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STOCHASTIC CALCULUS FOR BANACH SPACE VALUED REGULAR STOCHASTIC PROCESSES

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Abstract. We study the stochastic integral of an operator valued process against with a Banach space valued regular process. We establish the existence and uniqueness of solution of the stochastic differential equation for a Banach space valued regular process under the certain conditions. As an application of it, we study a noncommutative stochastic differential equation.

1. Introduction

Since the classical stochastic calculus initiated by Itô [4], the stochastic calculus for standard Brownian motion has been extensively developed with wide applications to fields with random phenomena. The most classical Itô integrator is real valued Brownian motion or martingale process, but it is naturally extended to Hilbert space or Banach space valued processes, and then the stochastic calculus for Hilbert (or Banach) space valued processes studied by many mathematician [1, 2, 3, 5, 7, 6, 9], etc. In particular, stochastic integrals for Hilbert space valued functions against with martingale-valued measures with values in the Hilbert space has been established in [2], and stochastic integrals for deterministic Banach space valued functions against with compensated Poisson random measures has been established in [1]. Also, a stochastic integration for Hilbert space valued martingales using projection operators was studied in [8]. On the other hand, Pettis-type stochastic integral of (deterministic) operator valued functions against with Wiener process was studied in [9], and, more generally, Pettis-type stochastic integral of Banach

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space valued deterministic functions against with Banach space valued Lévy processes has been established in [7]. In [3], the author constructed a stochastic integration for operator valued processes against with Banach space valued processes with independent increments by using the notions of p-smoothable Banach space and L^p-primitive process.

The aim of this paper is to study the noncommutative stochastic calculus in terms of the stochastic calculus for Banach space valued processes. For our purpose, we first construct the stochastic integrals of operator (on Banach space) valued processes against with a Banach space valued p-regular process, and establish the existence and uniqueness of solution of the stochastic differential equation for a Banach space valued p-regular process Z given by

$$\begin{aligned} dX(t) &= \sigma(t, X(t))d\nu(t) + \rho(t, X(t))dZ(t), \\ X(0) &= X_0, \end{aligned}$$

where ν is a Radon measure on $\mathbf{R}_+$ and Z is a Banach space valued p-regular process with respect to a Radon measure μ on $\mathbf{R}_+$. As an application of it, we study the noncommutative stochastic calculus.

The paper is organized as follows. In Section 2, we introduce a Banach space valued p-regular process, and then the stochastic integral of an operator valued process against with a Banach space valued p-regular process is constructed. In Section 3, we prove the existence and uniqueness of solution of the stochastic differential equation for a Banach space valued p-regular processes. Finally, in Section 4 we study the stochastic integral of an operator (on a Banach algebra) valued process against with a Banach algebra valued p-regular process.

2. Banach space valued stochastic integral

Let $(\Omega, \mathcal{F}, P)$ be a (complete) probability space and B a separable Banach space with norm $\|\cdot\|_B$. Let $\{\mathcal{F}_t \,|\, t \geq 0\}$ be a filtration of sub σ-algebra of $\mathcal{F}$, i.e., $\mathcal{F}_s \subseteq \mathcal{F}_t$ for any $0 \leq s \leq t$. We assume that the filtration $\{\mathcal{F}_t \,|\, t \geq 0\}$ is right continuous, i.e., for any $t \geq 0$,

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s.$$

Definition 2.1. Let $1 \leq p < \infty$ be given. A B-valued process X is said to be *p-regular* with respect to a Radon measure μ on $\mathbf{R}_+$ (or simply *p-regular*) if there exists a constant $C_1 \geq 0$ such that for any $0 \leq s < t$,

$$\mathbf{E}\left[\|X(t) - X(s)\|_B^p\right] \leq C_1 \mu((s,t]).$$

From now on, let $Z = \{Z(t)\,|\,t \geq 0\}$ be a B-valued p-regular process, which has independent increments and satisfies the following conditions:

(i) for any $t \geq 0$, $Z(t)$ is $\mathcal{F}_t$-measurable,
(ii) for any $0 \leq s \leq t$, $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$.

