271-300.

[7] K. Sakai, *Periodic points of positivly expansive maps*, Proc. Amer. Math. Soc. **94**(1985), 531-534.

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