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EXISTENCE OF MULTIPLE SOLUTIONS OF A SEMILINEAR BIHARMONIC PROBLEM WITH VARIABLE COEFFICIENTS

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Abstract. We obtain multiplicity results for the biharmonic problem with a variable coefficient semilinear term. We show that there exist at least three solutions for the biharmonic problem with the variable coefficient semilinear term under some conditions. We obtain this multiplicity result by applying the Leray-Schauder degree theory.

1. Introduction and statement of main result

Let Ω be a bounded domain in R^n with the smooth boundary $\partial\Omega$. Let $b(x)$ be *Hölder* continuous in Ω. Let $c \in R$, $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. In this paper we consider the multiplicity of the solutions for the following biharmonic equation with the variable coefficient semilinear term and the Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = b(x)u^+ + s\psi_1(x) \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where Δ is the Laplace operator and ψ_1 is the positive eigenfunction of $\Delta + c\Delta - b(x)$ with Dirichlet boundary condition. Choi and Jung [3] showed that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.2}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two solutions when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$, $s < 0$ and when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$, $s > 0$. They obtained these

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results by using the variational reduction method. They [5] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.2) has at least three nontrivial solutions by using degree theory. Tarantello [10] also studied the jumping problem

$$\Delta^2 u + c\Delta u = b((u+1)^+ - 1) \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

She show that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.3) has at least two solutions, one of which is a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [8] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then (1.3) has at least four solutions by the variational linking theorem and Leray-Schauder degree theory. Let $\lambda_k (k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k (k = 1, 2, \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \ \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem

$$\Delta^2 u + c\Delta u = \mu u \qquad \text{in } \Omega,$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \ \partial\Omega$$

has also infinitely many eigenvalues $\mu_k = \lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. We note that

$$\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) \leq \lambda_3(\lambda_3 - c) < \cdots .$$

The eigenvalue problem

$$\Delta^2 u + c\Delta u - b(x)u = \Lambda u \qquad \text{in } \Omega, \tag{1.4}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \ \partial\Omega$$

has also infinitely many eigenvalues Λ_k, $k \geq 1$, and ψ_k, $k \geq 1$ the corresponding eigenfunctions. We assume that the eigenfunctions are normalized with respect to H inner product (the space H is introduced in section 2). Standard eigenvalue theory gives that

$$\Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \cdots , \qquad \Lambda_k \to +\infty \qquad \text{as } \ k \to +\infty,$$
$$\psi_1(x) > 0 \qquad \text{in } \ \Omega.$$

Our main results are as follows:

THEOREM 1.1. *Let $c < \lambda_1$ and $b(x) < \lambda_1(\lambda_1 - c)$, $n \geq 1$. Then there exists $s_0 < 0$ such that for any s with $0 > s \geq s_0$ (1.1) has at least three solutions, one of which is a positive solution.*

For the proof of the main result we use the Leray-Schauder degree theory on the Hilbert space H (H will be introduced in section 2). The outline of the proof is that: In section 2 we investigate a priori estimate of the solutions of (1.1) and the non solvability condition. In section 3 we prove Theorem 1.1.

2. *A priori* estimate

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \psi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\Lambda_k| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\Lambda_k| h_k^2]^{\frac{1}{2}}.$$

Since $\Lambda_k \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u - b(x)u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.
Now we investigate the non solvability condition for (1.1):

LEMMA 2.1. *Assume that $\lambda_1 < c < \lambda_2$ and $b(x) < \lambda_1(\lambda_1 - c)$. Then we have:*
(i) If $s < 0$, then (1.1) has no solution.
(ii) If $s = 0$, then (1.1) has only trivial solution.

Proof. The conditions $\lambda_1 < c < \lambda_2$ and $b(x) < \lambda_1(\lambda_1 - c)$ imply that $\Lambda_1 > 0$ and $b(x) + \Lambda_1 \leq 0$. We rewrite (1.1) as

$$(\Delta^2 + c\Delta - b(x) - \Lambda_1)u = -\Lambda_1 u^+ + (b(x) + \Lambda_1)u^- + s\psi_1(x). \quad (2.1)$$

Taking the inner product of both sides of (2.1), we have

$$\begin{aligned} 0 &= ((\Delta^2 + c\Delta - b(x) - \Lambda_1)u, \psi_1(x)) \\ &= (-\Lambda_1 u^+ + (b(x) + \Lambda_1)u^- + s\psi_1(x), \psi_1(x)). \end{aligned} \quad (2.2)$$