Let $\mathcal{L}(B)$ be the space of all bounded linear operators on B with the operator norm $\|\cdot\|_{\text{op}}$ and $\mathcal{P}$ the σ-algebra generated by sets of the form:

$$(s,t] \times F \ \text{ for } 0 \leq s < t < \infty,\ F \in \mathcal{F}_s \text{ and } \{0\} \times F \text{ for } F \in \mathcal{F}_0,$$

and then $\mathcal{P}$ is called a *predictable σ-algebra*. A stochastic process X is said to be *predictable process* if X is $\mathcal{P}$-measurable. A $\mathcal{L}(B)$-valued process Φ is said to be *elementary* if there exists a sequence $0 = t_0 < t_1 < \cdots < t_k = T < \infty$ and a sequence $\Phi_{t_0}, \Phi_{t_1}, \cdots, \Phi_{t_{k-1}}$ of $\mathcal{L}(B)$-valued random variables such that Φ_{t_m} are $\mathcal{F}_{t_m}$-measurable and

$$\Phi(t) = \Phi_{t_m} \quad \text{for } t \in (t_m, t_{m+1}],\ m = 0, 1, \cdots, k-1.$$

Put

$$L^p_{\text{pred}}([0,T]) \equiv L^p_{\text{pred}}([0,T] \times \Omega, \mu \times P; \mathcal{L}(B)),$$

the space of all (predictable) $\mathcal{L}(B)$-valued processes, and so for any $\Phi \in L^p_{\text{pred}}([0,T])$, the following conditions:

(i) Φ is predictable process,
(ii) $\int_0^T \mathbf{E}\,[\|\Phi(t)\|^p_{\text{op}}]\,d\mu(t) < \infty$

hold, where μ is the Radon measure on $\mathbf{R}_+$ related to the B-valued p-regular process Z.

Let $\mathcal{E}^p([0,T], Z)$ be a linear space consisting of $\mathcal{L}(B)$-valued elementary (predictable) processes in $L^p_{\text{pred}}([0,T])$ such that for any

$$\Phi = \sum_{m=0}^{k-1} \Phi_{t_m} \mathbf{1}_{(t_m, t_{m+1}]} \in \mathcal{E}^p([0,T], Z) \tag{2.1}$$

for $0 = t_0 < t_1 < \cdots < t_k = T < \infty$, the following inequality is satisfied:

$$\mathbf{E}\left[\left\|\sum_{m=0}^{k-1} \Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m})\right\|_B^p\right] \leq C_2 \sum_{m=0}^{k-1} \mathbf{E}\left[\|\Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m})\|_B^p\right] \tag{2.2}$$

for some constant $C_2 \geq 0$, and let $\mathcal{I}^p_{\text{pred}}([0,T], Z)$ be the completion of $\mathcal{E}^p([0,T], Z)$ with respect to the norm on $L^p_{\text{pred}}([0,T])$. Note that if for any $\mathcal{L}(B)$-valued elementary process Φ given as in (2.1), (2.2) holds, then $\mathcal{E}^p([0,T], Z)$ becomes a linear space of all $\mathcal{L}(B)$-valued elementary processes and, since Z is a p-regular process, Z is a L^p-primitive [3].

Now, the stochastic integral of an elementary process Φ in $\mathcal{E}^p([0,T],Z)$ against with the Banach space valued p-regular process Z is defined by

$$I(\Phi) := \int_0^T \Phi(s)dZ(s) := \sum_{m=0}^{k-1} \Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m}).$$

Then the stochastic integral I is a linear operator from $\mathcal{E}^p([0,T],Z)$ into $L^p(\Omega;B)$. We have the following theorem.

THEOREM 2.2. *For any elementary process Φ in $\mathcal{E}^p([0,T],Z)$, $I(\Phi) \in L^p(\Omega;B)$ and*

$$\mathbf{E}\left[\|I(\Phi)\|_B^p\right] \le C \int_0^T \mathbf{E}[\|\Phi(t)\|_{\mathrm{op}}^p]d\mu(t) \tag{2.3}$$

for some $C \ge 0$.

