

ON SPECTRAL CONTINUITIES AND TENSOR PRODUCTS OF OPERATORS

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ABSTRACT. Let T be a bounded linear operator on a complex Hilbert space \mathcal{H} . An operator T is called class A operator if $|T^2| \geq |T|^2$, and is called class $A(k)$ operator if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$. In this paper, we show that σ is continuous when restricted to the set of class A operators and consider the tensor products of class $A(k)$ operators.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on a complex Hilbert space \mathcal{H} . Recall ([1], [3], [5], [7]) that an operator $T \in \mathcal{L}(\mathcal{H})$ is called p -hyponormal if

$$(T^*T)^p \geq (TT^*)^p \text{ for } p \in (0, 1].$$

If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. It is well known that q -hyponormal operators are p -hyponormal for $p \leq q$. An operator T is called *paranormal* if $\|T^2x\| \geq \|Tx\|^2$ for all unit vector $x \in \mathcal{H}$, and T is called *normaloid* if $\|T^n\| = \|T\|^n$ for $n \in \mathbb{N}$ (equivalently, $\|T\| = r(T)$, the spectral radius of T). Following ([10]) and ([9]) we say that $T \in \mathcal{L}(\mathcal{H})$ belongs to *class A* if

$$|T^2| \geq |T|^2.$$

It is well known that for $p \in (0, 1]$

$$\{p\text{-hyponormal}\} \subset \{\text{class } A\} \subset \{\text{paranormal}\} \subset \{\text{normaloid}\}.$$

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An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to *class* $A(k)$ for $k > 0$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2.$$

In this paper, we show that σ is continuous when restricted to the set of class A operators and consider the tensor products of class $A(k)$ operators.

2. Spectral continuity of class A operators

Let $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ be a polar decomposition, where U is a partial isometry with initial and final spaces $\overline{\text{ran}T^*}$ and $\overline{\text{ran}T}$, respectively. Note that if $T \in \mathcal{L}(\mathcal{H})$ then $\ker T = \ker |T|^\alpha$ for every $\alpha > 0$. Thus if $T = U|T|$ is a p -hyponormal operator then $\ker(|T|^{2p}) \subseteq \ker(|T^*|^{2p})$, so that $\ker T \subseteq \ker T^*$, which implies $\overline{\text{ran}T} \subseteq \overline{\text{ran}T^*}$. Thus, in the polardecomposition $T = U|T|$, the operator U can be extended to an isometry from \mathcal{H} to \mathcal{H} .

Let \mathfrak{G} denote the set, equipped with the Hausdorff metric, of all compact subsets of \mathbb{C} . If \mathfrak{U} is a unital Banach algebra then the spectrum can be viewed as a function $\sigma : \mathfrak{U} \rightarrow \mathfrak{G}$, mapping each $T \in \mathfrak{U}$ to its spectrum $\sigma(T)$. It is well known that the function σ is upper semicontinuous and that in noncommutative algebras, σ does have points of discontinuity. The work of J. Newburgh ([17]) contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel ([4]) have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^* -algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of points of spectral continuity and of classes \mathfrak{C} of operators for which σ becomes continuous when restricted to \mathfrak{C} . Recently Farenick and Lee ([8]) and Hwang and Lee ([12]) was considered the spectral continuity when restricted to certain subsets of the entire manifold of Toeplitz operators. The set of normal operators is perhaps the most immediate in the latter direction: σ is continuous on the set of normal operators. As noted in solution 105 of Hilbert space problem book, Newburgh's argument uses the fact that the inverses of normal resolvents are normaloid. This argument can be easily extended to the set of hyponormal operators because the inverses of hyponormal resolvents are also hyponormal and hence normaloid. Although class A operators are normaloid, class A operator is not translation-invariant. Thus the arguments of Newburgh cannot apply to show that σ is continuous when restricted to the set of class A operators. In this section, using the results of Hwang and Lee ([12]) and

Cho and Yamazaki ([6]), we show that σ is continuous when restricted to the set of class A operators.