It follows that from (2.2),

$$\begin{aligned} 0 &= ((\Delta^2 + c\Delta - b(x) - \Lambda_1)u, \psi_1(x)) \\ &= (-\Lambda_1 u^+ + (b(x) + \Lambda_1)u^- + s\psi_1(x), \psi_1(x)) \leq s. \end{aligned}$$

If $s < 0$, the left hand side of (2.2) is 0 and the right hand side of (2.2) is negative. Thus (1.1) has no solution. If $s = 0$, then the only possibility to hold the above equation is $u = 0$. □

LEMMA 2.2. *(a priori bound) Assume that $\Lambda_1 > \epsilon > 0$, $b(x) + \Lambda_1 \leq -\epsilon < 0$ and $b(x)$ is bounded in Ω. Then there exist a constant $C' > 0$ and $s_1 > 0$ such that if u is a solution of (1.1) with $0 < s \leq s_1$, then $\|u\| \leq C'$.*

Proof. From (2.2) we have

$$s = ((\Lambda_1 u^+ - (b(x) + \Lambda_1)u^-, \psi_1(x)).$$

Since $M \geq |(\Lambda_1 u^+ - (b(x) + \Lambda_1)u^-| \geq \epsilon|u_n|$ with $M = \sup\{|\Lambda_1|, |b(x) + \Lambda_1|\}$, we have

$$s \geq \epsilon \int_\Omega |u|\psi_1(x) \geq \epsilon| \int_\Omega u\psi_1(x)|.$$

Thus if u is a solution of (1.1), we have

$$|(u, \psi_1(x))| \leq \frac{1}{\epsilon} s, \tag{2.6}$$

where $s \geq 0$. We argue by contradiction. Suppose that there exists a sequence (u_n, s_n) such that $s_n \geq 0$, s_n is bounded, $\|u_n\| \to \infty$ and u_n satisfy the equations

$$(\Delta^2 + c\Delta - b(x) - \Lambda_1)u_n = -\Lambda_1 u_n^+ + (b(x) + \Lambda_1)u_n^- + s_n\psi_1(x).$$

Let $v_n = \frac{u_n}{\|u_n\|}$. By the compactness of $(\Delta^2 + c\Delta - b(x) - \Lambda_1)^{-1}$, there exists v such that $v_n \to v$. v satisfies $\|v\| = 1$ and

$$(\Delta^2 + c\Delta - b(x) - \Lambda_1)v + \Lambda_1 v^+ - (b(x) + \Lambda_1)v^- = 0. \tag{2.7}$$

Since, from (1.1), we have

$$(\Delta^2 + c\Delta)v_n = b(x)v_n^+ + s_n \frac{\psi_1(x)}{\|u_n\|},$$

(2.6) with u_n instead of u and the boundedness of s_n implies that

$$|(v_n, \psi_1(x))| \leq \frac{1}{\epsilon\|u_n\|}(s_n) \to 0 \qquad \text{as} \quad n \to \infty.$$

So we have that $|(v, \psi_1(x))| = 0$. By (2.7), we obtain

$$\int_\Omega (-\Lambda_1 v^+ + (b(x) + \Lambda_1)v^-)\psi_1(x) = 0. \tag{2.8}$$

Since $\Lambda_1 v^+ - (b(x) + \Lambda_1)v^- \geq \epsilon|v|$ and $\psi_1(x) > 0$, the only possibility to hold (2.8) is that $v = 0$, which is impossible, since $\|v\| = 1$. Thus we prove the lemma. $\square$

3. Proof of Theorem 1.1

Throughout this section we assume that $\Lambda_1 > 0$, $b(x) + \Lambda_1 < 0$ and $b(x)$ is bounded in Ω.

LEMMA 3.1. *Assume that $\lambda_1 < c < \lambda_2$, $b(x) < \lambda_1(\lambda_1 - c)$ and $b(x)$ is bounded in Ω. Then there exist a constant $R' > 0$ (depending on C' which is introduced in Lemma 2.2) and $s_1 > 0$ such that the Leray-Schauder degree*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_{R'}(0), 0) = 0$$

for $R' > C'$ and $s \leq s_1$.