Proof. For any elementary process $\Phi \in \mathcal{E}^p([0,T],Z)$ given as in (2.1), we obtain that

$$\begin{aligned}
\mathbf{E}\left[\|I(\Phi)\|_B^p\right] &= \mathbf{E}\left[\left\|\sum_{m=0}^{k-1} \Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m})\right\|_B^p\right] \\
&\le C_2 \sum_{m=0}^{k-1} \mathbf{E}\left[\|\Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m})\|_B^p\right] \\
&\le C_2 \sum_{m=0}^{k-1} \mathbf{E}\left[\|\Phi_{t_m}\|_{\mathrm{op}}^p \|Z_{t_{m+1}} - Z_{t_m}\|_B^p\right].
\end{aligned}$$

Since $\|\Phi_{t_m}\|_{\mathrm{op}}$ and $\|Z_{t_{m+1}} - Z_{t_m}\|_B$ are independent, and Z is p-regular with respect to a Radon measure μ on $\mathbf{R}_+$, we have

$$\begin{aligned}
\mathbf{E}\left[\|I(\Phi)\|_B^p\right] &\le C_2 \sum_{m=0}^{k-1} \mathbf{E}\left[\|\Phi_{t_m}\|_{\mathrm{op}}^p\right] \mathbf{E}\left[\|Z_{t_{m+1}} - Z_{t_m}\|_B^p\right] \\
&\le C_1 C_2 \sum_{m=0}^{k-1} \mathbf{E}\left[\|\Phi_{t_m}\|_{\mathrm{op}}^p\right] \mu((t_m, t_{m+1}]) \\
&= C \int_0^T \mathbf{E}\left[\|\Phi(t)\|_{\mathrm{op}}^p\right] d\mu(t)
\end{aligned}$$

for the constant $C = C_1C_2$, which implies (2.3). □

Now, we extend I to $\mathcal{I}^p_{\mathrm{pred}}([0,T],Z)$. By the construction, it is obvious that, if $\Phi \in \mathcal{I}^p_{\mathrm{pred}}([0,T],Z)$, then there exists a sequence $\{\Phi_n\}$ of elementary processes in $\mathcal{E}^p([0,T],Z)$ such that

$$\lim_{n\to\infty}\int_0^T \mathbf{E}\left[\|\Phi(t)-\Phi_n(t)\|_{\mathrm{op}}^p\right]d\mu(t)=0.$$

Thus by Theorem 2.2, the sequence $\{I(\Phi_n)\}$ is Cauchy sequence in $L^p(\Omega;B)$. Therefore, we define

$$I(\Phi)=\lim_{n\to\infty} I(\Phi_n) \quad \text{in } L^p(\Omega;B),$$

which is called the *stochastic integral* of the $\mathcal{L}(B)$-valued process Φ against with the Banach space valued p-regular process Z and denoted by $I(\Phi) := \int_0^T \Phi(s)dZ(s)$. Then the following theorem is obvious.

THEOREM 2.3. *For any $\mathcal{L}(B)$-valued process Φ in $\mathcal{I}^p_{\mathrm{pred}}([0,T],Z)$, we have*

$$\mathbf{E}\left[\|I(\Phi)\|_B^p\right]\le C\int_0^T \mathbf{E}[\|\Phi(t)\|_{\mathrm{op}}^p]d\mu(t) \ \textit{ for some } C\in\mathbf{R}_+.$$

By Theorem 2.3, the linear operator I from $\mathcal{I}^p_{\mathrm{pred}}([0,T],Z)$ into $L^2(\Omega;B)$ is continuous.

