JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 24, No. 1, March 2011

ON COCYCLIC MAPS AND COCATEGORY

YEON SOO YOON*

ABSTRACT. It is known [5] that the concepts of C_k-spaces and those can be characterized using by the Gottlieb sets and the LS category of spaces as follows; A space X is a C_k-space if and only if the Gottlieb set $G(Z, X) = [Z, X]$ for any space Z with $cat\ Z \leq k$. In this paper, we introduce a dual concept of C_k-space and obtain a dual result of the above result using the dual Gottlieb set and the dual LS category.

1. Introduction

A based map $g : B \to X$ is called *cyclic* [10] if there exist a map $G : X \times B \to X$ such that $Gj \sim \nabla(1 \vee g)$, where $j : X \vee B \to X \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. The *Gottlieb set* $G(B, X)$ is the set of all homotopy classes of cyclic maps from B to X. The loop space ΩX of any space X has a homotopy type of an associative H-space. A 0-connected space X is filtered by the projective spaces of ΩX by a result of Milnor [8] and Stasheff [9];

$$\Sigma\Omega X = P^1(\Omega X) \hookrightarrow P^2(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$

For each k, let $e_k^X : P^k(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusion. We write $e^X = e_1^X : \Sigma\Omega X = P^1(\Omega X) \to X$. It was shown [1] that X is a T-space if and only if $e = e_1 : \Sigma\Omega X \to X$ is cyclic. We see that $e_\infty^X \sim 1_X : X \to X$. A connected space X is called a *C_k-space* if the inclusion $e_k^X : P^k(\Omega X) \to X$ is cyclic [5]. In fact, T-spaces and C_1-spaces are the same. We showed [5] that the concept of a C_k-space can be characterized using by the Gottlieb set and the LS category as follows; A space X is a C_k-space if and only if the Gottlieb set $G(Z, X) = [Z, X]$ for any space Z with $cat\ Z \leq k$. In this paper, we introduce a dual

Received February 16, 2011; Accepted March 04, 2011.
2010 Mathematics Subject Classification: Primary 55P45, 55P35.
Key words and phrases: cocyclic maps, cocategory of spaces.
This work was supported by Hannam University Research Fund, 2010.

concept of C_k-space and obtain a dual result of the above result using the dual Gottlieb set and the dual LS category.

2. DC_k-spaces

We now recall the following Ganea's theorem [4].

THEOREM 2.1. [4] *Let $k \geq 1$ be an integer or $k = \infty$ and assume that X is a 0-connected space. The category cat $X \leq k$ if and only if $e_k^X : P^k(\Omega X) \to X$ has a right homotopy inverse.*

In [3], Ganea introduced the concept of cocategory of a space as follows; Let X be a any space. Define a sequence of cofibrations

$$\mathcal{C}_k : X \xrightarrow{e'_k} F_k \xrightarrow{s'_k} B_k \ (k \geq 0)$$

as follows, let $\mathcal{C}_0 : X \overset{e'_0}{\to} cX \overset{s'_0}{\to} \Sigma X$ be the standard cofibration. Assuming $\mathcal{C}_k$ to be defined, let F'_{k+1} be the fibre of s'_k and $e''_{k+1} : X \to F'_{k+1}$ lift e'_k. Define F_{k+1} as the reduced mapping cylinder of e''_{k+1}, let e'_{k+1} : $X \to F_{k+1}$ is the obvious inclusion map, and let $B_{k+1} = F_{k+1}/e'_{k+1}(X)$ and $s'_{k+1} : F_{k+1} \to F_{k+1}/e_{k+1}(X)$ the quotient map.

DEFINITION 2.2. [3] *The cocategory of X, cocat X, is the least integer $k \geq 0$ for which there is a map $r : F_k \to X$ such that $r \circ e'_k \sim 1$. If there is no such integer, cocat $X = \infty$.*

The following remark can easily obtained from the above definition.

REMARK 2.3.
(1) *cocat $X \leq k$ if and only if $e'_k : X \to F_k$ has a left homotopy inverse.*
(2) *cocat $X = 0$ if and only if X is contractible.*

A based map $g : X \to B$ is cocyclic [10] if there is a map $\theta : X \to X \vee B$ such that $j\theta \sim (1 \times g)\Delta$, where $j : X \vee B \to X \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. *The dual Gottlieb set*, denoted $DG(X, B)$, is the set of all homotopy classes of cocyclic maps from X to B.

