JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 1, March 2011

THE CONVERGENCE OF δ-FILTERS

Seung On Lee*, Ji Hyun Oh**, and Sang Min Yun***

Abstract. In this paper we define the convergence of δ-filters and study them. We show that δ-filters on a Hausdorff space X converge at most one point in X. We also show that in a P-space X, δ-filters on X converge at most one point in X if and only if X is a Hausdorff space.

1. Introduction

A topological space is an abstraction and generalization of a metric space, but there is a distinguished formal difference of the two in the way of thought. The notion of limits or continuity of a function is represented as convergence of sequences in a metric space, but it is calculated as a term of sets such as open sets or neighborhoods in a topological space.

It is well known [1, 6, 16] that the order structure plays the important role in the study of various mathematical structures. In analysis, we define continuity and the integral in terms of limits of sequences, and discover methods of determining the limit of an infinite series. In 1922, E. H. Moore and H. L. Smith were led to construct a general theory of convergence for analysis([15]). With the development of topology this Moore-Smith convergence was applied to topology by Birkhoff([2]) under a still more general form, using the concept of directed sets and nets. But this is not the only theory of convergence to be found in topology. The idea of a filter, a special kind of family of sets, was introduced into topology by Cartan in 1937([4]). This concept was studied and developed into a theory of convergence by Bourbaki and his colleagues([3]). The introduction of filters by H. Cartan([4]) has brought to topology a valuable instrument, usable in all sorts of applications (in which it

Received August 09, 2010; Revised November 10, 2010; Accepted February 15, 2011.
2010 Mathematics Subject Classification: Primary 54A20, 54E30.
Key words and phrases: filters, δ-filters, convergence of δ-filters.
Correspondence should be addressed to Seung On Lee, solee@chungbuk.ac.kr.

replaces to advantage the notion of "Moore-Smith convergence"([15])). Furthermore, the development of the theorem on ultrafilters has clarified and simplified the theory. For a further development and references in this field we refer to [7, 11, 17].

In [13], we introduced the concept of a c-net which is a special type of a net and we found some relations between topological spaces and the convergence of c-nets. In this paper, we define the concept of the convergence of δ-filters. And we study the properties of the convergence of δ-filters in topological spaces. We show that if X is a Lindelöf P-space, then every ultra δ-filter on X is convergent. Furthermore, we prove that if each δ-filter on a P-space X converges to at most one point in X, then X is a Hausdorff space.

2. Preliminaries

In this section, we introduce some definitions and theorems to be shown throughout [5, 10, 12]. For terminologies not introduced in this paper, we refer to [5, 10, 12].

Definition 2.1. Let X be a set. Then a nonempty collection $\mathcal{F}$ of nonempty subsets of X is called a filter on X if $\mathcal{F}$ has the following properties:
(F_1) $A_k \in \mathcal{F}$ for each $k \in K$, $\bigcap A_k \in \mathcal{F}$, where K is a finite set.
(F_2) $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Definition 2.2. Let $\mathcal{B}$ and $\mathcal{S}$ be nonempty collections of nonempty subsets of X. Then

(1) $\mathcal{B}$ is said to be a filter base on X provided that if $B_k \in \mathcal{B}$ for each $k \in K$, where K is a finite set, then there is $B \in \mathcal{B}$ such that $B \subseteq \bigcap B_k$.
(2) $\mathcal{S}$ is said to be a filter subbase on X if for every nonempty finite subcollection $\mathcal{S}^*$ of $\mathcal{S}$, $\bigcap \mathcal{S}^* \neq \emptyset$.

Definition 2.3. Let X be a topological space and $x \in X$. A subset V of X is called a neighborhood of x if there exists an open subset U of X such that $x \in U \subseteq V$.
Furthermore, if $\mathcal{N}_x$ is the collection of all neighborhoods of x, then $\mathcal{N}_x$ is called the neighborhood system of x.

Definition 2.4. Let $\mathcal{F}$ and $\mathcal{G}$ be filters in a topological space X. If each member of $\mathcal{F}$ is also a member of $\mathcal{G}$, then $\mathcal{G}$ is said to be finer than $\mathcal{F}$ and denoted by $\mathcal{F} \subseteq \mathcal{G}$.