EXAMPLE 2.4. Let m be a (complete) measure on $\mathbf{R}_+$ and $\{M(t)\,|\,t\ge 0\}$ a $\{\mathcal{F}_t\}$-adapted B-valued process which has independent increments and locally Bochner integrable with respect to the measure m. We assume that $M(0)=0$. Put

$$Z(t)=\int_0^t M(s)dm(s) \ \text{(Bochner integral)}.$$

Then $Z(t)$ is a $\{\mathcal{F}_t\}$-adapted B-valued process, 1-regular with respect to the measure μ on $\mathbf{R}_+$ defined by

$$\mu((s,t])=\int_s^t \mathbf{E}\left[\|M(u)\|_B\right]dm(u), \qquad s\le t.$$

In fact, we obtain that

$$\mathbf{E}\left[\|Z(t)-Z(s)\|_B\right]\le\int_s^t \mathbf{E}\left[\|M(u)\|_B\right]dm(u)=\mu((s,t]).$$

It is obvious that $\{Z(t)\,|\,t\ge 0\}$ has independent increments. Moreover, for any elementary $\mathcal{L}(B)$-valued processes $\Phi\in L^1([0,T]\times\Omega, m\times P;\mathcal{L}(B))$, we have

$$\mathbf{E}\left[\left\|\sum_{n=1}^{k-1}\Phi_{t_n}(Z_{t_{n+1}}-Z_{t_n})\right\|_B\right]\le\sum_{n=0}^{k-1}\mathbf{E}\left[\|\Phi_{t_n}(Z_{t_{n+1}}-Z_{t_n})\|_B\right].$$

Therefore, for the process $\Phi \in \mathcal{I}^p_{\text{pred}}([0,T],Z)$, the stochastic integral $\int_0^T \Phi(s)dZ(s)$ is well-defined.

EXAMPLE 2.5. Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and $\{B(t)\,|\,t \geq 0\}$ a H-valued Q-Brownian motion, where Q is a symmetric nonnegative operator in $\mathcal{L}(H)$. For each $t \geq 0$, let $\mathcal{F}_t = \sigma(\{B(s)\,|\,s \leq t\})$. Let $H_0 = Q^{1/2}(H)$ be the subspace of H with inner product $\langle h_1, h_2\rangle_0 = \langle Q^{1/2}h_1, Q^{1/2}h_2\rangle_H$ and Φ a $\mathcal{L}_2(H_0,H)$-valued process in $L^2([0,T]\times\Omega;\mathcal{L}_2(H_0,H))$, where $\mathcal{L}_2(H_0,H)$ is the space of all Hilbert-Schmidt operators from H_0 to H with norm $\|\Phi\|^2_{\mathcal{L}_2} = \mathrm{Tr}[\Phi Q\Phi^*]$. For any $\mathcal{L}_2(H_0,H)$-valued elementary process Φ in $L^2([0,T]\times\Omega;\mathcal{L}_2(H_0,H))$ given as in (2.1), the stochastic integral

$$I(\Phi) := \int_0^T \Phi(s)dB(s) := \sum_{n=0}^{k-1} \Phi_{t_n}(B_{t_{n+1}\wedge T} - B_{t_n\wedge T})$$

is well-defined, and then $\{B(t)\,|\,t \geq 0\}$ is 2-regular Hilbert space valued process with respect to Lebesgue measure m on $\mathbf{R}_+$ and we have

$$L^2([0,T]\times\Omega;\mathcal{L}_2(H_0,H)) \subset \mathcal{I}^2_{\text{pred}}([0,T],B).$$

For more study, we refer to [6].

3. Stochastic differential equation

In this section, we study the existence and uniqueness of solution of the stochastic differential equation:

$$\begin{cases} dX(t) = \sigma(t,X(t))d\mu_1(t) + \rho(t,X(t))dZ(t) \\ X(0) = X_0, \end{cases} \tag{3.1}$$

where μ_1 is a Radon measure on $\mathbf{R}_+$ and Z is a B-valued p-regular process with respect to a Radon measure μ_2 on $\mathbf{R}_+$. Let $1 \leq p < \infty$. We fix $T > 0$ and impose the following conditions for σ and ρ in the equation (3.1):

(C1) the map $\sigma : [0,T]\times B \to B$ is measurable,

(C2) the map $\rho : [0,T]\times B \to \mathcal{L}(B)$ is measurable, and for any $X \in \mathcal{I}^p_{\text{pred}}(B,Z)$, $\rho(\cdot,X) \in \mathcal{I}^p_{\text{pred}}(B,Z)$,