Proof. By Lemma 2.2, there exist a constant C and $s_1 > 0$ such that if u is a solution of (1.1) with s, $s \leq s_1$, then $\|u\| \leq C'$. Let us choose R' such that $R' > C'$. By Lemma 2.2, (1.1) has no solution when $s < 0$. Let us choose $s_* < 0$ such that (1.1) has no solution with s_*. Then the Leray-Schauder degree $d_{LS}(u-(\Delta^2+c\Delta)^{-1}(b(x)u^+ + s_*\psi_1(x)), B_{R'}(0), 0) = 0$. Since the Leray-Schauder degree is invariant under a homotopy, we have that the Leray-Schauder degree

$$\begin{aligned}
&d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_{R'}(0), 0)\\
&= d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x) + \lambda(s_* - s)\psi_1(x)), B_{R'}(0), 0)\\
&= d_{LS}(u - (u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s_*\psi_1(x)), B_{R'}(0), 0) = 0,
\end{aligned}$$

where $0 \leq \lambda \leq 1$ and $0 \leq s \leq s_1$. Thus we prove the lemma. $\square$

LEMMA 3.2. *Assume that $\lambda_1 < c < \lambda_2$ and $b(x) < \lambda_1(\lambda_1 - c)$. Then there exist $s_1 > 0$ and a small number $\eta' > 0$ such that for any s with $0 < s \leq s_1$ the Leray-Schauder degree*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_{\eta'}(y_2), 0) = 1,$$

where y_2 is the unique positive solution of the linear problem

$$(\Delta^2 + c\Delta)u = s\psi_1(x) \qquad \text{in } \Omega, \tag{3.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \textit{on } \partial\Omega.$$

Proof. The linear problem (3.1) has a unique solution y_2. (3.1) can be rewritten as

$$(\Delta^2 + c\Delta - b(x))u = s\psi_1(x) - b(x)u, \qquad \text{in } \Omega, \tag{3.2}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

Thus y_2 is the unique solution of (3.2). We can also rewrite (1.1) as

$$(\Delta^2 + c\Delta - b(x))u = b(x)u^+ + s\psi_1(x) - b(x)u, \qquad \text{in } \Omega, \tag{3.3}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

We claim that y_2 is positive. In fact, let u be a solution of (1.1). Since $b(x)u^+ + s\psi_1(x) - b(x)u \le s\psi_1(x) - b(x)u$ and the operator $\Delta^2 + c\Delta - b(x)$ is positive, $u \le y_2$. Since $u_2 = \frac{s}{\Lambda_1}\psi_1(x)$ with $s > 0$ is a positive solution of (1.1), we get that $u_2 \le y_2$. Thus y_2 is positive. Let K be the closure of $(\Delta^2 + c\Delta - b(x))^{-1}(\bar{B})$, where $\bar{B}$ is the closed unit ball in $L_2(\Omega)$. Let u be a solution of (1.1) which is different from the positive solution u_2 of (1.1). Since y_2 is positive, we can take $\eta' < \max|u_2(x) - y_2(x)|$ such that the ball $B_{\eta'}(y_2)$ with center y_2 and radius η' does not contain u_2. Let us write $u = y_2 + v$ and $\|v\| = \eta'$. Then v satisfies the equation

$$(\Delta^2 + c\Delta - b(x))v = b(x)(y_2 + v)^- + b(x)y_2 \tag{3.4}$$

or

$$v = (\Delta^2 + c\Delta - b(x))^{-1}(b(x)(y_2 + v)^- + b(x)y_2). \tag{3.5}$$

Let us set $\beta' = \max b(x)$. From (3.5) we get

$$v \in 2\beta'\eta' K. \tag{3.6}$$

It follows from (3.4) that

$$v + (\Delta^2 + c\Delta - b(x))^{-1}(-b(x)y_2) = (\Delta^2 + c\Delta - b(x))^{-1} b(x)(y_2 + v)^-. \tag{3.7}$$