Definition 2.5. Let $\mathcal{F}$ be a filter in a topological space X and $x \in X$. $\mathcal{F}$ is said to converge to x if each neighborhood V of x contains a member of $\mathcal{F}$. In particular the neighborhood filter of x converges to x.

Definition 2.6. Let $\mathcal{U}$ be a filter in a topological space X. $\mathcal{U}$ is said to be an ultrafilter if no other filter on X properly contains $\mathcal{U}$.

Theorem 2.7. *A topological space X is a Hausdorff space if and only if each filter on X converges to at most one point in X.*

Theorem 2.8. *If X is a set and $\mathcal{F}$ is a filter on X, then there exist an ultrafilter $\mathcal{U}$ on X that contains $\mathcal{F}$.*

Theorem 2.9. *A topological space X is compact if and only if every ultrafilter on X converges.*

3. The convergence of δ-filters

Definition 3.1. ([8, 9]) Let X be a set. Then a nonempty collection $\mathcal{F}$ of nonempty subsets of X is called a δ-filter on X if $\mathcal{F}$ has the following properties:

(F_1) $A_k \in \mathcal{F}$ for each $k \in K$, $\bigcap A_k \in \mathcal{F}$, where K is a countable set.
(F_2) $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Example 3.2. (1) *Let X be a topological space. Then $\mathcal{N}_x$ is a filter, but it need not be a δ-filter, where $\mathcal{N}_x$ is the neighborhood system of x. Thus a filter need not be a δ-filter in general.*
(2) *Let $[A] = \{F \subseteq X \mid A \subseteq F, A \neq \emptyset\}$. Then the principal filter $[A]$ is a δ-filter.*
(3) *Every δ-filter is a filter.*

Definition 3.3. ([14]) Let X be a topological space. Then X is said to be a P-space if every G_δ set in X is open, where G_δ set is a countable intersection of open subsets of X.

Remark 3.4. X is a P-space if and only if $\mathcal{N}_x$ is a δ-filter for any $x \in X$.

Proof. Let X be a P-space and $\mathcal{N}_x$ be the neighborhood system of $x \in X$. Take any $N_k \in \mathcal{N}_x$, for each $k \in K$, where K is a countable set. Then there is an open set G_k such that $x \in G_k \subseteq N_k$. Since $x \in \bigcap_{k\in K} G_k \subseteq \bigcap_{k\in K} N_k$ and $\bigcap_{k\in K} G_k$ is open, $\bigcap_{k\in K} N_k \in \mathcal{N}_x$. Let $N \in \mathcal{N}_x$ and $N \subseteq M$. Then $M \in \mathcal{N}_x$. Hence $\mathcal{N}_x$ is a δ-filter.

Conversely, suppose that $\mathcal{N}_x$ is a δ-filter. Take any open sets G_k, for each $k \in K$, where K is a countable set. If $\bigcap_{k\in K} G_k = \emptyset$, then $\bigcap_{k\in K} G_k$ is open. If $\bigcap_{k\in K} G_k \neq \emptyset$, then take any $x \in \bigcap_{k\in K} G_k$. Since $\mathcal{N}_x$ is a δ-filter, $\bigcap_{k\in K} G_k \in \mathcal{N}_x$. So there is an open set G such that $x \in G \subseteq \bigcap_{k\in K} G_k$. Hence $\bigcap_{k\in K} G_k$ is open. □

Definition 3.5. Let X be a P-space. Then for each $x \in X$, the neighborhood system $\mathcal{N}_x$ of x is called the neighborhood δ-filter.

Definition 3.6. Let $\mathcal{B}$ and $\mathcal{S}$ be nonempty collections of nonempty subsets of X.

(1) $\mathcal{B}$ is said to be a δ-filter base on X provided that if $B_k \in \mathcal{B}$ for each $k \in K$, where K is a countable set, then there is $B \in \mathcal{B}$ such that $B \subseteq \bigcap B_k$.
(2) $\mathcal{S}$ is said to be a δ-filter subbase on X if for every nonempty countable subcollection $\mathcal{S}^*$ of $\mathcal{S}$, $\bigcap \mathcal{S}^* \neq \emptyset$.