(C3) there exist a nonnegative function $K \in L^1([0,T],d\mu_1)\cap L^1([0,T],d\mu_2)$ such that for any $x,y \in B$ and $t \in [0,T]$,

$$\|\sigma(t,x)-\sigma(t,y)\|^p_B + \|\rho(t,x)-\rho(t,y)\|^p_{\text{op}} \leq K(t)\|x-y\|^p_B,$$
$$\|\sigma(t,x)\|^p_B + \|\rho(t,x)\|^p_{\text{op}} \leq K(t)(1+\|x\|^p_B).$$

The integral representation of the Equation (3.1) is given by

$$X(t) = X_0 + \int_0^t \sigma(s, X(s))d\mu_1(s) + \int_0^t \rho(s, X(s))dZ(s), \quad P-a.s.. \tag{3.2}$$

The main result of this section is the following.

THEOREM 3.1. *Assume that X_0 is a $\mathcal{F}_0$-measurable B-valued random variable with $\mathbf{E}[\|X_0\|_B^p] < \infty$. If the conditions* (C1) − (C3) *are satisfied, then there exists a unique solution X of* (3.1) *such that*

$$\mathbf{E}\left[\int_0^T K(s)\|X(s)\|_B^p d\mu_i(s)\right] < \infty, \qquad i = 1, 2, \tag{3.3}$$

where the function K is given in the condition (C3).

Proof. We first prove the uniqueness of the solution for (3.1). Let $X(t)$ and $\widehat{X}(t)$ be the solutions of (3.1) with same initial values X_0. Put

$$\widehat{\sigma}(s, X(s)) = \sigma(s, X(s)) - \sigma(s, \widehat{X}(s))$$

and

$$\widehat{\rho}(s, X(s)) = \rho(s, X(s)) - \rho(s, \widehat{X}(s)).$$

Then by (C3), Theorem 2.3 and the Hölder's inequality, for the q with $1/p + 1/q = 1$ we obtain that

$$\begin{aligned}\mathbf{E}\left[\|X(t) - \widehat{X}(t)\|_B^p\right] &\le \{2\mu_1([0,t])\}^{p/q}\mathbf{E}\left[\int_0^t \|\widehat{\sigma}(s, X(s))\|_B^p d\mu_1(s)\right] \\ &\quad + 2^{p/q}C\mathbf{E}\left[\int_0^t \|\widehat{\rho}(s, X(s))\|_{\mathrm{op}}^p d\mu_2(s)\right] \\ &\le 2^{p/q}\int_0^t K(s)\mathbf{E}\left[\|X(s) - \widehat{X}(s)\|_B^p\right] d\mu(s),\end{aligned}$$

where

$$d\mu(s) = \mu_1([0,t])^{p/q}d\mu_1(s) + Cd\mu_2(s) \tag{3.4}$$

and the constant C is given as in Theorem 2.3. Therefore, the function

$$V(t) = \mathbf{E}\left[\|X(t) - \widehat{X}(t)\|_B^p\right], \qquad t \in [0, T]$$

satisfies

$$V(t) \le 2^{p/q}\int_0^t K(s)V(s)d\mu(s).$$

Then by the Gronwall's inequality, we conclude that $V(t) = 0$ for all $t \ge 0$, and so $X(t) = \widehat{X}(t)$ P-a.s. for all $t \ge 0$, which implies the proof of the uniqueness.

For the proof of the existence, we use the Picard method. Define $Y^{(0)}(t) = X_0$ and $Y^{(k)}(t)$, inductively, as follows

$$Y^{(k+1)}(t) = X_0 + \int_0^t \sigma(s, Y^{(k)}(t))d\mu_1(s) + \int_0^t \rho(s, Y^{(k)}(t))dZ(s),$$

then the similar computations used for the uniqueness give

$$\mathbf{E}\left[Z(t;k)\right] \le 2^{p/q} \int_0^t K(s)\mathbf{E}\left[Z(t;k-1)\right] d\mu(s)$$

for $k \ge 1$ and $t \le T$, where $Z(t;k) = \|Y^{(k)}(t) - Y^{(k-1)}(t)\|_B^p$. By repeating this argument, we obtain that

$$\begin{aligned}\mathbf{E}[Z(t;k+1)] &\le 2^{p/q} \int_0^t K(t_1)\mathbf{E}\left[Z(t_1;k)\right] d\mu(t_1) \\ &\le 2^{pk/q} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} K(t_1)K(t_2)\cdots K(t_k) \\ &\qquad \times \mathbf{E}\left[Z(t_k;1)\right] d\mu(t_k)\cdots d\mu(t_2)d\mu(t_1).\end{aligned}$$