The function $w = \frac{v}{\eta'}$ has the properties $\|w\| = 1$ and $w \in 2\beta' K$. Since w is in compact set and different from zero and since $b(x)$ is not eigenvalue and $b(x) < 0$, $\inf_w \|w + \frac{1}{\eta'}(\Delta^2 + c\Delta - b(x))^{-1}(-b(x)y_2))\| = a' > 0$. Thus we get the estimate of the norm of the left hand side

$$\|v + (\Delta^2 + c\Delta - b(x))^{-1}(-b(x)y_2)\| \ge a'\eta'.$$

By Lemma 1 of [7], there exists a modulus of continuity $\delta(t)$ with $\delta(t) \to 0$ as $t \to 0$ such that $v \in K$ and $y_2 > 0$ satisfies $\|(tv + y_2)^-\| \le t\delta(t)$. It follows from (3.6) that

$$\|(v + y_2)^-\| \le 2\beta'\eta'\delta(2\beta'\eta').$$

On the other hand, we get the estimate of the norm of the right hand side of (3.7)

$$\begin{aligned}&(\Delta^2 + c\Delta - b(x))^{-1}b(x)(y_2+v)^- \\ &\quad \le \frac{\beta'}{\Lambda_1}(\Delta^2 + c\Delta - b(x))^{-1}\delta((\Delta^2 + c\Delta - b(x))^{-1}).\end{aligned}$$

We can choose $\eta' > 0$ so small that the right hand side is $< a'\eta'$ and $B_{\eta'}(y_2) \cap \{u_2\} = \emptyset$. Thus for this value of η', there is no solution of (1.1) of the form $u = y_2 + v$ with $\|v\| = \eta'$. That is,

$$u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)) \neq 0 \qquad \text{on} \ \ \partial B_{\eta'}(y_2).$$

We apply the similar argument to the equation

$$\begin{aligned}(\Delta^2 + c\Delta - b(x))u = \lambda b(x)u^- + (\lambda - 1)b(x)y_2 + s\psi_1(x),\\ 0 \le \lambda \le 1, \qquad \text{in} \ \ H,\end{aligned} \tag{3.8}$$

where u is of the form $u = y_2 + v$. Let u be a solution of the form $u = y_2 + v$ with $\|v\| = \eta'$. When $\lambda = 1$, (4.7) is equal to (1.1), while for any λ the function v satisfies the equation

$$v = (\Delta^2 + c\Delta - b(x))^{-1}(\lambda b(x)(y_2+v)^- + \lambda b(x)y_2). \tag{3.9}$$

From (4.8) we obtain

$$v \in 2\beta'\lambda\eta' K.$$

It follows from (3.9) that

$$\begin{aligned}&v + (\Delta^2 + c\Delta - b(x))^{-1}\lambda(-b(x)y_2)\\ &\quad = (\Delta^2 + c\Delta - b(x))^{-1}\lambda b(x)(y_2+v)^-.\end{aligned} \tag{3.10}$$

If w is the function $w = \frac{v}{\eta'}$, then $\inf_w \|w + \frac{1}{\eta'}(\Delta^2 + c\Delta - b(x))^{-1}\lambda(-b(x)y_2))\| = b' > 0$. Thus we get the estimate of the norm of the left hand side of (3.10)

$$\|v + (\Delta^2 + c\Delta - b(x))^{-1}\lambda(-b(x)y_2)\| \ge b'\eta'.$$

By a modulus of continuity $\delta(t)$, we get the estimate of the norm of the right hand side of (3.10)

$$\begin{aligned}\|(\Delta^2 + c\Delta - b(x))^{-1}\lambda b(x)(y_2+v)^-\| &\le \frac{\beta'}{\Lambda_1}2\beta'\lambda\eta'\delta(2\beta'\lambda\eta')\\ &\le \frac{\beta'}{\Lambda_1}2\beta'\lambda\eta'\delta(2\beta'\lambda\eta').\end{aligned}$$

We can choose η' so small that the right hand side of (3.10) is $< b\eta'$ and $B_{\eta'}(y_2) \cap \{u_2\} = \emptyset$. Thus for this value of η' there is no solution of (1.1) of the form $u = y_2 + v$ with $\|v\| = \eta'$. That is,

$$u - (\Delta^2 + c\Delta - b(x))^{-1}(\lambda b(x)u^- + (\lambda - 1)b(x)y_2 + s\psi_1(x)) \neq 0 \qquad \text{on} \quad \partial B_{\eta'}(y_2).$$