Example 3.7. *Let X be a nonempty set.*
Then $\{\{a\}\}$ is a δ-filter base for $\dot{a}$, where $\dot{a} = \{F \subseteq X \mid a \in F\}$.

Theorem 3.8. *Let X be a set. Then we have the followings:*

(a) *If $\mathcal{B}$ is a δ-filter base on X, then the collection $\{U \mid U \subseteq X, B \subseteq U$ for some $B \in \mathcal{B}\}$ is a δ-filter on X.*
(b) *If $\mathcal{S}$ is a δ-filter subbase on X, then the collection $\{U \mid U = \bigcap \mathcal{S}^*$ for a $\mathcal{S}^* \in Count(\mathcal{S})\}$ is a δ-filter base on X, where $Count(\mathcal{S}) = \{\mathcal{S}^* \mid \mathcal{S}^* : countable, \mathcal{S}^* \subseteq \mathcal{S}\}$.*

Proof. (a) Let $\mathcal{B}$ be a δ-filter base on X and $\mathcal{F} = \{U \mid U \subseteq X, B \subseteq U$ for some $B \in \mathcal{B}\}$. Then $\mathcal{F}$ is a nonempty collection of nonempty subsets of X. Take any $U_k \in \mathcal{F}$, for each $k \in K$, where K is a countable set. Then there is $B_k \in \mathcal{B}$ such that $B_k \subseteq U_k$, for all $k \in K$. So $\bigcap B_k \subseteq \bigcap U_k$. Since $\mathcal{B}$ is a δ-filter base, there is $C \in \mathcal{B}$ such that $C \subseteq \bigcap B_k \subseteq \bigcap U_k$. Hence $\bigcap U_k \in \mathcal{F}$. Let $U \in \mathcal{F}$ and $U \subseteq H$. Then there is $B \in \mathcal{B}$ such that $B \subseteq U \subseteq H$. Therefore $H \in \mathcal{F}$.

(b) Let $\mathcal{S}$ be a δ-filter subbase on X and $\mathcal{B} = \{U \mid U = \bigcap \mathcal{S}^*, \mathcal{S}^* \in Count(\mathcal{S})\}$. Take any $U_k \in \mathcal{B}$, $k \in K$, where K is a countable set. Then there is a countable subfamily $\mathcal{S}_k$ of $\mathcal{S}$ such that $U_k = \bigcap \mathcal{S}_k$ for each $k \in K$. Thus $\bigcap U_k = \bigcap_{k\in K}(\bigcap \mathcal{S}_k) \in \mathcal{B}$. So $\mathcal{B}$ is a δ-filter base.

□

Example 3.9. *Let* $(\mathbb{R}, \mathcal{C}_c)$ *be the co-countable space on the set of real numbers* $\mathbb{R}$ *and* $\mathcal{F} = \{F \subseteq \mathbb{R} \mid 0 \in F\}$. *Then* $\mathcal{F}$ *is a* δ*-filter and* $\mathcal{F} \to 0$. *But* $\mathcal{F} \nrightarrow 1$.

Proposition 3.10. *Let* X *be a* P*-space and* A *a nonempty subset of* X. *Then* $p \in X$ *is a limit point of* A *if and only if there exists a* δ*-filter* $\mathcal{F}$ *on* X *such that* $\mathcal{F} \longrightarrow p$ *and* $A - \{p\} \in \mathcal{F}$.

Proof. Let $p \in A'$ and $\mathcal{N}_p$ be the neighborhood system of p. Then for each $N \in \mathcal{N}_p$, $N \cap (A - \{p\}) \neq \emptyset$. Let $\mathcal{S} = \{A - \{p\}\} \cup \mathcal{N}_p$. Then $\mathcal{S}$ is a δ-filter subbase. Let $\mathcal{F}$ be a δ-filter generated by $\mathcal{S}$. Then $\mathcal{F} \to p$, and $A - \{p\} \in \mathcal{F}$.