On the other hand, by (C3) and the Hölder's inequality

$$\begin{aligned}\mathbf{E}\left[Z(t_k;1)\right] &\le 2^{p/q}\left\{\mu_1([0,t_k])^{p/q}\int_0^{t_k} K(s)\left(1+\mathbf{E}[\|X_0\|_B^p]\right)d\mu_1(s)\right. \\ &\qquad \left. + C\int_0^{t_k} K(s)(1+\mathbf{E}[\|X_0\|_B^p])d\mu_2(s)\right\} \\ &\le 2^{p/q} H\widetilde{K}(t_k),\end{aligned}$$

where $H \equiv \left(1 + \mathbf{E}\left[\|X_0\|_B^p\right]\right)$, $\widetilde{K}(t) = \int_0^t K(s)d\mu(s)$ and μ is given as in (3.4). Therefore, by induction on k, for $0 \le t \le T$, we have

$$\mathbf{E}\left[Z(t;k+1)\right] \le \frac{\left(2^{p/q}\widetilde{K}(t)\right)^{k+1}}{(k+1)!} H, \tag{3.5}$$

and so, for $i = 1, 2$, we obtain that

$$\begin{aligned}\mu_i \times P\left(Z(\cdot;k+1) \ge 2^{-k}\right) &\le 2^k \mathbf{E}\left[\int_0^T Z(t;k+1)d\mu_i(t)\right] \\ &\le 2^k \frac{\left(2^{p/q}\widetilde{K}(T)\right)^{k+1}}{(k+1)!} H\mu_i([0,T]).\end{aligned}$$

Therefore, by the Borel-Cantelli lemma, we have

$$\mu_i \times P\left(\|Y^{(k+1)} - Y^{(k)}\|_B^p \ge 2^{-k} \text{ for infinitely many k}\right) = 0.$$

Thus for almost all $(t,\omega) \in [0,T] \times \Omega$, there exists $k_0 = k_0(t,\omega)$ such that

$$\|Y^{(k+1)}(t) - Y^{(k)}(t)\|_B \leq 2^{-k/p} \text{ for } k \geq k_0.$$

Therefore, the sequence

$$Y^{(n)}(t,\omega) = Y^{(0)}(t,\omega) + \sum_{k=0}^{n-1} \Big(Y^{(k+1)}(t,\omega) - Y^{(k)}(t,\omega)\Big) \tag{3.6}$$

converges for almost all $(t,\omega) \in [0,T] \times \Omega$. Put

$$X(t,\omega) = \lim_{n\to\infty} Y^{(n)}(t,\omega), \qquad (t,\omega) \in [0,T] \times \Omega.$$

Then by using (3.5) and (3.6), the proof of (3.3) is straightforward. For $m \geq n \geq 0$, by (3.5) we obtain that

$$\begin{aligned}
&\mathbf{E}\left[\|Y^{(m)}(t) - Y^{(n)}(t)\|_B^p\right] \\
&\quad\leq \left(\sum_{k=n}^{m-1} \left\|Y^{(k+1)}(t) - Y^{(k)}(t)\right\|_{L^p(\Omega;B)}\right)^p \\
&\quad\leq \left(\sum_{k=n}^{\infty} \left[H\frac{\left(2^{p/q}\widetilde{K}(t)\right)^{k+1}}{(k+1)!}\right]\right)^p \\
&\quad\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.7}$$