Since the Leray-Schauder degree is invariant under a homotopy, we have

$$\begin{aligned} & d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_{\eta'}(y_2), 0) \\ &= d_{LS}(u - (\Delta^2 + c\Delta - b(x))^{-1}(\lambda b(x)u^- \\ &\qquad\qquad + (\lambda - 1)b(x)y_2 + s\psi_1(x)), B_{\eta'}(y_2), 0) \\ &= d_{LS}(u - (\Delta^2 + c\Delta - b(x))^{-1}(-b(x)y_2 + s\psi_1(x)), B_{\eta'}(y_2), 0) \\ &= d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u), B_{\eta'}(0), 0) \end{aligned}$$

The equation

$$u - (\Delta^2 + c\Delta)^{-1}(b(x)u) = \sigma u \tag{3.11}$$

is equivalent to

$$(\Delta^2 + c\Delta)u = rb(x)u, \text{ where } r = \frac{1}{1 - \sigma}. \tag{3.12}$$

We note that $\sigma < 0$ corresponds to $0 < r < 1$. we consider the eigenvalue problem

$$(\Delta^2 + c\Delta)u = r\lambda_1(\lambda_1 - c)\frac{b(x)}{\lambda_1(\lambda_1 - c)}u.$$

Since $\frac{b(x)}{\lambda_1(\lambda_1 - c)} > 1$, $r_k(\lambda_1(\lambda_1 - c)) < \lambda_k(\lambda_k - c)$, from which it follows that

$$r_k > \frac{\lambda_k(\lambda_k - c)}{\lambda_1(\lambda_1 - c)}.$$

Thus $r_1 > 1$ and $r_k < 0$, $k \geq 2$, which means that $\sigma_k = \frac{r_k - 1}{r_k} > 0$, $k \geq 1$. Thus

$$u - (\Delta^2 + c\Delta)^{-1}(b(x)u) = \sigma u$$

has only positive eigenvalues, which implies that

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u), N_1, 0) = +1.$$

Thus we prove the lemma. □

LEMMA 3.3. *Assume that $\lambda_1 < c < \lambda_2$ and $b(x) < \lambda_1(\lambda_1 - c)$. Then there exist $s_1 > 0$ and a small number $\nu > 0$ such that for any s with $0 < s \leq s_1$ the Leray-Schauder degree*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\nu(u_2), 0) = 1,$$

where $u_2 = \frac{s}{\Lambda_1}\psi_1(x)$ *is a positive solution of (1.1).*

Proof. The function $u_1 = \frac{s}{\Lambda_1}\psi_1(x)$ is a positive solution. Since the solutions of (1.1) is discrete, we can choose a small number $\tau > 0$ such that $B_\tau(u_1)$ does not contain the other solutions of (1.1) except u_1. Let $u \in B_\tau(u_1)$. Then u can be written as $u = u_1 + w$, $\|w\| < \tau$. Then the Leray-Schauder degree

$$\begin{aligned} d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\tau(u_1), 0) \\ = d_{LS}(u - u_1, B_\tau(u_1), 0) \\ = d_{LS}(u, B_\tau(0), 0) = 1 \end{aligned}$$

because $(\Delta^2 + c\Delta)u = b(x)u^+ + s\psi_1(x)$ has only one solution $u = u_1$ in $B_\tau(u_1)$. □

PROOF OF THEOREM 1.1

By Lemma 3.1, there exists a large number $R' > 0$ (depending on C') and $s_1 > 0$ such that the Leray-Schauder degree

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_{R'}(0), 0) = 0$$

for $R' > C'$ and $s \le s_1$. By Lemma 3.2, there exist $s_1 > 0$ and a small number $\eta' > 0$ such that for any s with $0 < s \le s_1$, the Leray-Schauder degree

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_{\eta'}(y_2), 0) = +1,$$

where y_2 is a unique positive solution of (3.2). By Lemma 3.3, there exist $s_1 > 0$ and a small number ν such that for any s with $0 < s \le s_1$ the Leray-Schauder degree

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\nu(u_2), 0) = 1,$$

where u_2 is the positive solution of (1.1). Thus the Leray-Schauder degree in the region $B_{R'}(0)\backslash\{B_{\eta'}(y_2) \cup B_\nu(u_2)\}$ is -2, so there exists the third solution in the region $B_{R'}(0)\backslash\{B_{\eta'}(y_2) \cup B_\nu(u_2)\}$. Therefore there exists at least three solutions of (1.1), one of which is a positive solution. Thus we complete the proof. □

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