Conversely, let $\mathcal{F}$ be a δ-filter with $\mathcal{F} \to p$ and $A - \{p\} \in \mathcal{F}$. Take any $N \in \mathcal{N}_p$. Since $N \in \mathcal{F}$ and $A - \{p\} \in \mathcal{F}$, $N \cap (A - \{p\}) \in \mathcal{F}$. Then $N \cap (A - \{p\}) \neq \emptyset$. Hence $p \in A'$. □

Since every δ-filter is a filter, we have the following by Theorem 2.7:

Corollary 3.11. *Let* $(X, \mathcal{T})$ *be a Hausdorff space. Then each* δ*-filter on* X *converges to at most one point in* X.

Theorem 3.12. *Let* $(X, \mathcal{T})$ *be a* P*-space. If each* δ*-filter on* X *converges to at most one point in* X, *then* X *is a Hausdorff space.*

Proof. Suppose that X is not a Hausdorff space, i.e. there are distinct points p and q in X such that for any $N \in \mathcal{N}_p$ and $M \in \mathcal{N}_q$, $N \cap M \neq \emptyset$. Let $\mathcal{B} = \{U \mid U = N \cap M,\ N \in \mathcal{N}_p \text{ and } M \in \mathcal{N}_q\}$. Then $\mathcal{B}$ is a δ-filter base. Therefore the δ-filter $\mathcal{F}$ generated by $\mathcal{B}$ includes both $\mathcal{N}_p$ and $\mathcal{N}_q$. Hence $\mathcal{F}$ converges two distinct points p and q. □

Definition 3.13. Let $\mathcal{F}$ be a δ-filter on a set X.

$\mathcal{F}$ is said to be an ultra δ-filter (or a maximal δ-filter) if no other δ-filter on X properly contains $\mathcal{F}$.

Example 3.14. $\dot{a}$ *is an ultra* δ*-filter* .

As we know, every filter $\mathcal{F}$ has an ultrafilter which contains $\mathcal{F}$, but we don't know whether a δ-filter $\mathcal{F}$ has an ultra δ-filter which contains $\mathcal{F}$ or not.

Lemma 3.15. *Let* $\mathcal{F}$ *be a* δ*-filter on a set* X *and* $A \subseteq X$. *Then there is a* δ*-filter* $\mathcal{G}$ *such that* $\mathcal{F} \subseteq \mathcal{G}$ *and* $A \in \mathcal{G}$ *if and only if each member of* $\mathcal{F}$ *meets* A.

Proof. Let $\mathcal{F}$ be a δ-filter on a set X and $A \subseteq X$. Suppose that there is a δ-filter $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$ and $A \in \mathcal{G}$. Take any $F \in \mathcal{F}$, then

$F \cap A \neq \emptyset$ because $\mathcal{G}$ is a δ-filter and $A \in \mathcal{G}$.
Conversely let $\mathcal{S} = \mathcal{F} \cup \{A\}$. Then $\mathcal{S}$ has the countable intersection property. Thus there is a δ-filter $\mathcal{G}$ containing $\mathcal{S}$. □

THEOREM 3.16. *Let $\mathcal{U}$ be a δ-filter on a set X, then the followings are equivalent.*

(1) *$\mathcal{U}$ is an ultra δ-filter.*
(2) *For each $A \subseteq X$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.*
(3) *If $A \cap U \neq \emptyset$ for any $U \in \mathcal{U}$ and $A \subseteq X$, then $A \in \mathcal{U}$*

Proof. (1)⇒(2)
Take any $A \subseteq X$. Suppose that $A^c \notin \mathcal{U}$. Then for any $F \in \mathcal{U}$, $F \nsubseteq A^c$. So $F \cap A \neq \emptyset$. By the Lemma 3.16, there is a δ-filter $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$ and $A \in \mathcal{G}$. Since $\mathcal{U}$ is an ultra δ-filter, $\mathcal{U} = \mathcal{G}$. Hence $A \in \mathcal{U}$.
(2)⇒(3)
Take any $A \subseteq X$ with $A \cap U \neq \emptyset$ for any $U \in \mathcal{U}$. If $A \notin \mathcal{U}$, then $A^c \in \mathcal{U}$ by (2). Hence $A \cap A^c \neq \emptyset$. This is a contradiction.
(3)⇒(1)
Let $\mathcal{G}$ be a δ-filter with $\mathcal{U} \subseteq \mathcal{G}$. It's enough to show that $\mathcal{G} \subseteq \mathcal{U}$. Take any $A \in \mathcal{G}$ and $F \in \mathcal{U}$, then $A \cap F \neq \emptyset$. By the hypothesis, $A \in \mathcal{U}$. Hence $\mathcal{G} \subseteq \mathcal{U}$. □