LEMMA 2.1. ([12], Theorem) The spectrum σ is continuous on the set of all p -hyponormal operators.

LEMMA 2.2. ([6], Theorem 2.1, 2.2) Let $T = U|T|$ be the polar decomposition of a class A operator. Then

(i) The polar decomposition of \hat{T} is given by

$$\hat{T} = WU|T^2|^{\frac{1}{2}}$$

is hyponormal, where $|T||T^*| = W | |T||T^*| |$ is the polar decomposition.

(ii) $\sigma(\hat{T}) = \sigma(T)$.

LEMMA 2.3. ([6], Theorem A) Let A and B are positive operators. Then for each $p \geq 0$ and $r \geq 0$

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Rightarrow A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Using the above lemmas we can show that following result:

THEOREM 2.4. *The spectrum σ is continuous when restricted to the set of class A operators.*

Proof. Suppose that T and T_n for $n \in \mathbb{N}$ are class A operators and $T = U|T|$ and $T_n = U_n|T_n|$ are polar decompositions of T and T_n , respectively. Suppose that T_n converges to T . By Lemma 2.1 and Lemma 2.2, it is sufficient to show that $|T|U|T|$ is semi-hyponormal. We claim that

$$(2.1) \quad (T^*|T|^2T)^{\frac{1}{2}} \geq |T|^2 \Leftrightarrow (|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2.$$

Let $T^* = U^*|T^*|$ be the polar decomposition of T^* . Firstly, suppose that $(T^*|T|^2T)^{\frac{1}{2}} \geq |T|^2$. Then

$$\begin{aligned} (|T^*||T|^2|T^*|)^{\frac{1}{2}} &= UU^* (|T^*||T|^2|T^*|)^{\frac{1}{2}} UU^* \\ &= U (U^*|T^*||T|^2|T^*|U)^{\frac{1}{2}} U^* \\ &= U (T^*|T|^2T)^{\frac{1}{2}} U^* \\ &\geq U|T|^2U^* = |T^*|^2. \end{aligned}$$

Secondly, suppose that $(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2$. Then

$$\begin{aligned} (T^*|T|^2T)^{\frac{1}{2}} &= (U^*|T^*||T|^2|T^*|U)^{\frac{1}{2}} \\ &= U^* (|T^*||T|^2|T^*|)^{\frac{1}{2}} U \\ &\geq U^*|T^*|^2U = |T|^2. \end{aligned}$$

By Lemma 2.3, we have

$$(2.2) \quad |T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}} = (|T|U|T|^2U^*|T|)^{\frac{1}{2}}.$$

Therefore by (2.1) and (2.2), $|T|U|T|$ is semi-hyponormal. \square

3. Tensor products

Given non-zero $T, S \in \mathcal{L}(\mathcal{H})$, let $T \otimes S$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{H}$. The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in \mathcal{L}(\mathcal{H})$, but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products (see [18, pp. 623]); again, whereas $T \otimes S$ is normal if and only if T and S are ([13], [20]), there exist paranormal operators T and S such that $T \otimes S$ is not paranormal ([2]). In [7], Duggal showed that for non-zero $T, S \in \mathcal{L}(\mathcal{H})$, $T \otimes S \in \mathcal{H}(p)$ if and only if $T, S \in \mathcal{H}(p)$. This result was extended to p -quasihyponormal operators and class A operators in [15] and [14], respectively.

In this section we consider the tensor products of class $A(k)$ operators. We need the following famous inequality as a useful tool.

LEMMA 3.1 (Hölder-McCarthy inequality [16]). *Let $A \geq 0$. Then the following properties hold.*

- (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for $r > 1$ and unit vector $x \in \mathcal{H}$.
- (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for $r \in [0, 1]$ and $x \in \mathcal{H}$.

The following key lemma is due to J. Stochel [20]

LEMMA 3.2. ([20], Proposition 2.2) Let $A_1, A_2 \in \mathcal{L}(\mathcal{H}), B_1, B_2 \in \mathcal{L}(\mathcal{K})$ non-negative operators. If A_1 and B_1 are non-zero, then the following assertions are equivalent:

- (i) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.
- (ii) There exists $c > 0$ for which $A_1 \leq cA_2$ and $B_1 \leq c^{-1}B_2$.