Then $\{Y^{(n)}(t)\}$ converges in $L^p(\Omega; B)$ and the limit is denoted by $Y(t)$. A subsequence of $\{Y^{(n)}(t,\omega)\}$ converges to $Y(t,\omega)$ almost all ω and so $Y(t) = X(t)$. It remains to show that $X(t)$ satisfies (3.2). For all n, we have

$$Y^{(n+1)}(t) = X_0 + \int_0^t \sigma(s, Y^{(n)}(s))d\mu_1(s) + \int_0^t \rho(s, Y^{(n)}(s))dZ(s). \tag{3.8}$$

Also, $\{Y^{(n+1)}(t,\omega)\}$ converges to $X(t,\omega)$ for almost all $(t,\omega) \in [0,T]\times\Omega$, by (3.7) and the Fatou's lemma, we have

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T \|X(t) - Y^{(n)}(t)\|_B^p d\mu_i(t)\right] \\
&\quad\leq \limsup_{m\to\infty} \mathbf{E}\left[\int_0^T \|Y^{(m)}(t) - Y^{(n)}(t)\|_B^p d\mu_i(t)\right] \\
&\quad\to 0
\end{aligned}$$

as $n \to \infty$ for $i = 1, 2$. Since

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left[\left\|\int_0^t [\sigma(s, Y^{(n)}(s)) - \sigma(s, X(s))]d\mu_1(s)\right\|_B^p\right] \\ \leq \lim_{n\to\infty} \{\mu([0,t])\}^{p/q} \int_0^t K(s)\mathbf{E}\left[\|Y^{(n)}(s) - X(s)\|_B^p\right] d\mu_1(s) \\ = 0,\end{aligned}$$

we have

$$\lim_{n\to\infty} \int_0^t \sigma(s, Y^{(n)}(s))d\mu_1(s) = \int_0^t \sigma(s, X(s))d\mu_1(s) \quad \text{in } L^p(\Omega; B).$$

Also, since

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left[\left\|\int_0^t [\rho(s, Y^{(n)}(s)) - \rho(s, X(s))]dZ(s)\right\|_B^p\right] \\ \leq \lim_{n\to\infty} C \int_0^t K(s)\mathbf{E}\left[\|Y^{(n)}(s) - X(s)\|_B^p\right] d\mu_2(s) \\ = 0,\end{aligned}$$

we have

$$\lim_{n\to\infty} \int_0^t \rho(s, Y^{(n)}(s))dZ(s) = \int_0^t \rho(s, X(s))dZ(s) \quad \text{in } L^p(\Omega; B).$$

Therefore, by taking the limit in the both sides of (3.8), we obtain (3.2) for $X(t)$. □

4. Noncommutative stochastic calculus

Let $\mathcal{A}$ be a Banach algebra with norm $\|\cdot\|_{\mathcal{A}}$ and $\{Z(t) \,|\, t \geq 0\}$ a $\mathcal{A}$-valued p-regular process with respect to a Radon measure μ on $\mathbf{R}_+$.

For an elementary process $\Phi \in \mathcal{E}^p([0,T], Z)$ given as in (2.1) with $\Phi_{t_m} \in \mathcal{L}(\mathcal{A})$, the stochastic integral of Φ against with the $\mathcal{A}$-valued p-regular process $Z(t)$ is defined by

$$I(\Phi) := \int_0^T \Phi(s)dZ(s) := \sum_{m=0}^{k-1} \Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m}),$$

and then we have

$$\mathbf{E}[\|I(\Phi)\|_{\mathcal{A}}^p] \leq C_2 \sum_{m=0}^{k-1} \mathbf{E}[\|\Phi_{t_m}(Z_{t_{m+1}} - Z_{t_m})\|_{\mathcal{A}}^p].$$

Therefore, by the construction in Section 2, the stochastic integral of a Banach algebra valued process $\Phi \in \mathcal{I}^p_{\text{pred}}([0,T],Z)$ against with the Banach algebra valued p-regular process Z is well-defined and denoted by $I(\Phi) := \int_0^T \Phi(s)dZ(s)$. Then the following theorem is obvious.