Every ultra δ-filter is an ultra filter, but the following example shows that an ultra filter need not be an ultra δ-filter in general.

EXAMPLE 3.17. *Let $\mathcal{F} = \{F \subseteq \mathbb{N} \mid F^c : finite\}$ be the Frechet filter. Then there is an ultra filter $\mathcal{U}$ with $\mathcal{F} \subseteq \mathcal{U}$. But $\mathcal{U}$ is not a δ-filter. For $\mathbb{N} - \{n\} \in \mathcal{U}$ for all $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}}(\mathbb{N} - \{n\}) = \mathbb{N} - \bigcup_{n\in\mathbb{N}}\{n\} = \emptyset$. Hence $\mathcal{U}$ is not an ultra δ-filter.*

THEOREM 3.18. *Let $(X, \mathcal{T})$ be a P-space. If X is a Lindelöf space, then every ultra δ-filter on X converges to a point $x \in X$.*

Proof. Let X be a Lindelöf space. Suppose that there is an ultra δ-filter $\mathcal{U}$ such that for all $x \in X, \mathcal{U} \nrightarrow x$. Then there is a neighborhood N of x such that $N \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultra δ-filter, $X - N \in \mathcal{U}$. We may assume that N is open in X. Let $\mathcal{G} = \{N \mid N$ is a neighborhood of x and $N \notin \mathcal{U}$, for all $x \in X\}$, then $\mathcal{G}$ is an open cover of X. Since X is a Lindelöf space, there is a countable subfamily $\{N_i\}$ of $\mathcal{G}$ with $X = \bigcup_{i\in\mathbb{N}} N_i$. Since $X - N_i \in \mathcal{U}$ for all $i \in \mathbb{N}$, $\bigcap(X - N_i) \in \mathcal{U}$. Thus $\bigcap_{i\in\mathbb{N}}(X - N_i) \neq \emptyset$. But $\bigcap_{i\in\mathbb{N}}(X - N_i) = X - \bigcup_{i\in\mathbb{N}} N_i = X - X = \emptyset$. This is a contradiction. □

PROPOSITION 3.19. *Let $\mathcal{F}$ be a δ-filter on X and $A \subseteq X$.*
Then $\mathcal{F}_A = \{F \cap A \mid F \in \mathcal{F}\}$ is a δ-filter on A if and only if for each $F \in \mathcal{F}, A \cap F \neq \emptyset$.

Proof. Suppose that $\mathcal{F}_A = \{F \cap A \mid F \in \mathcal{F}\}$ is a δ-filter on A. Since every δ-filter is a filter, for each $F \in \mathcal{F}, A \cap F \neq \emptyset$. Conversely let for each $F \in \mathcal{F}, A \cap F \neq \emptyset$. Clearly, $\mathcal{F}_A$ is a nonempty collection of nonempty subsets of A. Take any $F_k \cap A \in \mathcal{F}_A$ for each $k \in K$, where K is a countable set and $F_k \in \mathcal{F}$. Then $\bigcap_{k\in K}(F_k \cap A) = (\bigcap_{k\in K} F_k) \cap A \in \mathcal{F}_A$. Furthermore, let $F \cap A \in \mathcal{F}_A$ and $F \cap A \subseteq M \subseteq A$. Since $F \subseteq F \cup M$, $F \cup M \in \mathcal{F}$. Furthermore $(F \cup M) \cap A = (F \cap A) \cup (M \cap A) = (F \cap A) \cup M = M$. Hence $M \in \mathcal{F}_A$. □

If $\mathcal{F}_A$ is a δ-filter on A, then $\mathcal{F}_A$ is a trace filter of $\mathcal{F}$ on A because δ-filter is a filter.