THEOREM 3.3. *Let $T, S \in \mathcal{L}(\mathcal{H})$ be non-zero operators. Then $T \otimes S$ is class $A(k)$ operator if and only if one of the following holds:*

- (a) T and S are class $A(k)$ operators.
(b) $|T|^2 = 0$ or $|S|^2 = 0$.

Proof. By simple calculation we have

$$\begin{aligned}
& T \otimes S \text{ is class } A(k) \text{ operator} \\
\Leftrightarrow & \left[(T \otimes S)^* |T \otimes S|^{2k} (T \otimes S) \right]^{\frac{1}{k+1}} \geq |T \otimes S|^2 \\
\Leftrightarrow & \left[(T^* \otimes S^*) (|T|^{2k} \otimes |S|^{2k}) (T \otimes S) \right]^{\frac{1}{k+1}} - |T|^2 \otimes |S|^2 \geq 0 \\
\Leftrightarrow & (T^* |T|^{2k} T)^{\frac{1}{k+1}} \otimes (S^* |S|^{2k} S)^{\frac{1}{k+1}} - |T|^2 \otimes |S|^2 \geq 0 \\
\Leftrightarrow & \left[(T^* |T|^{2k} T)^{\frac{1}{k+1}} - |T|^2 \right] \otimes (S^* |S|^{2k} S)^{\frac{1}{k+1}} \\
& + |T|^2 \otimes \left[(S^* |S|^{2k} S)^{\frac{1}{k+1}} - |S|^2 \right] \geq 0.
\end{aligned}$$

Thus the sufficiency is easily proved. Conversely, suppose that $T \otimes S$ belongs to class $A(k)$. It suffices to show that if the statement (b) does not hold, then the statement (a) holds. Assume that $|S|^2$ and $|T|^2$ are non-zero operators. Without loss of generality, it is enough to show that T belongs to class $A(k)$. Since $T \otimes S$ is class $A(k)$ operator, we obtain

$$(T^* |T|^{2k} T)^{\frac{1}{k+1}} \otimes (S^* |S|^{2k} S)^{\frac{1}{k+1}} \geq |T|^2 \otimes |S|^2.$$

Therefore, by lemma 3.2, there exists a positive real number c for which

$$c(T^* |T|^{2k} T)^{\frac{1}{k+1}} \geq |T|^2 \text{ and } c^{-1}(S^* |S|^{2k} S)^{\frac{1}{k+1}} \geq |S|^2.$$

Consequently, for arbitrary $x, y \in \mathcal{H}$, using the lemma 3.1 we have

$$\begin{aligned}
\|T\|^2 &= \sup_{\|x\|=1} \langle |T|^2 x, x \rangle \\
&\leq \sup_{\|x\|=1} \langle c(T^* |T|^{2k} T)^{\frac{1}{k+1}} x, x \rangle \\
&\leq c \left(\sup_{\|x\|=1} \langle (T^* |T|^{2k} T) x, x \rangle \right)^{\frac{1}{k+1}} \\
&= c \left\| T^* |T|^{2k} T \right\|^{\frac{1}{k+1}} \\
&\leq c \|T\|^2.
\end{aligned}$$

and

$$\begin{aligned}
 \|S\|^2 &= \sup_{\|y\|=1} \langle |S|^2 y, y \rangle \\
 &\leq \sup_{\|y\|=1} \langle c^{-1} (S^* |S|^{2k} S)^{\frac{1}{k+1}} y, y \rangle \\
 &\leq c^{-1} \left(\sup_{\|y\|=1} \langle (S^* |S|^{2k} S) x, x \rangle \right)^{\frac{1}{k+1}} \\
 &= c^{-1} \left\| S^* |S|^{2k} S \right\|^{\frac{1}{k+1}} \\
 &\leq c \|S\|^2.
 \end{aligned}$$

Thus $c = 1$, and hence T is class $A(k)$ operator. \square

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