THEOREM 4.1. *Let Φ be a $\mathcal{L}(\mathcal{A})$-valued process in $\mathcal{I}^p_{\text{pred}}([0,T],Z)$ and Z a $\mathcal{A}$-valued p-regular process. Then $I(\Phi)$ in $L^p(\Omega;\mathcal{A})$ and*

$$\mathbf{E}[\|I(\Phi)\|^p_{\mathcal{A}}] \leq C\int_0^T \mathbf{E}[\|\Phi(t)\|^p_{\text{op}}]d\mu(t)$$

for some $C \geq 0$.

For each $\Phi \in \mathcal{I}^p_{\text{pred}}([0,T],Z)$, the *time-ordered exponential* is defined by

$$\begin{aligned}(4.1)\quad &\operatorname{Texp}\left(\int_0^t \Phi(s)dZ(s)\right)\\ &\equiv 1 + \sum_{n=1}^{\infty}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\Phi(t_1)\cdots\Phi(t_n)dZ(t_n)\cdots dZ(t_1)\end{aligned}$$

if the series converges.

PROPOSITION 4.2. *Let Φ be a $\mathcal{L}(\mathcal{A})$-valued process in $\mathcal{I}^1_{\text{pred}}([0,T],Z)$ satisfying the following property: for any n and $0 \leq t_i \leq T$, $i = 1,2,3,\cdots,n$*

$$\mathbf{E}\left[\overbrace{\|\Phi(\cdot)\|_{\text{op}}\cdots\|\Phi(\cdot)\|_{\text{op}}}^{n-times}\right] \in L^1([0,T]^n,\mu^n).$$

Then the time-ordered exponential converges in $L^1(\Omega;\mathcal{A})$.

Proof. To proof, we check the time-ordered exponential is absolutely convergent in $L^1(\Omega;\mathcal{A})$. By Fubini-Tonelli's theorem we obtain

$$\begin{aligned}&\left\|\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\Phi(t_1)\cdots\Phi(t_n)dZ(t_n)\cdots dZ(t_1)\right\|_{L^1(\Omega;\mathcal{A})}\\ &\quad\leq \frac{1}{n!}\mathbf{E}\left[\left(C\int_0^t\|\Phi(s)\|_{\text{op}}d\mu(s)\right)^n\right].\end{aligned}$$

Thus we have

$$1+\sum_{n=1}^{\infty}\left\|\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\Phi(t_1)\cdots\Phi(t_n)dZ(t_n)\cdots dZ(t_1)\right\|_{L^1(\Omega;\mathcal{A})}$$
$$\leq \mathbf{E}\left[\exp\left(C\int_0^t\|\Phi(s)\|_{\mathrm{op}}d\mu(s)\right)\right]<\infty,$$

which gives the proof. □

Let Φ be a $\mathcal{A}$-valued process. Then for each $t\geq 0$, $\Phi(t):\Omega\longrightarrow\mathcal{A}$ can be considered as the $\mathcal{L}(\mathcal{A})$-valued process as the left multiplication operator, i.e.,

$$\Phi(t)(a)=\Phi(t)a,\qquad a\in\mathcal{A},$$

and if $\Phi\in\mathcal{I}^p_{\mathrm{pred}}([0,T],Z)$, then the integral $I(\Phi)=\int_0^T\Phi(s)dZ(s)$ is well-defined and called a *noncommutative stochastic integral* of a Banach algebra valued process Φ against with a Banach algebra valued process Z.

COROLLARY 4.3. *Let $\mathbf{E}[\|X_0\|_{\mathcal{A}}]<\infty$ and $\sigma,\rho\in\mathcal{A}$. The unique solution of the linear stochastic differential equation:*

$$dX(t)=\sigma X(t)d\nu(t)+\rho X(t)dZ(t),\qquad X(0)=X_0$$

is given by

$$X(t)=X_0\mathrm{Texp}\left(\sigma\nu(\mathrm{t})+\rho\mathrm{Z}(\mathrm{t})\right),\qquad \mathrm{t}\geq 0,$$

where $\nu(t)=\int_0^t d\nu(s)$.

Proof. The proof is immediate from Theorem 3.1 and Proposition 4.2. □

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