We can easily show that F_1 and $\Omega\Sigma X$ have the same homotopy type. A space X is called [11] a *co-T-space* if $e' = e'_1 : X \to \Omega\Sigma X$ is cocyclic. Thus we can define DC_k-spaces as follows;

DEFINITION 2.4. *A space X is called a DC_k-space if the inclusion $e'_k : X \to F_k$ is cocyclic.*

Clearly, DC_1-spaces and co-T-spaces are the same.

The following theorem say that DC_k-spaces are closely related by the dual Gottlieb sets and cocategory of spaces.

THEOREM 2.5. *A space X is a DC_k−space if and only if $DG(X, Z) = [X, Z]$ for any space Z with cocat $Z \leq k$.*

Proof. Suppose X is a DC_k−space. Since $e'_k : X \to F_k$ is cocyclic, there is a map $\theta : X \to X \vee F_k$ such that $j\theta \sim (1 \times \theta)\Delta$, where $j : X \vee F_k \to X \times F_k$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Let Z be a space with *cocat* $Z \leq k$. Let $g : X \to Z$ be any map. Since *cocat* $Z \leq k$, there is a map $s : F_k \to Z$ such that $s \circ e'_k \sim 1_Z$. Interpreting F_k as a functor, we have the following homotopy commutative diagram;

$$\begin{array}{ccccc}
X & \xrightarrow{\quad g \quad} & Z & & \\
\downarrow{\scriptstyle e'_k} & & \downarrow{\scriptstyle e'_k} & \searrow{\scriptstyle 1} & \\
F_k(X) & \xrightarrow{F_k(g)} & F_k(Z) & \xrightarrow{\quad s \quad} & Z.
\end{array}$$

Also, we consider the following homotopy commutative diagram;

$$\begin{array}{ccccccc}
X \times X & \xrightarrow{(1 \times e'_k)} & X \times F_k(X) & \xrightarrow{(1 \times F_k(g))} & X \times F_k(Z) & \xrightarrow{(1 \times s)} & X \times Z \\
\Delta\uparrow & & j\uparrow & & j\uparrow & & j\uparrow \\
X & \xrightarrow{\quad \theta \quad} & X \vee F_k(X) & \xrightarrow{(1 \vee F_k(g))} & X \vee F_k(Z) & \xrightarrow{(1 \vee s)} & X \vee Z.
\end{array}$$

Thus we have a map $\phi = (1 \vee s)(1 \vee F_k(g))\theta : X \to X \vee Z$ such that $j\phi \sim (1 \times g)\Delta$, where $j : X \vee Z \to X \times Z$ is the inclusion. Thus $g : X \to Z$ is cocyclic. On the other hand, we assume that for any space Z with *cocat* $Z \leq k$, $DG(X, Z) = [X, Z]$. It is well known [3] that if $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, then *cocat* $F \leq$ *cocat* $E + 1$. From the fact that $F_k \simeq F'_k \to F_{k-1} \xrightarrow{s'_{k-1}} B_{k-1}$ is a fibration, we know that *cocat* $F_k \leq$ *cocat* $F_{k-1} + 1$. Then we have, by induction, *cocat* $F_k \leq k$. Thus we know, by our assumption, that $e'_k : X \to F_k$ is cocyclic and X is a DC_k-space. □

It is shown [2] that *cocat* $Z \leq 1$ if and only if Z can be dominated by a loop space. Thus we have the following corollary.

COROLLARY 2.6. [11] *A space X is a co-T-space if and only if $DG(X, \Omega B) = [X, \Omega B]$ for any space B.*

It is well known fact [7] that a space X is a co-H-spaces if and only if $1 : X \to X$ is cocyclic. Moreover, it is also known [10] that if $f : X \to Y$

is cocyclic and $g : Y \to Z$ is any map, then $gf : X \to Z$ is cocyclic. Thus we have the following corollary from the definition of cocategory and the above theorem.

COROLLARY 2.7.
(1) *If X is a DC_m−space, then X is a DC_n−space for any $n < m$.*
(2) *If X is a DC_k−space and cocat $X = k$, then X is a co-H-space.*

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Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: yoon@hannam.ac.kr