REMARK 3.20. (1) Let $\mathcal{B}$ be a δ-filter base for a δ-filter $\mathcal{F}$ on X and $A \subseteq X$. If $\mathcal{F}_A$ is a δ-filter on A, then $\{V \cap A \mid V \in \mathcal{B}\}$ is a δ-filter base for $\mathcal{F}_A$.

(2) Let $\mathcal{U}$ be an ultra δ-filter on X and $A \subseteq X$. Then $\mathcal{U}_A$ is a δ-filter on A if and only if $A \in \mathcal{U}$ because $\mathcal{U}_A$ is a δ-filter on A iff $A \cap F \neq \emptyset$ for all $F \in \mathcal{U}$ iff $A \in \mathcal{U}$ by Theorem 3.16. In case, $\mathcal{U}_A$ is an ultra δ-filter on A

PROPOSITION 3.21. *Let $\mathcal{B}$ be a δ-filter base for X and f a function of X into Y.*
Then $f(\mathcal{B})$ is a δ-filter base for Y, where $f(\mathcal{B}) = \{f(B) \mid B \in \mathcal{B}\}$.

Proof. Let $U_k \in f(\mathcal{B})$, $k \in K$, where K is a countable set. Then there is $B_k \in \mathcal{B}$ such that $U_k = f(B_k)$ for each $k \in K$. Since $\mathcal{B}$ is a δ-filter base for X, there is $B \in \mathcal{B}$ such that $B \subseteq \bigcap B_k$.
So $f(B) \subseteq f(\bigcap B_k) \subseteq \bigcap f(B_k) = \bigcap U_k$. Hence $f(\mathcal{B})$ is a δ-filter base for Y. □

THEOREM 3.22. *Let $\mathcal{F}$ be a δ-filter on X and $f : X \to Y$ a surjection. Then $\{f(F) \mid F \in \mathcal{F}\}$ is a δ-filter on Y.*

Proof. Let $\mathcal{G} = \{f(F) \mid F \in \mathcal{F}\}$. It is clear that each element of $\mathcal{G}$ is nonempty. Suppose first that $G \in \mathcal{G}$ and $G \subseteq H$. We shall show that $H \in \mathcal{G}$. Since $G \in \mathcal{G}$, there is $F \in \mathcal{F}$ such that $f(F) = G \subseteq H$. Then $F \subseteq f^{-1}(H)$, hence $f^{-1}(H) \in \mathcal{F}$. Since f is a surjection, $f(f^{-1}(H)) = H$. So $H \in \mathcal{G}$. Next suppose that $G_k \in \mathcal{G}$, $k \in K$, where K is a countable set. Then there is $F_k \in \mathcal{F}$ such that $G_k = f(F_k)$ for each $k \in K$. Since $\bigcap F_k \in \mathcal{F}$, $f(\bigcap F_k) \in \mathcal{G}$. Further, it follows from the first part of the proof that $\bigcap f(F_k) \in \mathcal{G}$ since $f(\bigcap F_k) \subseteq \bigcap f(F_k)$. This completes the proof. □

If $\mathcal{B}$ is a δ-filter base for $\mathcal{F}$, then $\mathcal{F}$ is denoted by $[\mathcal{B}]$.

Proposition 3.23. *Let $f : X \to Y$ be a function and $\mathcal{B}$ is a δ-filter base for an ultra δ-filter $\mathcal{U}$. Then $[f(\mathcal{B})]$ is an ultra δ-filter on Y.*

Proof. It is clear that each element of $[f(\mathcal{B})]$ is nonempty. Take any $V \subseteq Y$. Then $f^{-1}(V) \subseteq X$. Since $\mathcal{U}$ is an ultra δ-filter on X, $f^{-1}(V) \in \mathcal{U}$ or $f^{-1}(V)^c \in \mathcal{U}$. So there is $B \in \mathcal{B}$ such that $B \subseteq f^{-1}(V)$ or $B \subseteq f^{-1}(V)^c = f^{-1}(V^c)$. Further $f(B) \subseteq V$ or $f(B) \subseteq Y - V$. Since $f(B) \in f(\mathcal{B})$ and $f(\mathcal{B})$ is a δ-filter base on Y, $f(B) \in [f(\mathcal{B})]$. Hence $V \in [f(\mathcal{B})]$ or $Y - V \in [f(\mathcal{B})]$. Therefore $[f(\mathcal{B})]$ is an ultra δ-filter on Y. □

Proposition 3.24. *Let $f : X \to Y$ be a function and $\mathcal{B}$ a δ-filter base on Y. Then $f^{-1}(\mathcal{B}) = \{f^{-1}(V) \mid V \in \mathcal{B}\}$ is a δ-filter base on X if and only if $f^{-1}(V) \neq \emptyset$ for each $V \in \mathcal{B}$.*

Proof. Clearly, if $f^{-1}(\mathcal{B}) = \{f^{-1}(V) \mid V \in \mathcal{B}\}$ is a δ-filter base on X then $f^{-1}(V) \neq \emptyset$ for each $V \in \mathcal{B}$.
Next let $f^{-1}(V_k) \in f^{-1}(\mathcal{B})$, $k \in K$, where K is a countable set and $V_k \in \mathcal{B}$ for each $k \in K$. Since $\mathcal{B}$ is a δ-filter base on Y, there is $V \in \mathcal{B}$ such that $V \subseteq \bigcap V_k$. Then $f^{-1}(V) \subseteq f^{-1}(\bigcap V_k) = \bigcap f^{-1}(V_k)$ and $f^{-1}(V) \in f^{-1}(\mathcal{B})$. Hence $f^{-1}(\mathcal{B})$ is a δ-filter base on X. □

References

[1] D. W. Barnes and J. M. Mack, *An algebraic Introduction to Mathematical Logic,* Springer Verlag, New York, 1975.

[2] G. Birkhoff, *Moore-Smith Convergence in General Topology,* Ann. of Math. **3** (1937), 39-56.

[3] N. Bourbaki, *General Topology,* Part I, Hermann, Paris, 1966.

[4] H. Cartan, *Théorie des filtres; filtres et ultrafiltres,* C. R. Acad. Sc. Paris **205** (1937), 595-598 and 777-779.

[5] Charles O. Christenson and William L. Voxman, *Aspects of topology,* Marcel Dekker, INC., New York, 1997.

[6] B. A. Davey and H. A. Priestly, *Introduction to lattices and order,* Cambridge University Press, Cambridge, 1990.

[7] J. Dugundji, *Topology,* Allyn and Bacon., Boston, 1966.

[8] B. S. In, *A study on δ-ideals and δ-frames,* Ph. D. dissertation, Korea University, 1987.

[9] M. G. Kang, *A study on Lindelöf frames,* Ph. D. dissertation, Sogang University, 1993.

[10] R. H. Kasriel, *Undergraduate topology,* W. B. Saunders., London, 1971.

[11] J. L. Kelley, *General topology,* Springer Verlag., New York, 1955.
[12] S. O. Lee, *On countably approximating lattices,* J. of Korean Math. Soc. **25** (1988), 3-11.
[13] S. O. Lee, Y. J. Pi and J. H. Oh, *On the C-nets,* J. of Chungcheong Math. Soc. **23-1** (2010), 109-117.
[14] A. K. Misra, *A topological view of P-spaces,* Topology & Appl. **2** (1972), 349-362.
[15] E. H. Moore and H. L. Smith, *A General Theory of Limits,* American Journal of Mathematics. **44** (1922), 102-121.
[16] L. Nachbin, *Topology and Order,* D. Van Nostrand Co., Princeton, 1965.
[17] H. L. Royden, *Real Analysis,* Third edition, Macmillan, 1988.

*
Department of Mathematics
Chungbuk National University
Chungbuk 361-763, Republic of Korea
E-mail: solee@chungbuk.ac.kr

**
Department of Mathematics
Chungbuk National University
Chungbuk 361-763, Republic of Korea
E-mail: ojh507@hanmail.net

***
Department of Mathematics
Chungbuk National University
Chungbuk 361-763, Republic of Korea
E-mail: javesm@freechal.com